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ON THE BEHAVIOUR OF THE SOLUTIONS OF THE EQUATION $(a(t)x'(t))^{(n)}+f(t, x(t))=0$.

BY

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1. Introduction. In this paper, using an integral inequality, we study the behaviour at infinity of the solutions of the equation

$$(a(t)x'(t))^{(n)}+f(t, x(t))=0, \tag{1}$$

where $n\geq 1$, $t\in R_+$ and a, x, f are real-valued functions.

In order to establish the results, we shall be in need of the following result:

Lemma 1. ([1]). *Let the following conditions be satisfied:*

1. *$u, a: I=[0, h)\to R_+=[0, +\infty)$ are continuous functions, where $0<h\leq\infty$ and $a(t)\geq 1$ is nondecreasing on I;*

2. *the functions $f_i: D=\{(t, s): 0\leq s\leq t<h\}\to R_+$ are continuous and nondecreasing in t for fixed $s\in I$, $i=1, 2, \ldots, m$;*

3. *the following integral inequality*

$$u(t)\leq a(t)+\sum_{i=1}^{m}\int_0^t f_i(t, s)[u(s)]^{r_i}\,ds \tag{2}$$

holds for all $t\in I$, where r_i are numbers from $(0, 1]$. Then

$$u(t)\leq a(t)\prod_{i=1}^{m} G_i(t), \quad t\in I \tag{3}$$

where

$$G_i(t)=\begin{cases}\left[1+(1-r_i)\prod_{k=1}^{i-1} G_k(t)\int_0^t f_i(t, s)ds\right]^{\frac{1}{1-r_i}}, & \text{for } 0<r_i<1;\\ \exp\left\{\prod_{k=1}^{i-1} G_k(t)\int_0^t f_i(t, s)ds\right\}, & \text{for } r_i=1, \quad i=\overline{1, m}.\end{cases} \tag{4}$$

(Here it is supposed that $\prod_{k=1}^{m} G_k(t)=1$, $t\in I$).

Remark 1. If the functions

$$\int_0^t f_i(t, s)ds, \quad i=\overline{1, m} \text{ are bounded on } I,$$

then the functions G_i defined by relation (4) have the same property.

2. Main results. For the equation (1) we shall assume the following hypotheses :

(H) : $a : R_+ \to R_+$ is continuous, $a(t) \neq 0$ for $t \in R_+$, either $a(\,.\,)x'(\,.\,) \in C^n$, or $a \in C^n$;

$f : R_+ \times R \to R_+$ is continuous and satisfies the inequality

$$|f(t, x)| \leqslant \sum_{i=1}^{m} b_i(t) |x|^{r_i} + b_{m+1}(t) \tag{5}$$

for all $t \in R_+$ and $x \in R$, where $r_i \in (0, 1]$ are constant numbers and b_j, $j = \overline{1, m+1}$, are given nonnegative functions ;

the local existence and uniqueness of the solutions of (1) are valid, Then let us consider the functions

$$W_k(t, s) \overset{\Delta}{=} \int^t \frac{(z-s)^k}{a(z)}\,\mathrm{d}z, \quad k \in N, \tag{6}$$

for $t \geqslant s \geqslant 0$.

Proposition 1. *Under the hypotheses (H) the functions W_k, defined by the formula* (6), *have the following properties :*

(a) *for any $k \in N$, $W_k(t, s)$ is increasing in t for each fixed $s \in R_+$ and decreasing in s for each fixed $t \in R_+$;*

(b) *if $i < k$ and $\lim_{t\to\infty} W_k(t, s_0) = A_k < +\infty$, ($s_0 \geqslant 0$ fixed), then $\lim_{t\to\infty} W_i(t, s_0) = A_i < +\infty$;*

(c) *if $i < k$ and $\lim_{t\to\infty} W_i(t, s_0) = +\infty$, ($s_0 \geqslant 0$ fixed), then $\lim_{t\to\infty} W_k(t, s_0) = +\infty$ and, moreover, $W_i(t, s_0) \leqslant W_k(t, s_0)$ for sufficiently large t.*

Proof. The property (a) is an immediate consequence of the definition of W_k by the formula (6), because the function under the integral is positive. Then, applying the comparison criterion, we obtain

$$\lim_{z\to\infty} (z-s_0)^{i-k} = 0, \quad \text{for } i < k, \tag{7}$$

and hence the properties (b) and (c) follow. Q.E.D.

With regard to the equation (1) we start with the following global existence result.

Proposition 2. *In the hypotheses (H), every solution of the equation* (1), *can be extended on R_+.*

Proof. Let $x = x(t)$ be the solution of (1) satisfying the initial conditions

$$x(0) = x_0^0, \quad x'(0) = x_0^1, \ldots\ldots, x^{(n)}(0) = x_0^n \tag{8}$$

defined on the maximal interval $[0, L)$, $(0 < L \leqslant +\infty)$. Integrating both members of the equality (1) $k \leqslant n$ times and using Dirichlet's formula ([2]), we obtain

$$(a(t)x'(t))^{(n-k)} = \sum_{i=0}^{k-1} (a(t)\ x'(t))^{(n-k+i)}_{t=0} \frac{t^i}{i!} - \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} f(s, x(s))\mathrm{d}s, \quad t \in [0, L). \tag{9}$$

whence

$$x'(t) = \sum_{i=0}^{n-1} \frac{(a(t)x'(t))^{(i)}_{t=0}}{i!} \frac{t^i}{a(t)} - \int_0^t \frac{1}{(n-1)!} \frac{(t-s)^{n-1}}{a(t)} f(s, x(s))\mathrm{d}s, \quad t \in [0, L). \tag{10}$$

Integrating (10) from 0 to t and taking into account (6), we obtain

$$x(t) = x(0) + \sum_{i=0}^{n-} \frac{(a(t)x'(t))^{(i)}_{t=0}}{i} W_i(t, 0) - \frac{1}{(n-1)!} \int_0^t W_{n-1}(t, s) f(s, x(s))\mathrm{d}s, \quad t \in [0, L). \tag{11}$$

According to the hypotheses (H), from (11) we obtain

$$x(t) \leqslant A(t) + \sum_{j=}^{m} \int_0^t \frac{1}{(n-1)!} W_{n-1}(t, s) b_j(s) |x(s)|^{r_j} \mathrm{d}s, \ t \in [0, L) \tag{12}$$

where

$$A(t) = 1 + x(0) + \sum_{i=0}^{n-1} \frac{(a(t)x'(t))^{(i)}_{t=0}}{i!} W_i(t, 0) + \int_0^t \frac{1}{(n-1)!} W_{n-1}(t, s) b_{m+1}(s)\mathrm{d}s \tag{13}$$

Now we remark that the inequality (12) is of the form (2) and that all conditions of Lemma 1 are satisfied. Hence, we obtain

$$|x(t)| \leqslant A(t) \prod_{i=}^{m} G_i(t), \quad t \in [0, L). \tag{14}$$

where

$$G_i(t) = \begin{cases} \left[1 + (1-r_i) \prod_{k=0}^{i-1} G_k(t) \int_0^t \frac{1}{(n-1)!} W_{n-1}(t, s) b_i(s)\mathrm{d}s\right]^{\frac{1}{1-r_i}}, \\ \text{for } 0 < r_i \leqslant 1 \ ; \\ \exp\left\{\prod_{k=0}^{i-1} G_k(t) \int_0^t \frac{1}{(n-1)!} W_{n-1}(t, s) b_i(s)\mathrm{d}s\right\}, \quad \text{for } r_i = 1. \end{cases} \tag{15}$$

The inequality (14) involves $L = +\infty$. Indeed, if $L < +\infty$, then from the continuity of $x = x(t)$ and the inequality (14) we can assert that there exists $\lim_{\substack{t\to L\\ t<L}} x(t) = x_L^0$. Moreover, the relation (9) implies the existence of

$$\lim_{\substack{t\to L\\ t<L}} x^{(i)}(t) = x_L^i < +\infty, \quad \text{for } i = \overline{1, n}.$$

Now, if we consider the solution of the equation (1) with the initial condisions

$$x(L) = x_L^0, \quad x'(L) = x_L^1, \quad x^{(n)}(L) = x_L^n, \tag{16}$$

then, by the local existence and uniqueness for the equation (1), the solution $x=x(t)$ must exist on a certain interval $[0, L+M)$, where $M>0$, in contradiction with the fact that the maximal interval for existence of the solution is $[0, L)$. Q.E.D.

The inequality (14) used to prove the existence of solution on the half-axis, will be useful below, namely, we shall prove the following result:

Theorem 1. *If, in addition to the hypotheses (H), the following conditions:*

1° $\int_0^\infty W_{n-1}(t, s)b_j(s)ds$ *are bounded on* R_+, $j=\overline{1, m+1}$;

2° $\lim_{t\to\infty} W_{n-1}(t, 0)=A_{n-1}<+\infty$;

are satisfied, then any solution of the equation (1) is bounded on R_+.

Proof. We saw that, in the hypotheses (H), for any solution $x=x(t)$ of the equation (1), the inequality (14) holds. By the Remark 1, the condition 1° assures the boundedness of the functions G_i (defined by formula (15)). The condition 2° together with the condition $\int_0^\infty W_{n-1}(t, s)b_{m+1}(s)ds$-bounded, implies the boundedness of the function A defined by the formula (13) and taking into account (14), we obtain the boundedness of the solution $x=x(t)$. Q.E.D.

Next, we analyse the case when the condition 2° is not satisfied, viz., taking into consideration the monotonicity of $W_{n-1}(\cdot, 0)$, the case when $W_{n-1}(t, 0)\to+\infty$ for $t\to+\infty$.

Theorem 2. *The hypotheses (H) being satisfied, we assume in addition that*

3° $\int_0^\infty b_{m+1}(s)ds<+\infty$; $\int_0^\infty W^{r_j}_{n-1}(t, 0)b_j(s)ds<+\infty$; $j=\overline{1, m}$;

4° $\lim_{t\to\infty} W_{n-1}(t, 0)=+\infty$;

then for every solution $x=x(t)$ of the equation (1), we have the estimation

$$x(t)=O(W_{n-1}(t, 0)), \tag{17}$$

and

$$(a(t)x'(t))^{(i)}=O(t^{n-1-i}). \tag{17'}$$

Proof. By Proposition 2, every solution $x=x(t)$ of the equation (1) exists on R_+. Then, taking a value $t_0>0$ as the lower limit of the integrals and, following the same way as in the proof of Proposition 2, we find

$$x(t)=x(t_0)+\sum_{i=0}^{n-1}\frac{(a(t)x'(t))^{(i)}_{t=t_0}}{i!}W_i(t, t_0)-\int_{t_0}^t\frac{1}{(n-1)!}W_{n-1}(t, s)f(s, x(s))ds,\quad t\in R_+. \tag{18}$$

Since $W_{n-1}(t, 0)>0$, for $t\in[t_0, +\infty)$, we can divide the equality (18) by $W_{n-1}(t, 0)$ and taking into account the properties *a*) and *c*) given by Proposition 1, we obtain (for sufficiently large t_0) the inequality

$$\frac{|x(t)|}{W_{n-1}(t, 0)}\leqslant\frac{|x(t_0)|}{W_{n-1}(t, 0)}+\sum_{i=1}^{n-1}\frac{(a(t)x'(t))^{i}_{t=t_0}}{i!}+ \tag{19}$$
$$+\int_{t_0}^t\frac{1}{(n-1)!}f(s, x(s))ds,\quad t\in[t_0, +\infty).$$

By the hypotheses (H), this relation becomes

$$\frac{|x(t)|}{W_{n-1}(t, 0)}\leqslant\tilde{A}(t)+\sum_{j=1}^{m}\int_{t_0}^t\frac{W^{r_j}_{n-1}(t, 0)b_j(s)}{(n-1)!}\left[\frac{|x(t)|}{W_{n-1}(t, 0)}\right]^{r_j}ds,\quad t\in[t_0, +\infty), \tag{20}$$

where:

$$\tilde{A}(t)=1+\frac{|x(t_0)|}{W_{n-1}(t, 0)}+\sum_{i=0}^{n-1}\frac{(a(t)x'(t))^{(i)}_{t=t_0}}{i!}+\int_{t_0}^t\frac{b_{m+1}(s)}{(n-1)!}ds. \tag{21}$$

The inequality (20) has the form (2). All the conditions of Lemma 1 are fulfilled with $u(t)=|x(t)|/W_{n-1}(t, 0)$ so that we can write:

$$\frac{|x(t)|}{W_{n-1}(t, 0)}\leqslant\tilde{A}(t)\prod_{i=}^{m}G_i(t),\quad t\in[t_0, +\infty), \tag{22}$$

where

$$G_i(t)=\begin{cases}\left[1+(1-r_i)\prod_{k=1}^{i-1}G_k(t)\int_{t_0}^t\frac{1}{(n-1)!}W^{r_i}_{n-1}(s, 0)b_i(s)ds\right]^{\frac{1}{1-r_i}}, & \text{for } 0<r_i<1,\\ \exp\left\{\prod_{k=1}^{i-1}G_k(t)\int_{t_0}^t\frac{1}{(n-1)!}W^{r_i}_{n-1}(s, 0)b_i(s)ds\right\}, & \text{for } r_i=1.\end{cases}$$

The conditions (3°) and (4°) imply the fact that the right hand side of the inequality (22) is bounded, whence the relation (17) holds. Writting now the relation analogous with the identity (9), we obtain, by dividing both sides by t^{k-1}

$$\frac{|(a(t)x'(t))^{(n-k)}|}{t^{k-1}}\leqslant\sum_{=}^{k-1}(a(t)x'(t))^{(n-k+i)}_{t=\cdot}\frac{1}{i!}+ \tag{23}$$
$$+\frac{1}{(k-1)!}\int_{t_0}^t|f(s, x(s))|ds,\quad \text{for } t\in[t_0+1, +\infty),$$

which, in view of hypotheses (H), implies,

$$\frac{(a(t)x'(t))^{(n-k)}}{t^{k-1}} \leqslant \sum_{i=0}^{k-1} (a(t)x'(t))^{(n-k+i)}_{t=t_0} \frac{1}{i!} + \tag{24}$$

$$+\int_0^t b_{m+1}(s)ds + \frac{1}{(k-1)!}\sum_{j=1}^{m}\int_{t_0}^{t} W'^{r_j}_{n-1}(s,0)b_j(s)\left[\frac{x(s)}{W_{n-1}(s,0)}\right]^{r_j} ds.$$

Using now (17) we obtain

$$\frac{(a(t)x'(t))^{(n-k)}}{t^{k-1}} \leqslant A + \int_{t_0}^{\infty} b_{m+1}(s)ds + \tag{25}$$

$$+\frac{1}{(k-1)!}\sum_{j=1}^{m} B^{r_j}\int_{t_0}^{\infty} W^{r_j}_{n-1}(s,0)b_j(s)ds \leqslant C, \quad t \geqslant t_0+1$$

and hence $(a(t)x'(t))^{n-k} = O(t^{k-1})$. Putting here $n-k=i$ we obtain the relation (17'). Q.E.D.

3. Examples. 1) Let us now consider the equation

$$x^{(n)}(t) + f(t, x(t)) = 0, \quad t \in R_+. \tag{26}$$

Applying the Theorem 2 we obtain the following result:

Proposition 3. *Suppose that, with regard to the equation (26), the hypotheses (H) are satisfied. If, in addition, the following conditions are fulfilled*

5°, $\int_0^\infty b_{m+1}(t)dt < +\infty$. 6°, $\int_0^\infty t^{(n-1)r_j} b_j(t)dt < +\infty$, $j=\overline{1,m}$,

then, for every solution $x=x(t)$ of the equation (26) the following estimates

$$x(t) = O(t^{n-1}) \text{ and } x^{(k)}(t) = O(t^{n-1-k}), \ k=\overline{1,n}$$

hold.

2) Consider the equation

$$x^{(n)} + \sum_{j=1}^{m} C_j(t)x^{r_j} = C_{m+1}(t), \quad t \in R_+, \tag{27}$$

where $r_j \in (0, 1]$ and $C_j : R_+ \to R$ are continuous functions, $j=\overline{1, m+1}$. If the functions $C_{m+1}(t)$ and $t^{(n-1)r_j}|C_j(t)|$ belong to the class $L_1([0, +\infty))$, $j=\overline{1,n}$, then, by Proposition 3, all the solutions of the equation (27) exist on R_+ and satisfy the relations

$$|x(t)| = O(t^{n-1}) \text{ and } |x^{(k)}(t)| = O(t^{n-1-k}).$$

3) We consider the equation

$$(e^{ct}x'(t))^{(n)} + \sum_{j=1}^{m} p_j(t)x^{r_j} = p_{m+1}(t), \quad t \in R_+ \tag{28}$$

where $c>0$, $r_j \in (0, 1]$ are constant numbers, and p_j, $j=\overline{1, m+1}$ are polynomials.

The function $W_{n-1}(\cdot, 0)$, associated to the equation (28), has the form:

$$W_{n-1}(t, 0) = \int_0^t \frac{z^{n-1}}{e^{cz}} dz = q(t)e^{-ct}$$

where q is a polynominal of degree $n-1$. It is clear that the conditions of Theorem 1 are satisfied and thus every solution of the equation (28) exists and is bounded on R_+.

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