we get :

$$\begin{aligned}(29)\qquad &\frac{\partial A_i^j}{\partial y^k}+C_{kh}^j A_i^h-A_i^{*h}a_{hk}^j=\frac{\partial A_k^j}{\partial y^i}+C^j{}_{ih}A_k^h-A_k^{*h}a_{hi}^j\,;\\ &\frac{\partial A_i^{*j}}{\partial y^k}+b_{hk}^{*j}A_i^{*h}+b_{kh}^jA_i^h=\frac{\partial A_k^{*j}}{\partial y^i}+b_{hi}^{*j}A_k^{*h}+b_{ih}^jA_k^h\,;\end{aligned}$$

Similarly, from the condition $(\dot{D}_k\tilde{A})\left(\frac{\delta}{\delta x^i}\right)=(\dot{D}_i\tilde{A})\left(\frac{\partial}{\partial y^k}\right)$, we get :

$$\begin{aligned}(30)\qquad &\frac{\partial B_i^j}{\partial y^k}+C_{kh}^jB_i^h+A_h^ja_{ik}^h-a_{hk}^jB_i^{*h}=\frac{\delta A_k^j}{\delta x^i}+F_{ih}^jA_k^h-A_h^jF_{ik}^h-a_{hi}^{*j}A_k^{*h}\,;\\ &\frac{\partial B_i^{*j}}{\partial y^k}+b_{kh}^jB_i^h+b_{hk}^{*j}B_i^{*h}+A_h^{*j}a_{ik}^h=\frac{\delta A_k^{*j}}{\delta x^i}+b_{ih}^{*j}A_k^h+F_{ih}^{*j}A_k^{*h}-A_h^{*j}F_{ik}^h\end{aligned}$$

Finally, from the condition $(D_k\tilde{A})\left(\frac{\delta}{\delta x^i}\right)=(\dot{D}_i\tilde{A})\left(\frac{\delta}{\delta x^k}\right)$, we get :

$$\begin{aligned}(31)\qquad &\frac{\delta B_i^j}{\delta x^k}+F_{kh}^jB_i^h-a_{hk}^{*j}B_i^{*h}+A_h^ja_{ik}^{*h}=\frac{\delta B_k^j}{\delta x^i}+F_{ih}^jB_k^h-a_{hi}^{*j}B_k^{*h}+A_h^ja_{ki}^{*h}\\ &\frac{\delta B_i^{*j}}{\delta x^k}+F_{kh}^{*j}B_i^{*h}+b_{kh}^{*j}B_i^h+A_h^{*j}a_{ik}^{*h}=\frac{\delta B_k^{*j}}{\delta x^i}+F_{ih}^{*j}B_k^{*h}+b_{ih}^{*j}B_k^h+A_h^{*j}a_{ki}^{*h}\end{aligned}$$

If the Riemannian metric on TM, obtained in the theorem 3 and the morphism $\tilde{A}$ satisfy the conditions obtained from (26), then we can apply the local existence theorem for hypersurfaces to get a local embedding of TM in R^{2n+1}.

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HIGHER DIMENSIONAL NORMAL SECTIONS

BY

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1. Introduction. Let N be a (connected) n-dimensional $(2\leqslant n)$ submanifold of an m-dimensional Euclidean space $\mathbb{E}^m$. For each point p in N and each unit vector t tangent to N at p, the vector t and the normal space $T_p^\perp N$ to N at p determine an $(m-n+1)$-dimensional Euclidean subspace $E(p,t)$ of $\mathbb{E}^m$. The intersection of N and $E(p,t)$ gives locally rise to a (regular) curve γ which is called the *normal section of N at p determined by t* [2]. Submanifolds in Euclidean space with (pointwise) planar normal sections have been investigated by Chen [2, 3, 4] and by Houh and Wang [8]. Chen and one of the authors studied submanifolds N in $\mathbb{E}^m$ whose normal sections are geodesics in N [5, 6], which is equivalent to the study of helical submanifolds [12] (i.e. all geodesics of N, considered as curves in the ambient space $\mathbb{E}^m$, have the same constant curvatures, see also [11]). The only known examples of submanifolds with geodesic normal sections in $\mathbb{E}^m$ are given by the totally geodesic submanifolds of $\mathbb{E}^m$ and the standard immersions of compact rank on symmetric spaces.

In this paper we define k-dimensional normal sections of an n-dimensional submanifold N in $\mathbb{E}^m$ $(3\leqslant n,\ 1<k<n)$. When there exists a natural number k, $1<k<n$, such that all k-dimensional normal sections of N are totally geodesic in N, then this is true for all such numbers k and N is a space of constant sectional curvature which has geodesic (1-dimensional) normal sections. Conversely, all k-dimensional normal sections of a space of constant sectional curvature N which is immersed in $\mathbb{E}^m$ with geodesic normal sections are totally geodesic in N $(1<k<n)$. Submanifolds of dimension at least 2 and of codimension at most 4 in an Euclidean space having constant sectional curvature and geodesic normal sections are classified.

2. k-Dimensional Normal Sections. Let N be an n-dimensional submanifold of $\mathbb{E}^m$ $(3\leqslant n)$ and let k be a natural number such that $1<k<n$. For each point p in N and each k-dimensional subspace π of the tangent space T_pN to N at p, this subspace π and the normal space $T_p^\perp N$ to N at p determine a $(m-n+k)$-dimensional Euclidean subspace $E(N\,;\,p,\,\pi)$ of $\mathbb{E}^m$. The intersection of N and $E(N\,;\,p,\,\pi)$ locally gives a k-dimensional submanifold of N by considering the intersection of a maximal small enough open part U of N containing p with $E(N\,;\,p,\,\pi)$ which is supposed to be connected. This submanifold is called the *k-dimensional normal section of*

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N at p determined by the k-plane π and will be denoted by n.s. $(N ; p, \pi)$. We remark that this definition also applies for $2 \leq n$ and $1 \leq k \leq n$, where $k=1$ corresponds to the usual (1-dimensional) normal sections as in §1 (π is a 1-plane, i.e. a 1-dimensional subspace of T_pN, spanned by a unit vector t and $E(N ; p, \pi)=E(p, t)$ and where n.s. $(N ; p, T_pN)=N$ for $k=n$.

Lemma 1. *Let N be an n-dimensional submanifold of $\mathbb{E}^m$ $(2 \leq n)$ and let p be a point of N. Suppose $k, l \in \mathbb{N}_0$. For each k-plane π and each l-plane π' in T_pN which are transversal, we have the following:*

$$\text{n.s.}(\text{n.s.}(N ; p, \pi \wedge \pi') ; p, \pi)=\text{n.s.}(N ; p, \pi),$$

where $\pi \wedge \pi'$ denotes the $(k+l)$-plane spanned by π and π'.

Proof. Consider the following higher-dimensional normal sections:

$$N'=\text{n.s.}(N ; p, \pi), \quad M=\text{n.s.}(N ; p, \pi \wedge \pi')$$

and

$$M'=\text{n.s.}(M ; p, \pi).$$

Since

$$E(N ; p, \pi)=\pi \wedge T_p^{\perp}N \subset \pi \wedge \pi' T_p^{\perp}N=E(N ; p, \pi \wedge \pi'),$$

we have $N' \subset M$. Moreover

$$E(N ; p, \pi) \subset \pi \wedge T_p^{\perp}M=E(M ; p, \pi),$$

such that $N' \subset E(M ; p, \pi)$. This proves the inclusion $N' \subset M'$.

The inclusion $M' \subset N'$ follows from the fact that $M' \subset N$, from $M' \subset M \subset E(N ; p, \pi \wedge \pi')$ and from $M' \subset E(M ; p, \pi)$ because $E(N ; p, \pi)=E(N ; p, \pi \wedge \pi') \wedge E(M ; p, \pi)$.

As an application of Lemma 1 we mention the fact that a submanifold N of dimension $n \geq 2$, for which there exists a natural number k, $1 \leq k<n$, such that all k-dimensional normal sections are $(k+1)$-planar (i.e. contained in a $(k+1)$-plane of $\mathbb{E}^m$), has (2-) planar normal sections.

Let N^n be a submanifold of $\mathbb{E}^m$ $(2 \leq n)$. Let ∇ and $\tilde{\nabla}$ be the Riemannian connections of N and $\mathbb{E}^m$, respectively, and denote the second fundamental form of N in $\mathbb{E}^m$ by h, i.e.

$$h(X, Y)=\tilde{\nabla}_X Y-\nabla_X Y$$

for all vector fields X and Y tangent to N. For each normal section γ of N at $p=\gamma(0)$, parametrised by arc-length, we have $\nabla_{\gamma'(0)} \gamma'=0$ [2, 3]. We give the following generalization of this result for higher dimensional normal sections.

Proposition 1. *Let N be an n-dimensional submanifold of $\mathbb{E}^m (3 \leq n)$ and let N' be a k-dimensional normal section of N at p $(1<k<n)$. Then the second fundamental form of N' in N is zero at the point p.*

Proof. Suppose $N'=\text{n.s.}$ $(N ; p, \pi)$ for a k-plane π in T_pN. Denote the Riemannian connection of N' by ∇' and let $\bar{h}'$ (respectively h) be the second fundamental form of N' in N (respectively $\mathbb{E}^m$). Then for all vector fields X and Y tangent to N' we have

$$\bar{h}_p(X_p, Y_p)=h_p(X_p, Y_p)+(h'_p(X_p, Y_p),$$

where $\bar{h}_p(X_p, Y_p)$ and $h_p(X_p, Y_p)$ belong to the normal space of N at p because N' is contained in $E(N ; p, \pi)$ whereas $h'_p(X_p, Y_p)$ is tangent to M at p. This implies $h'_p=0$.

In the following we say that N^n has *totally geodesic k-dimensional normal sections* $(1<k<n)$ when each k-dimensional normal section of N is totally geodesic in N. When $k=1$ we say that N has geodesic normal sections if the normal sections of N are geodesics in N. The axiom of k-planes of Cartan [1] immediately implies the following result:

Proposition 2. *Let N be an n-dimensional submanifold in $\mathbb{E}^m$ $(3 \leq n)$ and let k be a natural number satisfying $1<k<n$. If N has totally geodesic k-dimensional normal sections, then N has constant sectional curvature.*

We remark that in Lemma 1, normal sections in N and M are taken in the same point p. The following states that totally geodesic k-dimensional normal sections are characterized by the fact that a k-dimensional normal section „does not change along itself" (compare with [5, 6]).

Theorem 1. *Let N^n be a submanifold in $\mathbb{E}^m$ and let $k \in \mathbb{N}_0$ such that $1<k<n$. Then N has totally geodesic k-dimensional normal sections if and only if for each k-dimensional normal section N' of N, the k-dimensional normal section of N at an arbitrary point q of N' determined by T_qN' is equal to N'.*

Proof. The if-part follows from Proposition 1 and the only if-part holds because for each point in a manifold there exists essentially no more than one totally geodesic submanifold with prescribed tangent space at that point.

3. Totally geodesic normal sections. Proposition 3. *Each submanifold N^n in $\mathbb{E}^m$ with totally geodesic k-dimensional normal sections for some k, $1<k<n$, has geodesic normal sections.*

Proof. The normal section of N at a point $p \in N$ determined by a vector $t \in T_pN \setminus \{0\}$ is given by the intersection of all k-dimensional normal sections of N at p determined by a k-plane containing t. Consider the geodesic γ in N satisfying $\gamma(0)=p$ and $\gamma'(0)=t$. Since k-dimensional normal sections are totally geodesic in N, im γ is contained in each n.s. $(N ; p, \pi)$ where π is a k-plane containing t. Hence γ is (locally) the normal section of N at p determined by t.

In order to prove the converse we certainly must add the condition that N has constant sectional curvature. In fact, this condition is also sufficient.

Theorem 2. *Let N^n be a submanifold with constant sectional curvature in $\mathbb{E}^m$ $(3 \leq n)$. If N has geodesic normal sections, then N has totally geodesic k-dimensional normal sections for each $k \in \mathbb{N}_0$ satisfying $1<k<n$.*

Proof. Let p be a point of N and π a k-plane in T_pN. If N has constant sectional curvature, then there exists a maximal totally geodesic submanifold M of N such that $p \in M$ and $T_pM=\pi$. We will prove that locally M coincides with the k-dimensional normal section N' of N at p determined by π.

Because N' is the union of all normal sections n.s.$(N ; p, V)$ where V is a 1-dimensional subspace of π and because N has geodesic normal sections, each point q of N' belongs to such a n.s. $(N ; p, V)$ which is a geodesic of N. This means that N' is contained in M. Conversely, for each point q in M there exists a geodesic γ in N with $\gamma(0)=p$ and $\gamma(a)=q$ for some $a \in \mathbb{R}^+$. This γ is locally a normal section of N at p determined by a vector in π which implies that $q=\gamma(a) \in N'$. This proves the theorem.

Corollary 1. *Let N^n be a submanifold of $\mathbb{E}^m$ $(3\leqslant n)$. If N has totally geodesic k-dimensional normal sections for some k satisfying $1<k<n$, then N has totally geodesic k-dimensional normal sections for each k such that $1<k<n$.*

From now on, by saying that a submanifold N^n $(3\leqslant n)$ of $\mathbb{E}^m$ has totally geodesic normal sections, we mean that N has totally geodesic k-dimensional normal sections for each k, $1<k<n$. This is equivalent to the fact that N is a space of constant sectional curvature which has geodesic normal sections.

Lemma 2. *Let N^n $(3\leqslant n)$ be a submanifold in $\mathbb{E}^m$ with totally geodesic normal sections. Then all k-dimensional normal sections of N have geodesic normal sections when $1<k<n$.*

Proof. Suppose p to be a point of N and let π be a k-plane in T_pN. Consider $N'=\text{n.s.}(N;p,\pi)$. Then N' is also a normal section of N at each point q of N' (Theorem 1), such that each normal section γ of N' at $q \in N'$ determined by a vector $t \in T_qN'$ can be considered as a normal section of N at q (Lemma 1). By Proposition 3, N has geodesic normal sections such that γ is a geodesic in N contained in N'. Consequently γ is a geodesic in N', which proves that each normal section of N' is a geodesic of N'.

Remark. Combining Lemma 2 and Theorem 2, it follows that each submanifold N^n $(4\leqslant n)$ of $\mathbb{E}^m$ with totally geodesic normal sections has k-dimensional normal sections with totally geodesic normal sections if $3\leqslant k<n$.

Lemma 3. *Let N^n $(3\leqslant n)$ be a submanifold of $\mathbb{E}^m$ with geodesic normal sections. Then for each geodesic γ of N and for all $k \in \mathbb{N}_0$ satisfying $1<k<n$, there exists a k-dimensional normal section N' of N such that γ is a geodesic of N'.*

Proof. Let γ be a geodesic in N. Since N has geodesic normal sections, γ is a normal section of N. Let γ be a normal section of N at p determined by $t \in T_pN$. Choose a k-plane π in T_pN containing t. Then γ is contained in the normal section N' of N at p determined by π, such that γ is a geodesic in N'.

4. Low Codimensional Submanifolds with Totally Geodesic Normal Sections.

We recall the following lemma from [6].

Lemma 4. *Let N^2 be a surface in $\mathbb{E}^m$ with geodesic normal sections. If there exists a geodesic of N which is properly contained in a $(m-2)$-plane, then $m=3$ and N is an open part of a 2-plane $\mathbb{E}^2$ in $\mathbb{E}^3$ or $m=4$ and N is an open part of a 2-sphere $S^2 \subset \mathbb{E}^3 \subset \mathbb{E}^4$.*

From this we derive the following result for submanifolds in $\mathbb{E}^m$ with codimension greater than two.

Proposition 4. *Let N^n $(3\leqslant n)$ be a submanifold of $\mathbb{E}^m$ with codimension $m-n\geqslant 3$. If N has totally geodesic normal sections, then all geodesics of N are $(m-n-1)$-planar.*

Proof. Let γ be a geodesic of N. Then there exists a 2-dimensional normal section N' such that γ is a geodesic of N'. N' is a surface with geodesic

normal sections in a $(m-n+2)$-dimensional Euclidean space. If γ is $(m-n)$-planar, then Lemma 4 implies that $m-n$ is equal to 1 or equal to 2, which contradicts the fact that $m-n\geqslant 3$. Hence all geodesics of N are $(m-n-1)$-planar.

Theorem 3. *Let N^n $(3\leqslant n)$ be a submanifold of $\mathbb{E}^m$ with totally geodesic normal sections. If $m-n\leqslant 4$, then N is totally umbilical in $\mathbb{E}^m$.*

Proof. Put $m-n=1$. Then N^n is a hypersurface in $\mathbb{E}^{n+1}$ and N^n has planar geodesics. Theorem 3 then follows from results of Hong [7], Little [9] and Sakamoto [10].

When $m-n=2$, there can never exist a geodesic which is proper 3-planar by Proposition 8 in [6]. Hence all geodesics of N are planar and the result follows from [7, 9, 10].

If $m-n=3$, then the geodesics of N are 2-planar by Proposition 4. Again [7, 9, 10] imply Theorem 3.

Finally, when $m-n=4$, the 2-dimensional normal sections of N are surfaces with geodesic normal sections (Lemma 2) in a 6-dimensional Euclidean space and they have constant Gaussian curvature. By Theorem 20 in [6] the geodesics of these 2-dimensional normal sections are planar. Consequently Lemma 3 implies that the geodesics of N are planar. Using [7, 9, 10], this proves Theorem 3.

Combined with Theorem 20 in [6], this yields the following result which we state for submanifolds with codimension 4 but which can be given in the (equivalent) form for codimension at most 4.

Theorem 4. *Let N^n $(2\leqslant n)$ be a submanifold with constant sectional curvature of codimension 4 in a Euclidean space $\mathbb{E}^{n+4}$. If N has geodesic normal sections, then N is an open part of one of the following submanifolds:*

(i) *a n-plane $\mathbb{E}^n$ in $\mathbb{E}^{n+4}$;*

(ii) *a n-sphere S^n in a $(n+1)$-plane of $\mathbb{E}^{n+4}$;*

(iii) *a Veronese surface in a hyperplane $\mathbb{E}^5$ of $\mathbb{E}^6$.*

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