$$
(2.13) \qquad \begin{cases} y'(t)+Ay(t)+\partial I_K(y(t)) \ni u(t) & \text{for } t\in[0, T] \\ y(0)=y_0 \end{cases}
$$

where

$$
A=\frac{b^2}{(\Delta x)^2}\begin{vmatrix} 2 & -1 & 0 & 0 \ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \ldots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 \ldots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 \ldots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \ldots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 \ldots & 0 & -1 & 2 \end{vmatrix}.
$$

The relation (2.13) can be written as:

$$
(2.14) \qquad \begin{cases} y_i'(t)+(Ay(t))_i=u_i(t) & \text{on } \{t\in[0, T]\ ;\ 0<y_i(t)<a\} \\ u_i(t)\leqslant 0 & \text{on } \{t\in[0, T]\ ;\ y_i(t)=0\} \\ u_i(t)\geqslant 0 & \text{on } \ t\in[0, T]\ ;\ y_i(t)=a\} \\ 0\leqslant y_i(t)\leqslant a & \text{on } [0, T] \\ y_i(0)=y^0(x_i)\in[0, a] \end{cases}
$$

for $i=2, 3, \ldots, n-1$.

We approximate the cost functional by:

$$
\Delta x\left[\int_0^T\sum_{i=1}^{n}(g(t, y_i(t))+h(u_i(t)))dt+\sum_{i=1}^{n}\varphi_0(y_i(T))\right].
$$

If $(y, u)\in W^{1,2}([0, T]\ ;\ R^n)$ is an optimal pair for the (P_2) problem then, from the maximum principle (see theorem 1.1), we have assured the existence of some elements $p\in BV([0, T]\ ;\ R^n)$ and $\xi\in L^\infty(0, T\ ;\ R^n)$ satisfying:

$$
(2.15) \qquad \begin{cases} p_i'(t)-(Ap(t))_i=\xi_i & \text{a.e. } \{t\in[0, T]\ ;\ 0<y_i(t)<a\} \\ \xi(t)\in\partial g(t, y(t)) \\ p_i(t)=0 & \text{a.e. } \{t\in[0, T]\ ;\ y_i(t)=0,\ u_i(t)\neq 0\} \\ p_i(t)=0 & \text{a.e. } \{t\in[0, T]\ ;\ y_i(t)=a,\ u_i(t)\neq 0\} \\ p(T)+\partial\varphi_0(y(T))\ni 0 \\ p(t)\in\partial h(u(t)) & \text{a.e. } t\in[0, T] \end{cases}
$$

where $i=\overline{2, n-1}$, $A^*=A$ (A is a symmetric matrix).

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ON THE EXISTENCE AND UNIQUENESS OF INSTATIONARY CONVECTION IN POROUS MEDIA

BY

DAN POLIȘEVSKI

1. Preliminaries. Let Ω be an open connected bounded set in $\mathbf{R}^n$ ($n=2$ or 3) locally located on one side of the boundary $\partial\Omega$, which is a Lipschitz manifold composed of a finite number of connected components; let real $\theta>0$ be a fixed and let us denote with Q the cylinder $\Omega\times(0, \theta)$.

With the assumptions and approximations which are frequently used for the thermal convection in a homogeneous porous medium saturated with an incompressible fluid, the governing evolution system of the Darcy-Boussinesq equations for the filtration velocity u, the pressure p and the temperature T may be written in a non-dimensional form as:

$$
(1.1) \qquad \operatorname{div} u=0 \ \text{ in } Q
$$

$$
(1.2) \qquad \gamma\partial_t u+u=-\nabla p+aTe \ \text{ in } Q \ \left(\partial_t=\frac{\partial}{\partial t}\right),
$$

$$
(1.3) \qquad \partial_t T+u\nabla T=\Delta T \ \text{ in } Q,
$$

where $\gamma>0$ and $a>0$ (the Rayleigh number) are two fixed real numbers, while e is the fixed versor of the gravitational acceleration. As we are in the non-dimensional case, we can assume that Ω can be included in a n-cube of edge-length 1.

Let Γ_0 be a subset of $\partial\Omega$ with positive surface measure and let us denote with $\Gamma_1=\partial\Omega\setminus\Gamma_0$; the boundary conditions are:

$$
(1.4) \qquad u\cdot\nu=0 \quad \text{on } \partial\Omega\times(0, \theta),
$$

$$
(1.5) \qquad \frac{\partial T}{\partial\nu}=0 \quad \text{on } \Gamma_1\times(0, \theta),
$$

$$
(1.6) \qquad T=\tau \quad \text{on } \Gamma_0\times(0, \theta),
$$

where ν denotes the unit outward normal on $\partial\Omega$ and $\tau=\tau(x, t)$ is the most important datum of our problem.

Also, u and T have to obey in some sense to the initial conditions

$$
(1.7) \qquad u(0)=u^\circ \quad \text{in } \Omega, \qquad\qquad (1.8) \qquad T(0)=T^\circ \quad \text{in } \Omega.
$$

Let us denote with

(1.9) $$H=\{u\in L^2(\Omega)/\operatorname{div} u=0 \text{ in } \Omega,\ u\cdot\nu=0 \text{ on } \partial\Omega\}$$

the closure of

$$\mathscr{X}(\Omega)=\{u\in\mathscr{D}(\Omega)\mid \operatorname{div} u=0 \text{ in } \Omega\}$$

in $L^2(\Omega)$, and with V the closure in $H^1(\Omega)$ of

$$\mathscr{O}(\Omega)=\{S\in\mathscr{E}(\bar\Omega)\mid S=0 \text{ on } \Gamma_0\}.$$

We assume that Γ_0 is sufficiently smooth as

(1.10) $$V=\{S\in H^1(\Omega)\mid S=0 \text{ on } \Gamma_0\}.$$

As usual the scalar products and norms in $L^2(\Omega)$, $H^m(\Omega)$ and $H_0^1(\Omega)$ are respectively denoted by

$$(u,v)=\int_\Omega u\cdot v,\quad |u|=(u,u)^{1/2},$$

$$((u,v))_m=\sum_{|i|\le m}\left(\frac{\partial u}{\partial x_i},\frac{\partial v}{\partial x_j}\right),\quad \|u\|_m=((u,u))_m^{1/2},$$

$$((u,v))=(\nabla u,\nabla v),\quad \|u\|=((u,u))^{1/2},$$

and the norm in $L^p(\Omega)$ $(p\neq 2)$ by $|\ |_p$. We agree to use the same notations for the scalar products and norms in the corresponding vector spaces, i.e. $L^2(\Omega)=[L^2(\Omega)]^n$ and so on.

Reminding here the Friedrichs' inequality (see [1])

(1.11) $$|S|\le C_0\|S\|\quad (\forall)\ S\in V,$$

we remark that $\|\ \|$ is a norm on V, also.

We start the study with the following hypotheses:

(1.12) $\tau\in H^1(0,\theta;H^{3/2}(\partial\Omega))$, (1.13) $u^0\in H\cap H^1(\Omega)$,

(1.14) $$T^0\in H^2(\Omega) \text{ with } (T^0-\tau(0))\in V.$$

Remark 1.1. From (1.12) it follows that τ is a.e. equal to a function of $C^0(0,\theta;H^{3/2}(\partial\Omega))$ and hence $\tau(0)$ in (1.14) makes sense.

Lemma 1.1. *For any $h>0$, there exists an element*

(1.15) $$\tau_h\in H^1(0,\theta;H^2(\Omega)),$$

satisfying a.e. in $(0,\theta)$

(1.16) $$\tau_h=\tau \text{ on } \partial\Omega,$$

(1.17) $$|S\nabla\tau_h|\le h\|S\|\quad (\forall)\ S\in V.$$

Proof. The construction of τ_h is the same as in the steady case, which can be found in [2]. The only new point is that the vector-valued function $L(\tau)\,t:=L(\tau)(t))$, (where $L:H^{3/2}(\partial\Omega)\to H^2(\Omega)$ is the linear and continuous „lifting of trace" operator) has the property

(1.18) $$L(\tau)\in H^1(0,\theta;H^2(\Omega)).$$

But noticing that $\partial_t(L(\tau))=L(\partial_t\tau)$, the property (1.18) is a straight consequence of (1.12).

Keeping the right to choose the parameter $h>0$ later, in a proper way, we introduce

(1.19) $$S=T-\tau_h.$$

Thus the system (1.1)—(1.8) becomes:

(1.20) $\operatorname{div} u=0$ in Q, (1.21) $\partial_t u+u=-\nabla p+a(S+\tau_h)\,e$ in Q

(1.22) $$\gamma\partial_t S-\Delta S+u\nabla(S+\tau_h)=(\Delta\tau_h-\gamma\partial_t\tau_h) \text{ in } Q$$

(1.23) $$u\cdot\nu=0 \text{ on } \partial\Omega\times(0,\theta),$$

(1.24) $\dfrac{\partial S}{\partial\nu}=-\dfrac{\partial\tau_h}{\partial\nu}$ on $\Gamma_1\times(0,\theta)$, (1.25) $S=0$ on Γ_0,

(1.26) $u(0)=u^0\in H\cap H^1(\Omega)$, (1.27) $S(0)=S^0:=T^0-\tau_h(0)\in V\cap H^2(\Omega)$.

2. The weak solutions. Let us suppose that u, p and S are smooth solutions of (1.20)—(1.27): then, making the dual product of (1.21) and (1.22) with $v\in\mathscr{X}(\Omega)$ and $T\in\mathscr{O}(\Omega)$, one can easily obtain:

(2.1) $$\frac{d}{dt}(u,v)+(u,v)=a(S+\tau_h,\ e\cdot v),$$

(2.2) $$\gamma\frac{d}{dt}(S+\tau_h,T)+((S+\tau_h,T))+(u\nabla(S+\tau_h),T)=0.$$

Taking in account that $\mathscr{X}(\Omega)$ and $\mathscr{O}(\Omega)$ are dense in H and respectively in V, the relations (2.1)—(2.2) suggest the following variational formulation of the problem (1.20)—(1.27):

Problem 2.1. *Find $u\in L^\infty(0,\theta;H\cap H^1(\Omega))$ and $S\in L^\infty(0,\theta;L^2(\Omega))\cap\cap L^2(0,\theta;V)$ satisfying the initial conditions (1.26)—(1.27) and the equations (2.1)—(2.2) for any $v\in H$, respectively $T\in V$.*

Theorem 2.1. *The Problem 2.1 has at least one solution.*

Proof. In order to prove the existence of the generalized solutions of the Problem 2.1 we employ the Galerkin method. Let

(2.3) $$\{v_j\}_{j\in N}\subset\mathscr{X}(\Omega)\ \text{ and }\ \{T_j\}_{j\in N}\subset\mathscr{O}(\Omega)$$

be two sequences which are free and total in H, respectively in V. For each $i\in N$, denoting with H_i the space spanned by the set $\{v_j\mid j=1,\dots,i\}$ and with V_i the space spanned by the set $\{T_j\mid j=1,\dots,i\}$, we define an approximate solution of the Problem 2.1 by

$$u_i(t)=\sum_{j=1}^{i}u_{ij}(t)\,v_j\ \text{ and }\ S_i(t)=\sum_{j=1}^{i}S_{ij}(t)\,T_j$$

satisfying for any $k\in\{1,\dots,i\}$ the system

(2.4) $$(\partial_t u_i, v_k) + (u_i, v_k) = a(S_i + \tau_h, e \cdot v_k),$$

(2.5) $$\gamma(\partial_t S_i, T_k) + ((S_i, T_k)) + (u_i \nabla(S_i + \tau_h), T_k) = -\gamma(\partial_t \tau_h, T_k) - (\nabla\tau_h, \nabla T_k),$$

(2.6) $u_i(0) = u_i^0$, (2.7) $S_i(0) = S_i^0$,

where $u_i^0 \in H_i$ and $S_i^0 \in V_i$ are chosen such that

(2.8) $u_i^0 \to u^0$ strongly in H, (2.9) $S_i^0 \to S^0$ strongly in V.

For any $k \in \{1, \dots, i\}$, let us write (2.4)–(2.7) in terms of its effective unknowns, that is $u_{ij}(t)$ and $S_{ij}(t)$:

(2.10) $$\sum_{j=1}^{i} (v_j, v_k)\, u'_{ij} + \sum_{j=1}^{i} (v_j, v_k)\, u_{ij} = a \sum_{j=1}^{i} (T_j, e v_k)\, S_{ij} + a(\tau_h, e \cdot v_k),$$

(2.11) $$\gamma \sum_{j=1}^{i} (T_j, T_k)\, S'_{ij} + \sum_{j=1}^{i} ((T_j, T_k))\, S_{ij} + \sum_{j=1}^{i} (v_j \nabla\tau_h, T_k)\, u_{ij} + \sum_{j=1}^{i}\sum_{l=1}^{i} (v_j \nabla T_l, T_k)\, u_{ij} S_{il} = -\gamma(\partial_t \tau_h, T_k) - (\nabla\tau_h, \nabla T_k).$$

(2.12) $u_{ik}(0) = u_{ik}^0 :=$ the k-th component in H_i of u_i^0,

(2.13) $S_{ik}(0) = S_{ik}^0 :=$ the k-th component in V_i of S_i^0.

As the $(2i \times 2i)$ matrix

$$\begin{Vmatrix} (v_j, v_k) & 0 \\ 0 & (T_j, T_k) \end{Vmatrix}$$

is obviously non-singular, by inverting it (2.10)–(2.11) take the canonical form of an ordinary differential system with (at least) continuous coefficients. Then there exists a local maximal solution of (2.10)–(2.13) defined on some interval $[0, \theta_i[$. For $t \in [0, \theta_i[$, we multiply (2.10)–(2.11) by $u_{ik}(t)$, respectively $S_{ik}(t)$, and sum correspondingly these equations over k from 1 to i; it follows

$$(\partial_t u_i, u_i) + (u_i, u_i) = a(S_i + \tau_h, e \cdot u_i)$$

$$\gamma(\partial_t S_i, S_i) + ((S_i, S_i)) + (u_i \nabla\tau_h, S_i) = -\gamma(\partial_t \tau_h, S_i) - (\nabla\tau_h, \nabla S_i),$$

which yield the next estimations:

$$\frac{1}{2}\frac{d}{dt}|u_i|^2 + |u_i|^2 \leq a(|S_i| + |\tau_h|) \cdot |u_i|,$$

$$\frac{\gamma}{2}\frac{d}{dt}|S_i|^2 + \|S_i\|^2 \leq h\|S_i\| \cdot |u_i| + \gamma|\partial_t\tau_h| \cdot |S_i| + |\nabla\tau_h| \cdot \|S_i\|,$$

where we have used property (1.17) of τ_h. With the Friedrichs' inequality (1.11), the preceding estimations become

(2.14) $$\frac{d}{dt}|u_i|^2 + |u_i|^2 \leq 2a^2 C_0^2 \|S_i\|^2 + 2a^2|\tau_h|^2,$$

(2.15) $$\gamma\frac{d}{dt}|S_i|^2 + \|S_i\|^2 \leq h\|S_i\|^2 + h|u_i|^2 + 2\gamma^2 C_0^2 |\partial_t\tau_h|^2 + 2|\nabla\tau_h|^2.$$

Multiplying (2.14) with $(2h)$ and adding it to (2.15) we receive

$$2h\frac{d}{dt}|u_i|^2 + \gamma\frac{d}{dt}|S_i|^2 + h|u_i|^2 + (1 - h - 4a^2C_0^2 h)\|S_i\|^2 \leq 4a^2 h|\tau_h|^2 + 2\gamma^2 C_0^2|\partial_t\tau_h|^2 + 2|\nabla\tau_h|^2 =: G(t).$$

Choosing $h < (2 + 8a^2C_0^2)^{-1}$ and integrating from 0 to some $s \in [0, \theta_i[$ it gives

(2.16) $$2h|u_i(t)|^2 + \gamma|S_i(t)|^2 + h\int_0^s |u_i(t)|^2\, dt + \frac{1}{2}\int_0^s \|S_i(t)\|^2\, dt \leq 2h|u_i^0|^2 + \gamma|S_i^0|^2 + \int_0^s G(t)\, dt.$$

As $\{u_i^0\}$ and $\{S_i^0\}$ are bounded and

$$\int_0^s G(t)\, dt \leq \int_0^\theta G(t)\, dt < \infty,$$

from (2.16) it follows that

(2.17) $\{u_i\}$ is bounded in $L^\infty(0, \theta; H)$.

(2.18) $\{S_i\}$ is bounded in $L^\infty(0, \theta; L^2(\Omega))$.

In particular this yields $\theta_i = \theta$ for any i and hence (u_i, S_i) is a global solution of (2.4)–(2.7). Also, by puting $s = \theta$ in (2.16) we see that

(2.19) $\{S_i\}$ is bounded in $L^2(0, \theta; V)$.

From (2.17)–(2.19) we learn that there exist $u \in L^\infty(0, \theta; H)$ and $S \in L^\infty(0, \theta; L^2(\Omega)) \cap L^2(0, \theta; V)$ for which, passing to some subsequence still denoted with $\{\ \}_i$, we have

(2.20) $u_i \to u$ weakly star in $L^\infty(0, \theta; H)$,

(2.21) $S_i \to S$ weakly star in $L^\infty(0, \theta; L^2(\Omega))$,

(2.22) $S_i \to S$ weakly in $L^2(0, \theta; V)$.

Let ψ be a real function defined on $[0, \theta]$ of class C^1 with $\psi(\theta) = 0$. Multiplying (2.4) and (2.5) with ψ and integrating on $[0, \theta]$ we obtain

(2.23) $$\psi(0)(u_i^0, v_k) - \int_0^\theta (u_i, v_k)\psi'\, dt + \int_0^\theta (u_i, v_k)\psi\, dt = a\int_0^\theta (S_i + \tau_h, e v_k)\psi\, dt$$

$$\gamma\psi(0)(S_i^0, T_k) - \gamma\int_0^\theta (S_i, T_k)\psi' dt + \gamma\int_0^\theta (\partial_t\tau_h, T_k)\psi dt +$$

(2.24)

$$+\int_0^\theta ((S_i, T_k))\psi dt + \int_0^\theta (\nabla\tau_h, \nabla T_k)\psi dt = -\int_0^\theta (u_i\nabla(S_i+\tau_h), T_k)\psi dt.$$

In the light of the convergences (2.8)–(2.9) and (2.20)–(2.22) we can try to pass (2.23) and (2.24) to the limit. All the terms converge easily except the right hand side of (2.24):

$$-\int_0^\theta (u_i\nabla(S_i+\tau_h), T_k)\psi dt = \int_0^\theta (u_i\nabla T_k, S_i+\tau_h)\psi dt.$$

But the injection of V in $L^2(\Omega)$ being compact from (2.22) we find also

(2.25) $\quad S_i \to S$ strongly in $L^2(0, \theta; L^2(\Omega))$,

and with (2.20) it implies

$$-\int_0^\theta (u_i\nabla(S_i+\tau_h), T_k)\psi dt \to \int_0^\theta (u\nabla T_k, S+\tau_h)\psi dt = -\int_0^\theta (u\nabla(S+\tau_h), T_k)\psi dt.$$

Consequently from (2.23) and (2.24) we get at the limit

(2.26) $$\psi(0)(u^0, v_k) - \int_0^\theta (u, v_k)\psi' dt + \int_0^\theta (u, v_k)\psi dt = a\int_0^\theta (S+\tau_h, ev_k)\psi\, dt,$$

$$\gamma\psi(0)(S^0, T_k) - \gamma\int_0^\theta (S, T_k)\psi' dt + \gamma\int_0^\theta (\partial_t\tau_h, T_k)\psi dt +$$

(2.27)

$$+\int_0^\theta ((S, T_k))\psi dt + \int_0^\theta (\nabla\tau_h, \nabla T_k)\psi dt = -\int_0^\theta (u\nabla(S+\tau_h), T_k)\psi dt.$$

All the terms in (2.26) are continuous in H with respect to the v_k-argument. Thus, choosing $\psi\in\mathcal{D}((0,\theta))$ we remark that (2.26) becomes exactly (2.1) for any $v\in H$, in the distribution sense. Moreover, it is easy to see that

$$\frac{d}{dt}(u, v) = -(u, v) + a(S+\tau_h, e\cdot v)\in L^2((0,\theta)),\ (\forall) v\in H.$$

This means $\partial_t u\in L^2(0, \theta; H)$ and

$$(\partial_t u + u - a(S+\tau_h)e, v) = 0 \quad (\forall)\ v\in H.$$

As the orthogonal complement of H in $L^2(\Omega)$ is

$$H^\perp = \{w\in L^2(\Omega) \mid (\exists)\ p\in H^1(\Omega) \text{ such that } w = \nabla p\},$$

(for a proof see [3]), there exists $p(t)\in H^1(\Omega)$ such that

(2.28) $$\partial_t u + u - a(S+\tau_h)e = \nabla p.$$

If we choose $\psi\in C^1([0,\theta])$ with $\psi(\theta)=0$ and $\psi(0)\neq 0$, multiply (2.1) with it and integrate on $[0,\theta]$, then we obtain

(2.29) $$\psi(0)(u(0), v) - \int_0^\theta (u, v)\psi' dt + \int_0^\theta (u, v)\psi dt = a\int_0^\theta (S+\tau_h, e\cdot v)\psi dt.$$

Thus (1.26) follows by subtracting (2.29) from (2.26); as we have proved that $u\in H^1(0,\theta; H)$ we see that, at least, $u\in C^0(0,\theta; H)$ and hence (1.26) makes sense. Moreover, from (2.28) we find that $v(t) := \exp(t)\cdot\mathrm{rot}\, u(t)$ verify the problem

$$\begin{cases} v' = -ae\otimes\nabla(S+\tau_h)\in L^2(0,\theta; L^2(\Omega)) \ (\otimes \text{ denotes the} \\ \text{vectorial product}) \\ v(0) = \mathrm{rot}\ u^0\in L^2(\Omega), \end{cases}$$

which obviously has a unique global solution in $L^\infty(0,\theta; L^2(\Omega))$. This implies rot $u\in L^\infty(0,\theta; L^2(\Omega))$ and as the space

$$\{u\in L^2(\Omega) \mid \mathrm{div}\ u\in L^2(\Omega),\ \mathrm{rot}\ u\in L^2(\Omega);\ u\cdot\nu = 0 \text{ on } \partial\Omega\}$$

is isomorphic to $H^1(\Omega)$ (see [54]), it follows $u\in L^\infty(0,\theta; H^1(\Omega))$.

This last result enables us to pass to the limit over $k\to\infty$ in (2.27), all the terms being continuous in V with respect to the T_k-argument; then choosing $\psi\in\mathcal{D}((0,\theta))$ we see that (2.27) is equivalent with (2.2) for any $T\in V$, in the distribution sense.

Finally we choose again $\psi\in C^1([0,\theta])$ with $\psi(\theta)=0$ and $\psi(0)\neq 0$. Multiplying (2.2) with it and integrating on $[0,\theta]$ we obtain

$$\gamma\psi(0)(S(0), T) - \gamma\int_0^\theta (S, T)\psi' dt + \gamma\int_0^\theta (\partial_t\tau_h, T)\psi dt +$$

(2.30)

$$+\int_0^\theta ((S, T))\psi dt + \int_0^\theta (\nabla\tau_h, \nabla T)\psi dt = -\int_0^\theta (u\nabla(S+\tau_h), T)\psi dt$$

and (1.27) is given by the substraction of (2.30) from (2.27).

In order to precise the sense of (2.4), let us consider for any (u, S), solution of the Problem 2.1, the functions defined a.e. in $(0,\theta)$ by

$$\langle A(S), T\rangle = ((S+\tau_h, T)),\quad \langle B(u, S), T\rangle = (u\nabla(S+\tau_h), T),\ (\forall)\ T\in V,$$

where $\langle\ ,\ \rangle$ denotes the duality product between V and V'. It is easy to check that $A(S)\in L^2(0,\theta; V')$. As we have proved that $u\in L^\infty(0,\theta; H^1(\Omega))$, from the straight estimation

$$\|B(u, S)\|_{V'} \leq C\|u\|_1\cdot\|S+\tau_h\|,$$

it follows $B(u, S)\in L^2(0,\theta; V')$. Thus from (2.2) we receive

$$\gamma\frac{d}{dt}(S, T) = \langle -A(S) - B(u, S) - \gamma\partial_t\tau_h, T\rangle\in L^2((0,\theta)),$$

and therefore $\partial_t S L^2(0,\theta; V')$; this also yields

$$\gamma\partial_t S = -A(S) - B(u, S) - \gamma\partial_t\tau_h, \tag{2.31}$$

in the weak generalized sense. Moreover S is a.e. equal to a function of $C^0(0, \theta; L^2(\Omega))$, and this is the way in which we understand (1.27).

3. A weak maximum principle and the uniqueness. In this section we assume an additional condition for our main datum τ, i.e.

$$\tau \in L^\infty(0, \theta; L^\infty(\partial\Omega)), \tag{3.1}$$

which is in the same time more and less than (1.12).

The weak maximum principle will be formulated in terms of inequality in the sense of $H^1(\Omega)$. That is why we proceed by recalling this notion and some related properties, following [5].

Let $u \in H^1(\Omega)$ and $E \subseteq \bar{\Omega}$; we say that u is nonnegative on E in the sense of $H^1(\Omega)$, or briefly, $u \geqslant 0$ on E in $H^1(\Omega)$, if there exists a sequence $u_n \in W^{(1)}_\infty(\Omega)$ such that $u_n(x) \geqslant 0$ for $x \in E$ and $u_n \to u$ in $H^1(\Omega)$. Let $v \in H^1(\Omega)$; naturally, we say that $u \leqslant v$ on E in $H^1(\Omega)$ if $v - u \geqslant 0$ on E in $H^1(\Omega)$. As v may be a constant, we define

$$\sup_E u = \inf\{m \in \mathbf{R} \mid u \leqslant m \text{ on } E \text{ in } H^1(\Omega)\}.$$

Also, for any $x \in \Omega$, we say that $u(x) > 0$ in (the sense of) $H^1(\Omega)$ if there exist $B(x)$, a neighbourhood of x, and φ, a function from $W^{(1)}_\infty(B(x))$ with $\varphi \geqslant 0$ and $\varphi(x) > 0$, such that $u \geqslant \varphi$ on $B(x)$ in $H^1(\Omega)$. Let us remark that the set $\{x \in \Omega \mid u(x) > 0 \text{ in } H^1(\Omega)\}$ is open.

Proposition 3.1. *If $u \geqslant 0$ on E in $H^1(\Omega)$, then $u \geqslant 0$ a.e. on E.*

Proposition 3.2. *If $\sup_{\partial\Omega} u < \infty$ then for any $M \geqslant \sup_{\partial\Omega} u$ we have*

$$\max\{u - M, 0\} \in H^1_0(\Omega) \text{ and } \max\{u - M, 0\} \geqslant 0 \text{ on } \Omega \text{ in } H^1(\Omega).$$

Proposition 3.3. *Let $u \in W^{(1)}_p(\Omega)$ $(p \geqslant 1)$; then $v = \max\{u, 0\} \in W^{(1)}_p(\Omega)$ and we have in the sense of distributions*

$$\nabla v = \begin{cases} \nabla u \ \text{ in } \{x \in \Omega \mid u(x) > 0 \text{ in } H^1(\Omega)\}, \\ 0 \ \text{ otherwise.} \end{cases}$$

Now we are able to prove our main results: the maximum principle and, based on it, the uniqueness.

Theorem 3.1. *If (u, S) is a solution of Problem 2.1, then*

$$\| S + \tau_h \|_{L^\infty(Q)} \leqslant C(\tau, T^0), \tag{3.2}$$

where $C(\tau, T^0) = \max\{\|\tau\|_{L^\infty(0,\theta;L^\infty(\partial\Omega))}, \|T^0\|_\infty\}$.

Proof. The property (1.16) implies:

$$S(t) + \tau_h(t) = \tau(t) \text{ on } \Gamma_0 \text{ a.e. in } (0, \theta).$$

As $S \in L^2(0, \theta; V)$ from Proposition 3.2 we have

$$R(t) = \max\{S(t) + \tau_h(t) - C(\tau, T^0), 0\} \in L^2(0, \theta; V).$$

Apealing to Proposition 3.3 we get also

$$\nabla R = \begin{cases} \nabla(S + \tau_h) & \text{when } R \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Noticing, via Theorem 2.1, that $\partial_t R \in L^2(0, \theta; V')$ and hence

$$\frac{1}{2}\frac{d}{dt}|R|^2 = (\partial_t R, R) = \begin{cases} (\partial_t S + \partial_t\tau_h, R) & \text{when } R \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

and taking the duality product of (2.31) with $R(t)$, we find

$$0 = \frac{\gamma}{2}\frac{d}{dt}|R|^2 + \|R\|^2 + (u\nabla R, R) = \frac{\gamma}{2}\frac{d}{dt}|R|^2 + \|R\|^2. \tag{3.3}$$

In particular we have

$$\frac{d}{dt}|R|^2 \leqslant 0,$$

and by integration on $(0, t)$, for some t, it implies

$$|R(t)|^2 \leqslant |R(0)|^2 \text{ a.e. on } (0, \theta). \tag{3.4}$$

As $S(0) + \tau_h(0) = T^0$ in Ω it follows that $R(0) = 0$, and (3.4) gives $|R(t)| = 0$. Recalling (3.3) we see that $\|R(t)\| = 0$, that is

$$S(t) + \tau_h(t) \leqslant C(\tau, T^0) \text{ on } \Omega \text{ in } H^1(\Omega), \text{ a.e. in } (0, \theta).$$

According to Proposition 3.1 this implies $S(t) + \tau_h(t) \leqslant C(\tau, T^0)$ a.e. in Q. Analogously, with $R(t) = \min\{S(t) + \tau_h(t) + C(\tau, T^0), 0\}$ we get $S(t) + \tau_h(t) \geqslant -C(\tau, T^0)$ a.e. in Q, and the proof is completed.

Let (u_1, S_1) and (u_2, S_2) be two solutions of the Problem 2.1 corresponding to the initial data (u^0_1, T^0_1), respectively (u^0_2, T^0_2). Denoting with

$$u = u_1 - u_2, \quad S = S_1 - S_2$$

we obtain from (2.28) and (2.31) by substraction

$$\partial_t u + u - aSe = \nabla(p_1 - p_2) \tag{3.5}$$

$$\gamma\partial_t S + A(S) + B(u, S_1 + \tau_h) + B(u_2, S) = 0. \tag{3.6}$$

Taking the duality product of (3.5)–(3.6) with u, respectively S, we get

$$\frac{1}{2}\frac{d}{dt}|u|^2 + |u|^2 = (aSe, u), \quad \frac{\gamma}{2}\frac{d}{dt}|S|^2 + \|S\|^2 = (u\nabla S, S_i + \tau_h) \quad (i = 1 \text{ or } 2)$$

which yield the following estimations

$$\frac{d}{dt}|u|^2 + 2|u|^2 \leqslant 2a|S| \cdot |u| \leqslant |u|^2 + a^2|S|^2, \tag{3.7}$$

$$\gamma\frac{d}{dt}|S|^2 + 2\|S\|^2 \leqslant 2K_0|u| \cdot \|S\| \leqslant \|S\|^2 + K_0^2|u|^2, \tag{3.8}$$

where $K_0=\min\{C(\tau, T_1^0), C(\tau, T_0^0)\}$. Using also the Friedrichs' inequality (1.11) from (3.7)—(3.8) we receive

$$\frac{d}{dt}\begin{pmatrix}|u|^2\\ |S|^2\end{pmatrix}\leq \mathcal{M}\begin{pmatrix}|u|^2\\ |S|^2\end{pmatrix}, \tag{3.9}$$

where $\mathcal{M}$ is the following (2×2) matrix

$$\mathcal{M}=\begin{Vmatrix}-1 & a^2\\ C_0^2\gamma^{-1} & -K_0^{-2}\gamma^{-1}\end{Vmatrix}.$$

Integrating (3.21) from 0 to some t, we are lead to

$$\begin{pmatrix}|u(t)|^2\\ |S(t)|^2\end{pmatrix}\leq \exp(\mathcal{M}t)\begin{pmatrix}|u_1^0-u_2^0|^2\\ |T_1^0-T_2^0|^2\end{pmatrix}. \tag{3.10}$$

In particular, with $(u_1^0, T_1^0)=(u_2^0, T_2^0)$ in (3.10), we have proved:

Theorem 3.2. *The Problem 2.1 has a unique solution.*

Finally, the relation (3.10) permit us to establish also a stability result, that is

Theorem 3.3. *Any perturbation of the initial data (1.7)—(1.8) decreases exponentially in time if*

$$\max\{\|\tau\|_{L^\infty(0,\theta;L^\infty(\partial\Omega))}, \min\{\|T_1^0\|_\infty, \|T_2^0\|_\infty\}\}\leq a^{-1}C_0^{-1}. \tag{3.11}$$

Proof. If condition (3.11) is satisfied, then $\mathcal{M}$ has distinct eigenvalues which are also strictly negative and the proof is completed using classical results on spectral decomposition.

Remark 3.1. The relation (3.11) is equivalent to the unicity criterion (see [2]), in the steady case.

The present article was preceded by the note [6] which was presented on the Int. Spring School on Evolution Equations, Dobrichovice by Prague, May 21—25, 1984. After that it appeared the note [7] in which the same problem is considered in a particular case. We were surprised to learn from this note that the Galerkin method does not work in the tridimensional case.

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A NOTE ON INTERVAL GRAPHS

BY

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1. Let $G=(X, E)$ be a simple nonoriented graph, where X is the set of vertices and E is the set of the edges for G. G is an interval graph if there exists a set $\mathcal{J}$ of open intervals on real line such that G and the intersection graph associated to $\mathcal{J}$ are isomorphic. Necessary and sufficient conditions for interval graphs are given by C. C. Lekkerkerker and J. Ch. Boland [2], P. C. Gilmore and J. J. Hoffman [1], B. G. Mirkin [3]. An algorithm for solving the problem whether a given graph is an interval graph is described by B. G. Mirkin and S. N. Rodin [4,5].

Theorem A. (Gilmore-Hoffman). *A graph G is an interval graph if and only if its complementary graph is transitively oriented and every cycle of length four in G has at least one triangulator.*

For an oriented graph $G=(X, A)$, where X is the set of the vertices and A is the set of the arcs of G, for $x\in X$ we denote by $I^+(x, G)$, $I^-(x, G)$, or simply $I^+(x)$, $I^-(x)$, the sets defined by

$$I^+(x, G)=\{y \mid y\in X,\ (x, y)\in A\},$$

$$I^-(x, G)=\{y \mid y\in X,\ (y, x)\in A\}.$$

Theorem B (Mirkin). *A graph $G=(X, E)$ is an interval graph if and only if its complementary graph, $\bar{G}$, is transitively oriented and for any transitive orientation of $\bar{G}$ and any $x, y\in X$ we have*

$$I^+(x, \bar{G})\subset I^+(y, \bar{G}) \text{ or } I^+(x, \bar{G})\supset I^+(y, \bar{G}).$$

2. In this note we give some new structural properties of interval graphs. If $G=(X, E)$ is a simple nonoriented graph and $x\in X$, then we denote by $I(x, G)$, or simply $I(x)$, the set of the vertices of G which are adjacent to x, i.e.

$$I(x, G)=\{y \mid y\in X,\ [x, y]\in E\}.$$

Theorem 1. *Let $G=(X, E)$ be the complementary graph of an interval graph. If $a, b\in X$, $a\notin I(b)$ then*

$$I(x)\supset I(b)\setminus I(a), \text{ for all } x\in I(a)\setminus I(b). \tag{1}$$

Proof: Suppose $a, b\in X$, $a\notin I(b)$; if $x\in I(a)\setminus I(b)$ and $I(b)\setminus I(a)=\varnothing$ it is clear that (1) holds. Now consider $y\in I(b)\setminus I(a)$; if $y\notin I(x)$ then in the