Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXI, s. I a, Matematică, 1985, f. 3.

EMBEDDINGS OF THE TANGENT BUNDLE IN RIEMANNIAN MANIFOLDS

BY

CORNELIA-LIVIA BEJAN and VASILE OPROIU

In the last few years, the differential geometry of Finsler spaces has been interpreted as the geometry of the vector bundle over the tangent bundle induced from the tangent bundle itself by the natural projection [2]. Alternatively the same vector bundle is isomorphic to the vertical subbundle in the tangent bundle to the tangent bundle. In [5], the second author adopted this point of view to study the properties of the Finsler connections.

In this paper we study the differential geometry of the tangent bundle to a differentiable manifold when this tangent bundle is embedded in a Riemannian manifold. It appears that the non-linear connection is defined naturally by the condition that the vertical and horizontal distributions are orthogonal to each other. Using the decomposition in the vertical and horizontal parts of the covariant derivative with respect to the Levi-Civita connection on the tangent bundle, we obtain the so called Gauss-Weingarten formulae. From the Riemannian metrics induced on the vertical and horizontal distributions and from the expression of the corresponding non-linear connection, we obtain explicitly the induced connections on the vertical and horizontal distributions as well as the corresponding second fundamental forms.

Next we get the corresponding Gauss-Codazzi compatibility conditions. Finally, in the case when TM is embedded as a hypersurface in R^{2n+1}, we obtain the conditions under which, the well known existence theorem for hypersurfaces, is valid.

1. The metric on TM. Let M be an n-dimensional C^∞-differentiable manifold and TM its tangent bundle. Then TM can be endowed with a $2n$-dimensional C^∞-differentiable structure. Recall that if $\tau : TM \to M$ is the natural projection and (U, φ) is a local chart on M with local coordinates $(x^1, \ldots, x^n)$, then the local chart $(\tau^{-1}(U), \Phi)$ is defined on TM, where $\Phi : \tau^{-1}(U) \to \varphi(U) \times R^n \subset R^n \times R^n$ is determined as follows. For $v_p \in T_pM \subset \tau^{-1}(U)$, $p = \tau(v_p) \in U$, we have $\Phi(v_p) = (\varphi\tau(v_p), T\varphi_p(v_p))$, where $T\varphi_p : T_pM \to T_{\varphi(p)}R^n = R^n$ is the tangent mapping in p of $\varphi : U \to R^n$. If the coordinate expression of v_p in the natural basis of (U, φ) is $v_p = y^i(\partial/\partial x^i)_p$, then the local coordinates of v_p in the local chart $(\tau^{-1}(U), \Phi)$ are $(x^i(\tau(p)), y^i)$. Usually, we write simply x^i instead of $x^i \circ \tau$. Recall that a coordinate transformation $\bar{x}^i = \bar{x}^i(x^1, \ldots, x^n)$ on M induces the following coordinate transformation on TM:

$$\tilde{x}^i=\tilde{x}^i(x^1, \dots, x^n)\ ;\ \tilde{y}^i=\frac{\partial \tilde{x}^i}{\partial \tilde{x}^j} y^j .$$

Suppose $f: TM \to \widetilde{M}$ is an embedding of the tangent bundle TM in the Riemannian manifold $\widetilde{M}$ with the Riemannian metric $\tilde{g}$. The Riemannian metric $f^*\tilde{g}$ is induced on TM and let us study the properties of TM related to this induced Riemannian metric. Denote:

(1) $$z^a=f^a(x^i, y^i),\ a=1, \dots ,2n+p,$$

the local expression of f in the local chart $(\tau^{-1}(U), \Phi)$ on TM and the local chart (V, ψ) on $\widetilde{M}$. From now on, consider the indices taking the following values:

$$i, j, k, l, h, m, r=1, \dots, n\ ;\ a, b, c=1, \dots, 2n+p\ ;\ \alpha, \beta, \gamma=1, \dots, p,$$

whenever other specification is omitted.

Denote by VTM the vertical subbundle of TTM over TM, defined by $VTM=\operatorname{Ker}\tau$. Recall that $(\partial/\partial y^1, \dots, \partial/\partial y^n)$ is a local frame of VTM over $\tau^{-1}(U)$. Consider the orthogonal complement HTM to VTM in TTM with respect to the induced Riemannian metric $f^*\tilde{g}$ on TM. We have:

(2) $TTM=VTM\oplus HTM$, thus: (2′) $X=X^V+X^H$,

for every $X\in TTM$, where $X^V\in VTM$, $X^H\in HTM$.

From the local natural frame $(\partial/\partial y^i, \partial/\partial x^i)$ on TM, we can obtain a local adapted frame $(\partial/\partial y^i, \delta/\delta x^i)$ such that $(\delta/\delta x^1, \dots, \delta/\delta x^n)$ is a local frame in HTM. Writing as usual:

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-N_i^j\frac{\partial}{\partial y^j},\quad i=1, \dots, n$$

for the vector fields of HTM, we obtain from the orthogonality condition $f^*\tilde{g}(\partial/\partial x^i, \delta/\delta x^j)=0$ the quite natural expression:

(3) $$N_i^j=N_{ik}g^{kj}$$

involving the local expression of $f^*\tilde{g}$ with respect to $(\partial/\partial y^i, \partial/\partial x^i)$, where:

(3′) $$g_{ij}=f^*\tilde{g}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right),\quad N_{ik}=f^*\tilde{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^k}\right),$$

and g^{jk}'s are defined as the entries of the inverse matrix of $[g_{ij}]$, thus $g_{ij}g^{jk}=\delta_i^k$.

Remark. The matrix $[g_{ij}]$ gives the local expression of the restriction $f^*\tilde{g}|VTM$ of the metric $f^*\tilde{g}$ to VTM, thus we have a generalized Finsler metric on M. The existence of the orthogonal complement HTM to VTM in TTM gives the existence of the non-linear connection on M whose local coeficients N_i^j are determined by (3).

The local coframe in T^*TM dual to the frame $(\partial/\partial y^i, \delta/\delta x^i)$ adapted to the complementary orthogonal distributions VTM, HTM is $(\delta y^i, dx^i)$, where:

(4) $$\delta y^i=dy^i+N_j^i dx^j.$$

The local expression of the Riemannian metric $f^*\tilde{g}$ in this adapted local coframe is:

(5) $$f^*\tilde{g}=g_{ij}\delta y^i\delta y^j+g^*_{ij}dx^i dx^j,\quad \text{where: } g^*_{ij}=f^*\tilde{g}\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right).$$

Remark. $[g^*_{ij}]$ is the matrix giving the local expression of the restriction $f^*\tilde{g}|_{HTM}$ of $f^*\tilde{g}$ to HTM. It could be also considered as a Finsler metric on M.

2. The Gauss-Weingarten formulae on TM. Let $\tilde{\nabla}$ be the Levi-Civita connection of $\tilde{g}$ on $\widetilde{M}$. If TM is embedded by f in $\widetilde{M}$, then the connection induced on TM from $\tilde{\nabla}$ is just the Levi-Civita connection D on TM of the induced Riemannian metric $f^*\tilde{g}$. Denote by $\tilde{b}$ the second fundamental form of TM and by $D^\perp$ the connection induced from $\tilde{\nabla}$ on the normal bundle $T^\perp TM$ to TM. Then we have the well known Gauss-Weingarten formulae:

(6) $$\tilde{\nabla}_X Y=D_X Y+\tilde{b}(X, Y),\quad \tilde{\nabla}_X\mathfrak{N}=-\tilde{A}_{\mathfrak{N}}+D^\perp_X\mathfrak{N},$$
$$\forall X, Y\in\Gamma(TTM),\ \mathfrak{N}\in\Gamma(T^\perp TM)$$

where $\tilde{A}$ is the Weingarten mapping associated with $\tilde{b}$. Recall $\tilde{b}$ is symmetric and:

(7) $$\tilde{g}(\tilde{b}(X, Y), \mathfrak{N})=\tilde{g}(\tilde{A}_{\mathfrak{N}}X, Y).$$

Further we consider the decomposition of the covariant derivative on TM with respect to D into vertical and horizontal parts. Just like in the case of the embedded manifolds, we obtain the induced connections on VTM and HTM as well as the corresponding second fundamental forms. The obtained formulae will be called *Gauss-Weingarten formulae* of TTM with the Riemannian metric $f^*\tilde{g}$.

Let $Y\in\Gamma(VTM)$ be a vertical vector field on TM and $X\in\Gamma(TTM)=\mathcal{X}(TM)$ an arbitrary vector field on TM. Then:

(8) $$D_X Y=\nabla_X Y+b(X, Y),$$

where $\nabla_X Y=(D_X Y)^V\in\Gamma(VTM)$, $b(X, Y)=(D_X Y)^H\in\Gamma(HTM)$. Then, just like in the case of the embedded manifolds we obtain:

Proposition 1. *∇ is a linear connection in VTM such that:*

(9) $$\nabla_X(f^*\tilde{g}|_{VTM})=0,\ \forall X\in\Gamma(TTM)\ \textit{ and}$$

(10) $$\nabla_X Y-\nabla_Y X=[X, Y],$$

whenever $X, Y \in \Gamma(VTM)$. *Moreover,* b *is a bilinear morphism of vector bundles* $b : TTM \times VTM \to HTM$ *and :*

$$b(X, Y) = b(Y, X), \text{ whenever } X, Y \in \Gamma(VTM). \tag{11}$$

In a similar way, we deal with the case of the sections in HTM. So, if $Z \in \Gamma(HTM)$, $X \in \Gamma(TTM)$, we have :

(12) $D_X Z = -A_Z X + \nabla^{\perp}_X Z$, where $-A_Z X = (D_X Z)^V \in \Gamma(VTM)$, $\nabla^{\perp}_X Z = (D_X Z)^H \in \Gamma(HTM)$. Then :

Proposition 2. $\nabla^{\perp}$ *is a linear connection in* HTM *such that :*

$$\nabla^{\perp}_X (f^* \tilde{g}|_{HTM}) = 0, \quad \forall X \in \Gamma(TTM), \text{ and :} \tag{13}$$

$$\nabla^{\perp}_X Y - \nabla^{\perp}_Y X = [X, Y]^H, \text{ whenever } X, Y \in \Gamma(HTM). \tag{14}$$

Moreover, A *is a bilinear morphism of vector bundles* $A : HTM \times TTM \to VTM$ *and satisfies the relation :*

$$A_X Y - A_Y X = [X, Y]^V \text{ whenever } X, Y \in \Gamma(HTM). \tag{15}$$

The bilinear vector bundle morphisms b *and* A *are related by the formula :*

$$(f^*\tilde{g})(b(X, Y), Z) = (f^*g)(A_Z X, Y), \quad \forall X \in \Gamma(TTM),\ Y \in \Gamma(VTM),\ Z \in \Gamma(HTM). \tag{16}$$

Next, let us give the local expressions of the formulae written above, with respect to the local adapted frame $(\partial/\partial y^i, \delta/\delta x^i)$. First, we denote :

$$\dot{D}_i = D_{\frac{\partial}{\partial y^i}}, \quad D_i = D_{\frac{\delta}{\delta x^i}}, \quad \dot{\nabla}_i = \nabla_{\frac{\partial}{\partial y^i}}, \quad \nabla_i = \nabla_{\frac{\delta}{\delta x^i}}, \quad \dot{\nabla}^{\perp}_i = \nabla^{\perp}_{\frac{\partial}{\partial y^i}}, \quad \nabla^{\perp}_i = \nabla^{\perp}_{\frac{\delta}{\delta x^i}}.$$

Then the connection coeficients of ∇, $\nabla^{\perp}$ with respect to the adapted frame $(\partial/\partial y^i, \delta/\delta x^i)$ are given by :

$$\dot{\nabla}_i \frac{\partial}{\partial y^j} = C^k_{ij} \frac{\partial}{\partial y^k}, \quad \nabla_i \frac{\partial}{\partial y^j} = F^k_{ij} \frac{\partial}{\partial y^k}, \quad \dot{\nabla}^{\perp}_i \frac{\delta}{\delta x^j} = C^{*k}_{ij} \frac{\delta}{\delta x^k}, \quad \nabla^{\perp}_i \frac{\delta}{\delta x^j} = F^{*k}_{ij} \frac{\delta}{\delta x^k}, \tag{17}$$

Similarly, we obtain the local expressions corresponding to the bilinear vector bundle morphisms b and A :

$$\begin{gathered} b\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = b^k_{ij} \frac{\delta}{\delta x^k}, \quad b\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = b^{*k}_{ij} \frac{\delta}{\delta x^k}, \\ A_{\frac{\partial}{\partial x^k}} \frac{\delta}{\partial y^i} = a^j_{ki} \frac{\partial}{\partial y^i}, \quad A_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^i} = a^{*j}_{ki} \frac{\partial}{\partial y^j}. \end{gathered} \tag{18}$$

With these relations, we obtain the local expressions of the Gauss-Weingarten formulae (8) and (12) :

$$\dot{D}_i \frac{\partial}{\partial y^j} = C^k_{ij} \frac{\partial}{\partial y^k} + b^k_{ij} \frac{\delta}{\delta x^k}, \quad D_i \frac{\partial}{\partial y^j} = F^k_{ij} \frac{\partial}{\partial y^k} + b^{*k}_{ij} \frac{\delta}{\delta x^k} \tag{8'}$$

$$\dot{D}_i \frac{\delta}{\delta x^j} = C^{*k}_{ij} \frac{\delta}{\delta x^k} - a^k_{ji} \frac{\partial}{\partial y^k}, \quad D_i \frac{\delta}{\delta x^j} = F^{*k}_{ij} \frac{\delta}{\delta x^k} - a^{*k}_{ji} \frac{\partial}{\partial y^k}. \tag{12'}$$

The conditions from the proposition 1, satisfied by ∇ and b, can be expressed locally by :

$$g_{hk} C^h_{ij} + g_{jh} C^h_{ik} = \frac{\partial g_{jk}}{\partial y^i}, \quad g_{hk} F^h_{ij} + g_{jh} F^h_{ik} = \frac{\delta g_{jk}}{\delta x^i}, \tag{9'}$$

$$C^k_{ij} = C^k_{ji}, \quad b^k_{ij} = b^k_{ji}. \tag{10', 11'}$$

Similarly, the conditions from the proposition 2, satisfied by $\nabla^{\perp}$ and A, can be expressed locally by :

$$g^*_{hk} C^{*h}_{ij} + g^*_{jh} C^{*h}_{ik} = \frac{\partial g^*_{jk}}{\partial y^i}, \quad g^*_{hk} F^{*h}_{ij} + g^*_{jh} F^{*h}_{ik} = \frac{\delta g^*_{jk}}{\delta x^i}, \tag{13'}$$

$$F^{*k}_{ij} = F^{*k}_{ji}, \quad a^{*k}_{ij} - a^{*k}_{ji} = R^k_{ji}. \tag{14', 15'}$$

Recall that R^k_{ji} is the local expression of the complete integrability tensor field for the distribution HTM :

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R^k_{ji} \frac{\partial}{\partial y^k}. \tag{19}$$

The relation (16) satisfied by b and A is expressed locally by :

$$g^*_{ih} b^h_{jk} = g_{hk} a^h_{ji}, \quad g^*_{ih} b^{*h}_{jk} = g_{hk} a^{*h}_{ji}. \tag{16'}$$

Finally, since D is torsion free we get :

$$C^{*k}_{ij} = b^{*k}_{ji}, \quad F^k_{ij} + a^k_{ij} = (\partial N^k_i)/(\partial y^j). \tag{20}$$

The main result of this section will be a theorem giving the explicit expression of the connexions ∇, $\nabla^{\perp}$ and bilinear vector bundle morphisms b, A. The result is true without the hypotesis that TM is embedded in $\tilde{M}$. However, in this case the situation looks more natural.

Theorem 3. *Let* N *be a non-linear connection and let* g, g^* *be two generalized Finsler metrics on the manifold* M. *Consider the Riemannian metric defined on* TM *such that its restrictions to* VTM *and* HTM *are* g *and* g^* *respectively, and the distributions* VTM *and* HTM *are orthogonal to each other.*

Then the connections ∇, $\nabla^{\perp}$ *and the morphisms* b, A *from proposition 1 and 2 are uniquelly determined such that (20) is true.*

Proof, A carefull study of the above conditions shows that they are equivalent to the following solution :

$$C^k_{ij} = \frac{1}{2} g^{kh} \left(\frac{\partial g_{hj}}{\partial y^i} + \frac{\partial g_{ih}}{\partial y^j} - \frac{\partial g_{ij}}{\partial y^h}\right), \quad b^k_{ij} = \frac{1}{2} g^{*kh} \left(-\frac{\partial g_{ij}}{\partial x^h} + g_{il} \frac{\partial N^l_h}{\partial y^j} + g_{jl} \frac{\partial N^l_h}{\partial y^i}\right) \tag{21}$$

$$F^k_{ij} = \frac{1}{2} \frac{\partial N^k_i}{\partial y^j} + \frac{1}{2} g^{kh} \left(\frac{\delta g_{ih}}{\delta x^i} - g_{jl} \frac{\partial N^l_i}{\partial y^h}\right), \quad b^{*k}_{ij} = \frac{1}{2} g^{*kh} \left(g_{jl} R^l_{ih} + \frac{\partial g^*_{ih}}{\partial y^j}\right) = C^{*k}_{ji} \tag{22}$$

$$a^k_{ij}=\frac{1}{2}\frac{\partial N^k_i}{\partial y^j}-\frac{1}{2}g^{kh}\left(\frac{\delta g_{jh}}{\delta x^i}-g_{jl}\frac{\partial N^l_i}{\partial y^h}\right), \tag{23}$$

$$F^k_{ij}=\frac{1}{2}g^{*kh}\left(\frac{\delta g^*_{hj}}{\delta x^i}+\frac{\delta g^*_{ih}}{\delta x^j}-\frac{\delta g^*_{ij}}{\delta x^h}\right);\quad a^{*k}_{ji}=\frac{1}{2}\left(R^k_{ij}+g^{kh}\frac{\partial g^*_{ij}}{\partial y^h}\right) \tag{24}$$

where g^{*ij}'s are the entries of the inverse matrix of $[g^*_{ij}]$.

3. The Gauss-Codazzi conditions. Denote by K the curvature tensor field of the Levi-Civita connection D on TM. Consider its components with respect to the complementary distributions VTM and HTM. We should prefer expressing the formulae in local coordinates. Use the definition :

$$K(X, Y)Z=D_XD_YZ-D_YD_XZ-D_{[X,Y]}Z,$$

of K and the following notations :

$$G^l_{hij}\frac{\partial}{\partial y^l}+G^{*l}_{hij}\frac{\delta}{\delta x^l}=K\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^h};$$

$$H^l_{hij}\frac{\partial}{\partial y^l}+H^{*l}_{hij}\frac{\delta}{\delta x^l}=K\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)\frac{\delta}{\delta x^h};$$

$$I^l_{hij}\frac{\partial}{\partial y^l}+I^{*l}_{hij}\frac{\delta}{\delta x^l}=K\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^h};$$

$$J^l_{hij}\frac{\partial}{\partial y^l}+J^{*l}_{hij}\frac{\delta}{\delta x^l}=K\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\right)\frac{\delta}{\delta x^h};$$

$$K^l_{hij}\frac{\partial}{\partial y^l}+K^{*l}_{hij}\frac{\delta}{\delta x^l}=K\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^h};$$

$$L^l_{hij}\frac{\partial}{\partial y^l}+L^{*l}_{hij}\frac{\delta}{\delta x^l}=K\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^h}.$$

Recall that in the theory of Finsler connections three curvature tensor fields R^l_{hij}, P^l_{hij}, S^l_{hij} are known [3,5]. Then :

$$G^l_{hij}=-S^l_{hij}-a^l_{hi}b^k_{jh}+a^l_{kj}b^k_{ih};\quad G^{*l}_{hij}=\frac{\partial b^l_{jh}}{\partial y^i}-\frac{\partial b^l_{ih}}{\partial y^j}+ \tag{25}$$

$$+b^l_{ik}C^k_{jh}-b^l_{jk}C^k_{ih}+b^{*l}_{ki}b^k_{jh}-b^{*l}_{kj}b^k_{ih};$$

$$H^l_{hij}=-\frac{\partial a^l_{hj}}{\partial y^i}+\frac{\partial a^l_{hi}}{\partial y^j}-C^l_{ik}a^k_{hj}+C^l_{jk}a^k_{hi}+a^l_{kj}b^{*k}_{hi}-a^l_{ki}b^{*k}_{hj};$$

$$H^{*l}_{hij}=\frac{\partial b^{*l}_{hj}}{\partial y^i}-\frac{\partial b^{*l}_{hi}}{\partial y^j}+b^{*l}_{ki}b^{*k}_{hj}-b^{*l}_{kj}b^{*k}_{hi}-b^l_{ik}a^k_{hj}+b^l_{jk}a^k_{hi};$$

$$I^l_{hij}=-P^l_{hij}+a^l_{kj}b^{*k}_{ih}-a^{*l}_{ki}b^k_{jh};$$

$$I^{*l}_{hij}=\frac{\delta b^l_{jh}}{\delta x^i}-\frac{\partial b^{*l}_{ih}}{\partial y^j}+F^{*l}_{ik}b^k_{jh}-b^l_{jk}F^k_{ih}-b^{*l}_{kj}b^{*k}_{ih}+b^{*l}_{ik}C^k_{jh}-b^l_{kh}\frac{\partial N^k_i}{\partial y^j};$$

$$J^l_{hij}=-\frac{\delta a^l_{hj}}{\delta x^i}+\frac{\partial a^{*l}_{hi}}{\partial y^j}+a^l_{kj}F^{*k}_{ih}-F^l_{ik}a^k_{hj}-a^{*l}_{ki}b^{*k}_{hj}+C^l_{jk}a^{*k}_{hi}+a^l_{hk}\frac{\partial N^k_i}{\partial y^j}$$

$$J^{*l}_{hij}=\frac{\delta b^{*l}_{hj}}{\delta x^i}-\frac{\partial F^{*l}_{ih}}{\partial y^j}+F^{*l}_{ik}b^{*k}_{hj}-b^{*l}_{kj}F^{*k}_{ih}-b^{*l}_{ik}a^k_{hj}+b^l_{jk}a^{*k}_{hi}-b^{*l}_{hk}\frac{\partial N^k_i}{\partial y^j}$$

$$K^l_{hij}=-R^l_{hij}-a^{*l}_{kj}b^{*k}_{ih}-a^{*l}_{ki}b^{*k}_{jh};$$

$$K^{*l}_{hij}=\frac{\delta b^{*l}_{jh}}{\delta x^i}-\frac{\delta b^{*l}_{ih}}{\delta x^j}+b^{*l}_{ik}F^k_{jh}-b^{*l}_{jk}F^k_{ih}+F^{*l}_{ik}b^{*k}_{jh}-F^{*l}_{jk}b^{*k}_{ih}+b^l_{kh}R^k_{ij};$$

$$L^l_{hij}=-\frac{\delta a^{*l}_{hj}}{\delta x^i}+\frac{\delta a^{*l}_{hi}}{\delta x^j}-F^l_{ik}a^{*k}_{hj}+F^l_{jk}a^{*k}_{hi}-a^{*l}_{ki}F^{*k}_{jh}+a^{*l}_{kj}F^{*k}_{ih}-a^l_{hk}R^k_{ij}$$

$$L^{*l}_{hij}=\frac{\delta F^{*l}_{jh}}{\delta x^i}-\frac{\delta F^{*l}_{ih}}{\delta x^j}+F^{*l}_{ik}F^{*k}_{jh}-F^{*l}_{jk}F^{*k}_{ih}+b^{*l}_{hk}R^k_{ij}-b^{*l}_{ik}a^{*k}_{hj}+b^{*l}_{jk}a^{*k}_{hi}$$

4. The case when TM is a hypersurface in $\mathbf{R}^{2n+1}$. If TM is a hypersurface in R^{2n+1}, then we have the well known relations between the curvature tensor field K of D and the Weingarten mapping $\tilde{A}$ from the formulae (6), (see [1]).

$$K(X, Y)Z=g(\tilde{A}Y, Z)\tilde{A}X-g(\tilde{A}X, Z)\tilde{A}Y;\quad (D_X\tilde{A})Y=(D_Y\tilde{A})X. \tag{26}$$

Use the following notations for the local components of $\tilde{A}$:

$$\tilde{A}\left(\frac{\partial}{\partial y^i}\right)=A^j_i\frac{\partial}{\partial y^j}+A^{*j}_i\frac{\delta}{\delta x^j};\quad \tilde{A}\left(\frac{\delta}{\delta x^i}\right)=B^j_i\frac{\partial}{\partial y^j}+B^{*j}_i\frac{\delta}{\delta x^j}. \tag{27}$$

Then, using the relations of the section 3, we obtain :

$$\begin{aligned}
&G^l_{hij}=g_{hk}(A^l_iA^k_j-A^l_jA^k_i); && G^{*l}_{hij}=g_{hk}(A^{*l}_iA^k_j-A^{*l}_jA^k_i);\\
&H^l_{hij}=g^*_{hk}(A^l_iA^{*k}_j-A^l_jA^{*k}_i); && H^{*l}_{hij}=g^*_{hk}(A^{*l}_iA^{*k}_j-A^{*l}_jA^{*k}_i);\\
&I^l_{hij}=g_{hk}(B^l_iA^k_j-A^l_jB^k_i); && I^{*l}_{hij}=g_{hk}(B^{*l}_iA^k_j-A^{*l}_jB^k_i);\\
&J^l_{hij}=g^*_{hk}(B^l_iA^{*k}_j-A^l_jB^{*k}_i); && J^{*l}_{hij}=g^*_{hk}(B^{*l}_iA^{*k}_j-A^{*l}_jB^{*k}_i);\\
&K^l_{hij}=g_{hk}(B^l_iB^k_j-B^l_jB^k_i); && K^{*l}_{hij}=g_{hk}(B^{*l}_iB^k_j-B^{*l}_jB^k_i);\\
&L^l_{hij}=g^*_{hk}(B^l_iB^{*k}_j-B^l_jB^{*k}_i); && L^{*l}_{hij}=g^*_{hk}(B^{*l}_iB^{*k}_j-B^{*l}_jB^{*k}_i);
\end{aligned} \tag{28}$$

From the condition :

$$(\dot{D}_k\tilde{A})\left(\frac{\partial}{\partial y^i}\right)=(\dot{D}_i\tilde{A})\left(\frac{\partial}{\partial y^k}\right),$$

we get :

$$
(29)\qquad
\begin{aligned}
&\frac{\partial A_i^j}{\partial y^k}+C_{kh}^j A_i^h-A_i^{*h}a_{hk}^j=\frac{\partial A_k^j}{\partial y^i}+C^j{}_{ki}A_k^h-A_k^{*h}a_{hi}^j\,;\\
&\frac{\partial A_i^{*j}}{\partial y^k}+b_{hk}^{*j}A_i^{*h}+b_{kh}^j A_i^h=\frac{\partial A_k^{*j}}{\partial y^i}+b_{hi}^{*j}A_k^{*h}+b_{ih}^j A_k^h\,;
\end{aligned}
$$

Similarly, from the condition $(\dot{D}_k\tilde{A})\left(\frac{\delta}{\delta x^i}\right)=(\dot{D}_i\tilde{A})\left(\frac{\partial}{\partial y^k}\right)$, we get :

$$
(30)\qquad
\begin{aligned}
&\frac{\partial B_i^j}{\partial y^k}+C_{kh}^j B_i^h+A_h^j a_{ik}^h-a_{hk}^j B_i^{*h}=\frac{\delta A_k^j}{\delta x^i}+F_{ih}^j A_k^h-A_h^j F_{ik}^h-a_{hi}^{*j}A_k^{*h}\,;\\
&\frac{\partial B_i^{*j}}{\partial y^k}+b_{kh}^j B_i^h+b_{hk}^{*j}B_i^{*h}+A_h^{*j}a_{ik}^h=\frac{\delta A_k^{*j}}{\delta x^i}+b_{ih}^{*j}A_k^h+F_{ih}^{*j}A_k^{*h}-A_h^{*j}F_{ik}^h
\end{aligned}
$$

Finally, from the condition $(D_k\tilde{A})\left(\frac{\delta}{\delta x^i}\right)=(\dot{D}_i\tilde{A})\left(\frac{\delta}{\delta x^k}\right)$, we get :

$$
(31)\qquad
\begin{aligned}
&\frac{\delta B_i^j}{\delta x^k}+F_{kh}^j B_i^h-a_{hk}^{*j}B_i^{*h}+A_h^j a_{ik}^{*h}=\frac{\delta B_k^j}{\delta x^i}+F_{ih}^j B_k^h-a_{hi}^{*j}B_k^{*h}+A_h^j a_{ki}^{*h}\\
&\frac{\delta B_i^{*j}}{\delta x^k}+F_{kh}^{*j}B_i^{*h}+b_{kh}^{*j}B_i^h+A_h^{*j}a_{ik}^{*h}=\frac{\delta B_k^{*j}}{\delta x^i}+F_{ih}^{*j}B_k^{*h}+b_{ih}^{*j}B_k^h+A_h^{*j}a_{ki}^{*h}
\end{aligned}
$$

If the Riemannian metric on TM, obtained in the theorem 3 and the morphism $\tilde{A}$ satisfy the conditions obtained from (26), then we can apply the local existence theorem for hypersurfaces to get a local embedding of TM in R^{2n+1}.

REFERENCES

1. G h e o r g h i e v Gh., O p r o i u V. – *Varietăţi diferenţiabile finit şi infinit dimensionale*, vol. II, Ed. Acad. R.S.R., 1979.
2. M a t s u m o t o M. – *Foundation of Finsler geometry and special Finsler spaces*, Kyoto Univ., 1982.
3. M i r o n R. – *Introduction to the theory of Finsler spaces*, Proc. Nat. Sem. on Finsler spaces, Univ. Braşov (1980), 130–183.
4. M i r o n R. – *Metrical Finsler structures and metrical Finsler connections*, Journal of Math. of Kyoto Univ., 23 (1983), 213–224.
5. O p r o i u V. – *Some properties of the tangent bundle related to the Finsler geometry*, Proc. Nat. Sem. on Finsler spaces, Braşov, 1980, 195–207.

Received 19.II.1985

Faculty of Mathematics
University of Iaşi
6600 Iaşi, Romania

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXI, s. I a, Matematică, 1985, f. 3.

HIGHER DIMENSIONAL NORMAL SECTIONS

BY

J. DEPREZ*, P. VERHEYEN and L. VERSTRAELEN

1. Introduction. Let N be a (connected) n-dimensional $(2\leqslant n)$ submanifold of an m-dimensional Euclidean space $\mathbb{E}^m$. For each point p in N and each unit vector t tangent to N at p, the vector t and the normal space $T_p^\perp N$ to N at p determine an $(m-n+1)$-dimensional Euclidean subspace $E(p, t)$ of $\mathbb{E}^m$. The intersection of N and $E(p, t)$ gives locally rise to a (regular) curve γ which is called the *normal section of N at p determined by t* [2]. Submanifolds in Euclidean space with (pointwise) planar normal sections have been investigated by C h e n [2, 3, 4] and by H o u h and W a n g [8]. C h e n and one of the authors studied submanifolds N in $\mathbb{E}^m$ whose normal sections are geodesics in N [5, 6], which is equivalent to the study of helical submanifolds [12] (i.e. all geodesics of N, considered as curves in the ambient space $\mathbb{E}^m$, have the same constant curvatures, see also [11]). The only known examples of submanifolds with geodesic normal sections in $\mathbb{E}^m$ are given by the totally geodesic submanifolds of $\mathbb{E}^m$ and the standard immersions of compact rank on symmetric spaces.

In this paper we define k-dimensional normal sections of an n-dimensional submanifold N in $\mathbb{E}^m$ $(3\leqslant n,\ 1<k<n)$. When there exists a natural number k, $1<k<n$, such that all k-dimensional normal sections of N are totally geodesic in N, then this is true for all such numbers k and N is a space of constant sectional curvature which has geodesic (1-dimensional) normal sections. Conversely, all k-dimensional normal sections of a space of constant sectional curvature N which is immersed in $\mathbb{E}^m$ with geodesic normal sections are totally geodesic in N $(1<k<n)$. Submanifolds of dimension at least 2 and of codimension at most 4 in an Euclidean space having constant sectional curvature and geodesic normal sections are classified.

2. k-Dimensional Normal Sections. Let N be an n-dimensional submanifold of $\mathbb{E}^m$ $(3\leqslant n)$ and let k be a natural number such that $1<k<n$. For each point p in N and each k-dimensional subspace π of the tangent space T_pN to N at p, this subspace π and the normal space $T_p^\perp N$ to N at p determine a $(m-n+k)$-dimensional Euclidean subspace $E(N\,;\ p,\ \pi)$ of $\mathbb{E}^m$. The intersection of N and $E(N\,;\ p,\ \pi)$ locally gives a k-dimensional submanifold of N by considering the intersection of a maximal small enough open part U of N containing p with $E(N\,;\ p,\ \pi)$ which is supposed to be connected. This submanifold is called the *k-dimensional normal section of*

---

* Aspirant N.F.W.O. (Belgium).