Analele științifice ale Universității „Al. I. Cuza" din Iași
Tomul XXXI, s. I a, Matematică, 1985, f. 3.

NONLINEAR INTEGRAL INEQUALITIES IN n INDEPENDENT VARIABLES

BY

ADRIAN CORDUNEANU

1. The integral inequality

$$u(x)\leqslant f(x)+\int_0^x g(x, s)u(s)\mathrm{d}s, \qquad x\geqslant 0 \tag{1}$$

where u, f are continuous nonnegative functions defined for $x\geqslant 0$, f is nondecreasing and g is continuous nonnegative for $x\geqslant s\geqslant 0$, was considered in last time by many authors. Here it is assumed that x and s are n-dimensional variables, i.e. $x=(x_1, x_2,\ldots,x_n)$, $s=(s_1, s_2,\ldots,s_n)$. The inequality $x\geqslant s$ is equivalent to $x_i\geqslant s_i$ for $i=\overline{1, n}$ and $\int_0^x \mathrm{d}s$ stands for $\int_0^{x_1}\mathrm{d}s_1\int_0^{x_2}\mathrm{d}s_2\ldots\ldots\int_0^{x_n}\mathrm{d}s_n$. The case when $g=g(x, s)$ is nondecreasing with respect to the first variable was recently studied by J. Conlan and C. L. Wang [1]; these authors call such a function *good kernel*. Let $x>0$ be fixed and denote $\varphi_x(s)=g(x, s)$ for $0\leqslant s\leqslant x$; we have from (1) that

$$u(t)\leqslant f(t)+\int_0^t \varphi_x(s)u(s)\mathrm{d}s, \quad 0\leqslant t\leqslant x \tag{2}$$

and it is well known (see for example [2], Theorem 1) that from (2) it follows

$$u(t)\leqslant f(t) \exp \left(\int_0^t \varphi_x(s)\, \mathrm{d}s\right), \qquad 0\leqslant t\leqslant x, \tag{3}$$

and particularly

$$u(x) \leqslant f(x) \exp \left(\int^x g(x, s)\mathrm{d}s\right). \tag{4}$$

Because $x>0$ was taken arbitrary, we conclude that under the used here conditons, the bound (4) holds for all $x\geqslant 0$. We can resume the above situation under the form of the

Proposition 1. *Suppose that all the functions in* (1) *are continuous nonnegative, f is nondecreasing and g is nondecreasing in the first variable. Then u satisfies for $x\geqslant 0$ the bound* (4).

If $f\geqslant 0$ is not nondecreasing, all the other hypotheses remaining true, puting

$$\hat{f}(x)=\sup f(s) \quad \text{for} \quad 0\leqslant s\leqslant x, \tag{5}$$

we obtain from (1) the cruder bound

$$u(x)\leqslant\hat{f}(x)\exp\left(\int_0^x g(x,s)ds\right),\quad x\geqslant 0 \tag{6}$$

In particular, from the inequality

$$u(x)\leqslant f(x)+g(x)\int_0^x h(s)u(s)ds,\quad x\geqslant 0, \tag{7}$$

where all the involved functions are continuous nonnegative for $x\geqslant 0$ and f, g are nondecreasing, it follows

$$u(x)\leqslant f(x)\exp\left(g(x)\int_0^x h(s)\,ds\right),\quad x\geqslant 0. \tag{8}$$

The inequality (7), in which g is not monotone but $g(x)\geqslant 1$, was considered in [2], where for the function u it is obtained the bound

$$u(x)\leqslant f(x)g(x)\exp\left(\int_0^x g(s)h(s)ds\right)\quad x\geqslant 0. \tag{9}$$

Of course, the bounds (8) and (9) are independent, even when g satisfies both the restrictions $g(x)\geqslant 1$ and g=nondecreasing. For example, if $g(0)>1$, the bound (8) is better than (9), at least for some $x\geqslant 0$.

Denoting

$$E(x)=\min\left\{g(x)\exp\left(\int_0^x g(s)h(s)\,ds\right),\quad \exp\left(g(x)\int_0^x h(s)\,ds\right)\right\} \tag{10}$$

we can state the following result, combining Theorem 1 of the paper [2] with the above remarks

Proposition 2. *Suppose that all the functions in* (7) *are continuous nonnegative and that* $f, g,$ *are nondecreasing,* $g(x)\geqslant 1$ *for* $x\geqslant 0$. *Then* u *satisfies the majoration*

$$u(x)\leqslant f(x)E(x),\quad x\geqslant 0. \tag{11}$$

Remark. If in (7) f and g are not monotone, we consider $\hat{f}$ given by (5) and the corresponding function $\hat{g}$. Substituting in (10) g by $\hat{g}$ and writting $\hat{E}(x)$ instead of $E(x)$, we obtain from (7) the bound

$$u(x)\leqslant\hat{f}(x)\hat{E}(x),\quad x\geqslant 0. \tag{12}$$

2. The nonlinear integral inequality

$$u(x)\leqslant f(x)+g(x)G\left(\int_0^x h(s)Q(u(s))\,ds\right),\quad x\geqslant 0, \tag{13}$$

was also considered by many authors (see the works [2] of C. C. Yeh and [3] of P. R. Beesack). Of course, we maintain the hypothesis of continuity and nonnegativity of the involved here functions, but we assume that G and Q belong to a class of functions, which seems not to be used in other similar papers on this subject.

Definition 1. *We say that a function* $p=p(u)$ *of one variable* $(u\geqslant 0)$ *belongs to the class* $\mathcal{F}$, *if* p *and* p' *are continuous nonnegative functions for* $u\geqslant 0$ *and if*

$$v^{-1}p(u)\leqslant p(v^{-1}u)\quad\text{for}\quad 0<v\leqslant 1,\quad u\geqslant 0 \tag{14}$$

We remark that a function $p\in\mathcal{F}$ is nondecreasing on the real semiaxis $u\geqslant 0$; an example of function $p\in\mathcal{F}$ is $p(u)=u^{\alpha}$ with $\alpha\geqslant 1$. In what follows, we shall use the function $H=H(u)$ given by

$$H(u)=\int_0^u ds/[Q(1+G(s))],\quad u\geqslant 0. \tag{15}$$

It is assumed in (15) that $Q(u)>0$ for $u>0$. We can state the following result, analogous to the Theorem 3 of [2] and to the Theorem A of [3], namely

Proposition 3. *In the inequality* (13), *let* f, g, h, u *be continuous for* $x\geqslant 0$ *functions of* n *independent variables and suppose that*

(i) $f=f(x)$ *is nondecreasing and satisfies one of the conditions* $0<f(x)\leqslant 1$ *or* $0\leqslant f(x)<1$

(ii) $g(x)\geqslant 1$, $h(x)\geqslant 0$, $u(x)\geqslant 0$ *and the functions* $G, Q\in\mathcal{F}$

(iii) Q *is submultiplicative and* $Q(u)>0$ *for* $u>0$

Then, from (13) *we obtain the bound*

$$u(x)\leqslant f(x)g(x)\left\{1+G\left[H^{-1}\left(\int_0^x h(s)Q(g(s))\,ds\right)\right]\right\}, \tag{16}$$

provided that

$$H\left(\int_0^x h(s)Q(g(s))\,ds\right)<H(\infty) \tag{17}$$

where H *is given by* (15).

Proof. We shall use the method of Yeh [2], supposing firstly that $0<f(x)\leqslant 1$. Taking into account that $G, Q\in\mathcal{F}$ and $g(x)\geqslant 1$, we deduce from (13) that

$$u(x)/f(x)\leqslant g(x)\left[1+G\left(\int_0^x h(s)Q(u(s)/f(s))\,ds\right)\right] \tag{18}$$

Denoting

$$r(x)=\int_0^x h(s)Q(u(s)/f(s))\,ds,\quad x\geqslant 0 \tag{19}$$

and using the notations $D_i=\partial/\partial x_i$, we have from (19), using the fact that Q is nondecreasing and submultiplicative

$$D_1D_2\ldots\ldots D_n r(x)=h(x)Q(u(x)/f(x))\leqslant h(x)Q(g(x))Q(1+G(r(x))) \tag{20}$$

from which we arrive to the inequality

$$D_n[D_1\ldots\ldots D_{n-1}r(x)/Q(1+G(r(x)))]\leqslant h(x)Q(g(x)) \tag{21}$$

Integrating the preceding inequality with respect to x_n, we obtain for $x\geqslant 0$

$$D_{n-1}[D_1\ldots\ldots D_{n-2}r(x)/Q(1+G(r(x)))]\leqslant\int_0^{x_n} h(x_1,\ldots,x_{n-1},s_n)\cdot Q(g(x_1,\ldots,x_{n-1},s_n))\,ds_n. \tag{22}$$

Starting from (22), after $n-1$ integrations with respect to $x_{n-1},\ldots, x_2, x_1$ we conclude that

$$H(r(x))\leqslant \int_0^x h(s)Q(g(s))\,ds, \tag{23}$$

for all $x\geqslant 0$ satisfying (17). Obviously, we have denoted by $H(\infty)$ the limit (finite or infinite) of the function $H=H(u)$ when $u\to\infty$. Now, the desired bound (16) follows from (18), (19) and

$$r(x)\leqslant H^{-1}(\int_0^x h(s)Q(g(s))\,ds). \tag{24}$$

If we suppose that f satisfies the restriction $0\leqslant f(x)<1$ then, fixing $x_0>0$ we have $f(x)+\varepsilon\leqslant 1$ for $0\leqslant x\leqslant x_0$ if $\varepsilon>0$ is sufficiently small. Together with (13), we consider for $0\leqslant x\leqslant x_0$ the inequality

$$u(x)\leqslant\tilde{f}(x)+g(x)G(\int_0^x h(s)Q(u(s))\,ds), \tag{25}$$

where $\tilde{f}(x)=f(x)+\varepsilon$ clearly satisfies the condition $0<\tilde{f}(x)\leqslant 1$. We can repeat the above argument, obtaining for $0\leqslant x\leqslant x_0$ a bound similar to (16), where in the right hand side it appears $\tilde{f}(x)$ instead of $f(x)$. For fixed x, we obtain (16) letting $\varepsilon\to 0$ in this inequality and because $x_0>0$ was taken arbitrary, the s ated conclusion holds and in this case.

Corollary 3.1. *Assume that the hypotheses of the Proposition 3 are fulfiled. If $H(\infty)=\infty$, then the bound (16) holds for all $x\geqslant 0$.*

Example 3.1. Consider the inequality

$$u(x)\leqslant f(x)+\int_0^x h(s)\,(u(s))^2 ds,\qquad x\geqslant 0. \tag{26}$$

where u, f, h are continuous nonnegative functions of n independent variables, defined for $x\geqslant 0$, $f=f(x)$ is nondecreasing and

$$0\leqslant f(x)<1\quad\text{or}\quad 0<f(x)\leqslant 1,\quad I=\int_0^\infty h(s)\,ds<1. \tag{27}$$

Then, from (26) it follows for $x\geqslant 0$

$$u(x)\leqslant f(x)[1+(\int_0^x h(s)\,ds)/(1-\int_0^x h(s)\,ds)], \tag{28}$$

and in particular

$$u(x)\leqslant Kf(x),\quad\text{where}\quad K=1+I(1-I)^{-1}. \tag{29}$$

To justify the above afirmations, it suffices to take in (13) $g(x)\equiv 1$, $G(u)=u$, $Q(u)=u^2$ and to make use of (16), the condition (17) being satisfie d for all $x\geqslant 0$.

Now, consider an inequality more general than (13), namel y

$$u(x)\leqslant f(x)+\int_0^x k(x, s)u(s)\,ds+g(x)G(\int_0^x h(s)Q(u(s))\,ds) \tag{30}$$

and denote by f_1 the function of n independent variables

$$f_1(x)=f(x)\exp(\int_0^x k(x, s)\,ds),\qquad x\geqslant 0 \tag{31}$$

and by g_1 the function

$$g_1(x)=g(x)\exp(\int_0^x k(x, s)\,ds),\qquad x\geqslant 0 \tag{32}$$

$H=H(u)$ is again the function given by (15).

Proposition 4. *Assume that in (30) the functions u, g, h, G, Q satisfy the same conditions as in the Proposition 3 and $f(x)\geqslant 0$ is continuous nondecreasing, $k(x, s)\geqslant 0$ is continuous and nondecreasing with respect to x. Assume also that $0<f_1(x)\leqslant 1$ or $0\leqslant f_1(x)<1$ for $x\geqslant 0$. Then we have from (30) the bound*

$$u(x)\leqslant f_1(x)g_1(x)\{1+G[H^{-1}(\int_0^x h(s)Q(g_1(s))\,ds)]\}, \tag{33}$$

provided that

$$H(\int_0^x h(s)Q(g_1(s))\,ds)<H(\infty). \tag{34}$$

Proof. Denoting by F the nondecreasing function

$$F(x)=f(x)+g(x)G(\int_0^x h(s)Q(u(s))\,ds), \tag{35}$$

we have from (30) that

$$u(x)\leqslant F(x)+\int_0^x k(x, s)u(s)\,ds,\quad x\geqslant 0, \tag{36}$$

and consequently, accordingly to the Proposition 1, it follows

$$u(x)\leqslant F(x)\exp(\int_0^x k(x, s)\,ds),\quad x\geqslant 0 \tag{37}$$

or, under the equivalent form

$$u(x)\leqslant f_1(x)+g_1(x)G(\int_0^x h(s)Q(u(s))\,ds). \tag{38}$$

Now, we are precisely in the same situation as in the Proposition 3 and thus the bound (33) is proved.

Corollary 4.1. *Assume that the continuous nonnegative function of n independent variables satisfies for $x\geqslant 0$ the inequality*

$$u(x)\leqslant\int_0^x \tilde{k}(x, s)u(s)ds+\tilde{g}(x)G(\int_0^x h(s)Q(u(s))ds) \tag{39}$$

where the functions h, G, Q satisfy the same restrictions as in the Proposition 4 and $\tilde{k}(x, s)\geqslant 0$, $\tilde{g}(x)\geqslant 0$ are continuous. Then it follows $u(x)\equiv 0$.

For the proof, it suffices to put $f(x)=0$, $k(x, s)=\sup\tilde{k}(t, s)$ for $0\leqslant t\leqslant x$ (s =fixed), $g(x)=\max\{1, \tilde{g}(x)\}$ for $x\geqslant 0$ and to make use of the obtained b ound (33).

Remark. The preceding Proposition 4 may be used to obtain a bound for the solution $u=u(x_1, x_2)$ of the equation

(40) $$\partial^2 u/\partial x_1 \partial x_2 = k(x_1, x_2)u + h(x_1, x_2)Q(u), \qquad x_1, x_2 \geq 0$$

with initial conditions $u(x_1, 0)=\varphi_1(x_1)$, $u(0, x_2)=\varphi_2(x_2)$ $(\varphi_1(0)=\varphi_2(0))$, where k, h are continuous for $x_1, x_2 \geq 0$ and $Q=Q(u)$ is a continuous function, whose restriction to the interval $[0, \infty)$ is of the class $\mathcal{F}$ and moreover, $|Q(u)| \leq Q(|u|)$ for every real u.

3. Recently, studying the boundedness of the solutions of a certain ordinary differential equation, E. H. Yang [4] obtained an interesting integral inequality for the functions of one variable. Our aim is to prove the generalization of Yang's result to the case when the function depends of n independent variables.

Consider the nonlinear inequality

(41) $$u(x) \leq f(x) + \sum_{i=1}^{m} \int_0^x g_i(x, s)\,(u(s))^{r_i}\,ds, \qquad x \geq 0$$

and suppose that

(α) $u(x) \geq 0$ and $f(x) \geq 1$ are continuous functions for $x \geq 0$, $f(x)$ being nondecreasing

(β) the nonnegative functions $g_i(x, s)$ defined for $x \geq s \geq 0$ are continuous and nondecreasing with respect to x, $i=\overline{1, m}$

(γ) $r_i \in (0, 1]$ for $i=\overline{1, m}$

Proposition 5. *Under the above conditions* (α), (β) *and* (γ), *we obtain from* (41) *the bound*

(42) $$u(x) \leq f(x) \prod_{i=1}^{m} G_i(x), \qquad x \geq 0,$$

where $G_1(x)$ *is given by*

(43) $$G_1(x) = \begin{cases} \exp\left(\int_0^x (g_1(x, s)\,ds\right), & r_1 = 1 \\ \left[1+(1-r_1)\int_0^x g_1(x, s)\,ds\right]^{\frac{1}{1-r_1}}, & 0 < r_1 < 1, \end{cases}$$

and $G_i(x)$ *for* $i=\overline{2, m}$ *are given by*

(44) $$G_i(x) = \begin{cases} \exp\left[\left(\prod_{k=1}^{i-1} G_k(x)\right) \int_0^x g_i(x, s)\,ds\right], & r_i = 1 \\ \left[1+(1-r_i)\left(\prod_{k=1}^{i-1} G_k(x)\right) \int_0^x g_i(x, s)\,ds\right]^{\frac{1}{1-r_i}}, & 0 < r_i < 1 \end{cases}$$

Proof. Firstly, we consider $m=1$ assuming that for $x \geq 0$ we have

(45) $$u(x) \leq f(x) + \int_0^x g_1(x, s)\,(u(s))^{r_1}\,ds.$$

If $r_1 = 1$, we make use of the Proposition 1 to conclude

(46) $$u(x) \leq f(x) \exp\left(\int_0^x g_1(x, s)\,ds\right), \qquad x \geq 0.$$

If $0 < r_1 < 1$, we fix $x > 0$ and denote $\varphi_x(s) = g_1(x, s)$ for $0 \leq s \leq x$, writing

(47) $$u(t) \leq f(t) + \int_0^t \varphi_x(s)\,(u(s))^{r_1}\,ds, \qquad 0 \leq t \leq x.$$

Now, applying a well known result of Bihari type ([2], pp. 317 ; [3] pp. 403), we can write for $0 \leq t \leq x$

(48) $$u(t) \leq f(t)\left[1+(1-r_1)\int_0^t \varphi_x(s)\,ds\right]^{\frac{1}{1-r_1}}$$

Using (46) and (48), in which we put $t=x$, we conclude that

(49) $$u(x) \leq f(x) G_1(x), \qquad x \geq 0,$$

where $G_1(x)$ is given by (43). The proof may be continued in the same way as in ([4], pp. 542), assuming that the inequality (42) holds for some $m=k$ and concluding that it remains true and for $m=k+1$.

Remark. If the hypotheses of the Proposition 5 are satisfied and all $r_i = 1$, using the Proposition 1 we have form (41) the bound

(50) $$u(x) \leq f(x) \exp\left(\sum_{i=1}^{m} \int_0^x g_i(x, s)\,ds\right), \qquad x \geq 0$$

which is considerably better that (42). Moreover, the condition $f(x) \geq 1$ may be replaced in this case by $f(x) \geq 0$. Consequently, the Proposition 5 is applicable only if in (41) there exist exponents $r_i < 1$. The principal conclusion which we can derive from Proposition 5 is that if all the functions $\int_0^x g_i(x, s)\,ds$ are bounded for $x \geq 0$, then the function u satisfying (41) is bounded for $x \geq 0$.

Finally, let us consider the more general inequality

(51) $$u(x) \leq f(x) + \sum_{i=1}^{m} \int_0^x g_i(x, s)\,(u(s))^{r_i}\,ds + g(x) G\left(\int_0^x h(s) Q(u(s))\,ds\right)$$

where u, f, g_i are as in the Proposition 5, but the functions G, Q belong to another class, different from those used in the Proposition 4 (see [3] and the comments given there about this class).

Definition 2. *We say that a function* $p=p(u)$ *of one variable* $(u \geq 0)$ *belongs to the class* F, *if* p *is continuous nonnegative and nondecreasing for* $u \geq 0$, p' *is continuous for* $u > 0$ *and*

(52) $$v^{-1} p(u) \leq p(v^{-1} u) \qquad \text{for} \quad v \geq 1, \quad u \geq 0$$

A typical example of function $p \in F$ is $p(u) = u^{\alpha}$, $\alpha \in (0, 1]$. In what follows, we shall use the notations

(53) $$\tilde{f}(x) = f(x) \prod_{i=1}^{m} G_i(x), \qquad \tilde{g}(x) = g(x) \prod_{i=1}^{m} G_i(x)$$

where $G_i(x)$ are given by (43)–(44) ; $H=H(u)$ is defined by (15).

Proposition 6. *Assume that in* (51) *the conditions* (α), (β), (γ) *of the Proposition 5 are satisfied*, $g(x) \geq 1$, $h(x) \geq 0$, G *and* Q *belong to the class* F, Q *is submultiplicative and* $Q(u) > 0$ *for* $u > 0$. *Then we have the bound*

(54) $$u(x)\leqslant \tilde{f}(x)\tilde{g}(x)\{1+G[H^{-1}(\int_0^x h(s)Q(\tilde{g}(s))\mathrm{d}s)]\}$$

provided that

(55) $$H(\int_0^x h(s)Q(\tilde{g}(s))\mathrm{d}s)<H(\infty).$$

Proof. Denoting

(56) $$F(x)=f(x)+g(x)G(\int_0^x h(s)Q(u(s))\mathrm{d}s), \quad x\geqslant 0,$$

we obtain from (51)

(57) $$u(x)\leqslant F(x)+\sum_{i=1}^{m}\int_0^x g_i(x,s)\,(u(s))^{r_i}\mathrm{d}s, \quad x\geqslant 0,$$

which, according to the Proposition 5, implies

(58) $$u(x)\leqslant F(x)\prod_{i=1}^{m} G_i(x), \quad x\geqslant 0.$$

From (56) and (58), it follows

(59) $$u(x)\leqslant \tilde{f}(x)+\tilde{g}(x)\,G(\int_0^x h(s)Q(u(s))\mathrm{d}s), \quad x\geqslant 0$$

where $\tilde{f}$ and $\tilde{g}$ are given by (53). We make use of Yeh's result ([2], Theorem 3) to obtain the majoration (54). A similar result with the preceding Proposition 6 is Theorem 3 of the paper [3].

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Received 23.IV.1985

Department of Mathematics
Polytechnical Institute
6600 Iaşi, Romania

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXI, s. I. a, Matematică, 1985, f. 3

SUR DE L'ETAT DE CONTRAINTE PLAN DES CORPS ORTHOTROPES AVEC ECROUISSAGE

PAR

V. DIACONIŢA

Dans ce travail on montre l'existence et l'unicité de la solution faible du problème de l'état de contrainte plan des corps orthotropes avec écrouissage. On considère que les limites d'écoulement augmentent proportionnellement avec un paramètre qui dépend de l'histoire de la déformation.

1. Considérations sur la plasticité des corps orthotropes avec écrouissage. Si l'on considère que la pression hydrostatique n'influence pas l'écoulement, alors la condition de plasticité pour les corps orthotropes sans écrouissage se présente sous la forme [2] :

(1.1) $$K_{23}(\sigma_{22}-\sigma_{33})^2+K_{31}(\sigma_{33}-\sigma_{11})^2+K_{12}(\sigma_{11}-\sigma_{22})^2+K_{44}\sigma_{23}^2+ +K_{55}\sigma_{31}^2+K_{66}\sigma_{12}^2=1,$$

où K_{ij} sont les paramètres qui caractérisent l'état d'écoulement anisotrope et σ_{ij} les composantes du tenseur des contraintes.

En désignant par X_{ij} les limites d'écoulement à l'extension et au glissement par rapport aux axes d'anisotropie nous avons :

(1.2) $$2K_{23}=\frac{1}{X_{22}^2}+\frac{1}{X_{33}^2}-\frac{1}{X_{11}^2}, \quad 2K_{31}=\frac{1}{X_{33}^2}+\frac{1}{X_{11}^2}-\frac{1}{X_{22}^2},$$
$$2K_{12}=\frac{1}{X_{11}^2}+\frac{1}{X_{22}^2}-\frac{1}{X_{33}^2}, \quad K_{44}=\frac{1}{X_{23}^2}, \quad K_{55}=\frac{1}{X_{31}^2}, \quad K_{66}=\frac{1}{X_{12}^2}.$$

De (1.2) on observe que les paramètres K_{44}, K_{55}, K_{66} sont positifs, et

(1.3) $$K_{23}+K_{31}>0,\ K_{31}+K_{12}>0,\ K_{12}+K_{23}>0,\ K_{23}+K_{31}+K_{12}>0.$$

Nous considérons, comme en [2], [1], que dans le processus de chargement, les limites d'écoulement augmentent proportionnellement, c'est-à-dire $X_{ij}=hX_{ij}^0$, où nous avons noté avec l'indice zéro les valeurs initiales des limites d'écoulement et où h est un paramètre monotone croissant à partir de 1 et qui exprime la grandeur de l'écrouissage. Les paramètres K_{ij} décroissent comme $1/h^2$. En tenant compte de ce fait, de (1.1) nous déduisons :

(1.4.) $$K_{23}^0(\sigma_{22}-\sigma_{33})^2+K_{31}^0(\sigma_{33}-\sigma_{11})^2+K_{12}^0(\sigma_{11}-\sigma_{22})^2+K_{44}^0\sigma_{23}^2+ +K_{55}^0\sigma_{31}^2+K_{66}^0\sigma_{12}^2=h^2,$$