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ON A STRONGER BERTRAND'S POSTULATE

BY

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This paper deals with some prime number inequalities related with Bertrand's postulate. An improvement of a result of V. S. Udrescu [13] is given and the first hundred values for a certain number-theoretical function are tabulated, thus extending a result due to R. E. Dressler [3].

1. Introduction. Let p_n be the n-th prime and let us denote by $d(k)$ the least positive integer n for which

$$p_{n+1}<2p_n-k \tag{1}$$

is valid (see [3], [2]), i.e. $d(k)=M$ if (1) is true for every $n\geqslant M$ but $p_M\geqslant 2p_{M-1}-k$ (for $k=0$ (1) reduces to Bertrand's postulate [6], [7, p. 343]). A general (but crude) upper bound for this number-theoretical function was given, using an inequality due to Rosser [12, Theorem 29], by Udrescu [13] and an application of the equality $d(10)=7$ on partitions with prime summands was given by Dressler [3] (see also [8], [4], [5] for other theorems of this type).

2. An upper bound for $d(k)$. The following theorem is an improvement of a result of Udrescu [13]. Here $[x]$ is the greatest integer $\leqslant x$.

Theorem 1. *For every integer $k\geqslant 1$ we have*

$$d(k)\leqslant (1/4)\,\{M_k+2+(M_k^2+12M_k+4)^{1/2}\}, \tag{2}$$

where $M_k=\max\,(118,\ [13k/12]+1)$.

Proof. Let $k\geqslant 1$ be a fixed integer and let $M:=M_k$. A theorem of Rohrbach and Weis [10] states that for $x\geqslant 118$, in $(x,\ 14x/13]$ occurs at least a prime number (we must also note that 118 is the best integer constant in the above theorem of Rohrbach and Weis).

Because for $x>13k/12$ we have $2x-k>14x/13$ it follows from the aforementioned result of Rohrbach and Weis that for

$$x>\max\,(118,\ 1+[13k/12])=M,$$

between $(x,\ 2x-k)$ occurs at least a prime number.

Hence for every integer n for which $p_n>M$ between p_n and $2p_n-k$ there exists at least a prime number, i.e. we have

$$p_{n+1}<2p_n-k.$$

From a theorem of J. B. Rosser [11] we have $p_n > n \log n$. But

$$\log (1+1/x) > 2/(2x+1),$$

for each $x>0$ (see D. S. Mitrinović [9, p. 274]) so, for every $y>1$, we have

$$y \log y > 2y(y-1)/(y+1),$$

Thus, if the following inequality

$$2n(n-1)/(n+1) \geq M \tag{3}$$

holds, then $p_n > M$.

The inequality (3) is equivalent with

$$2n^2-(M+2)\,n-M \geq 0, \tag{4}$$

and furthermore with

$$n \geq (1/4)\ \{M+2+(M^2+12M+4)^{1/2}\}. \tag{5}$$

Thus if (5) is valid then $p_n > M$ and hence

$$p_{n+1} < 2p_n - k.$$

From the definition of the number-theoretical function d it follows that

$$d(k) \leq (1/4)\ \{M+2+(M^2+12M+4)^{1/2}\},$$

which completes the proof.

If instead of the inequality

$$y \log y > 2y(y-1)/(y+1)$$

we had used in the proof of Theorem 1 the crude inequality

$$y \log y > (\log y)^2,$$

which was used by Udrescu [13], we should have obtained the following estimation

$$d(k) \leq \exp\ \{\max\ (118,\ 1+[13k/12])\},$$

which is still better than Udrescu's inequality

$$d(k) \leq \exp\ \{[1+\exp\ (k+10)]^{1/2}\}.$$

3. The first hundred precise values of $d(k)$. In the remainder of our present paper we shall find the precise values of $d(k)$, $k=1, 2, \ldots, 100$ (by Bertrand's postulate we have $d(0)=1$). It is almost sufficient to find $d(100)$ because it is easy to show that the number-theoretical function d is nondecreasing, i.e. we have $d(k-1) \leq d(k)$.

Theorem 2. *We have* $d(100)=31$.

A proof of Theorem 2 can be performed using the above Theorem 1 because the upper bound (2) is not very large for $k=100$. However, for the sake of brevity in computations, we shall use a slightly different method based upon the proof of Theorem 1.

Proof of Theorem 2. It is easy to see that $M_{100}=118$. From the proof of Theorem 2 we know that

$$p_{n+1} < 2p_n - 100 \tag{6}$$

for $p_n > 118$. Because $p_{31}=127>118$ it follows that (6) is valid for $n \geq 31$. Because

$$p_{31}=127>126=2\cdot 113-100=2p_{30}-100,$$

we get $d(100)=31$.

Now the proof of Theorem 2 is complete.

Using a table for primes (see for instance C. L. Backer and F. J. Gruenberger [1]) and the monotonicity property $d(k-1) \leq d(k)$ we find that the first hundred of the terms of the infinite sequence $d(1)$, $d(2)$, ... are the following

3, 3, 5, 5, 5, 5, 5, 7, 7, 7,
7, 7, 7, 7, 9, 9, 10, 10, 10, 10,
10, 10, 10, 10, 12, 12, 12, 12, 12, 12,
12, 12, 13, 13, 13, 13, 13, 13, 15, 15,
16, 16, 16, 16, 16, 16, 17, 17, 17, 17,
17, 17, 17, 17, 19, 19, 19, 19, 19, 19,
19, 19, 20, 20, 20, 20, 22, 22, 22, 22,
22, 22, 22, 22, 23, 23, 24, 24, 24, 24,
25, 25, 25, 25, 25, 25, 25, 25, 25, 25,
25, 25, 26, 26, 26, 26, 26, 26, 31, 31.

Using a greater amount of computations, we are able to find other precise values for $d(k)$. For example we have $d(101)=d(102)=31$.

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