where $K_0 = \min\{C(\tau, T_1^0), C(\tau, T_0^0)\}$. Using also the Friedrichs' inequality (1.11) from (3.7)–(3.8) we receive

$$\frac{d}{dt}\begin{pmatrix} |u|^2 \\ |S|^2 \end{pmatrix} \leqslant \mathcal{M}\begin{pmatrix} |u|^2 \\ |S|^2 \end{pmatrix}, \tag{3.9}$$

where $\mathcal{M}$ is the following (2×2) matrix

$$\mathcal{M} = \begin{Vmatrix} -1 & a^2 \\ C_0^2\gamma^{-1} & -K_0^{-2}\gamma^{-1} \end{Vmatrix}.$$

Integrating (3.21) from 0 to some t, we are lead to

$$\begin{pmatrix} |u(t)|^2 \\ |S(t)|^2 \end{pmatrix} \leqslant \exp(\mathcal{M}t)\begin{pmatrix} |u_1^0 - u_2^0|^2 \\ |T_1^0 - T_2^0|^2 \end{pmatrix}. \tag{3.10}$$

In particular, with $(u_1^0, T_1^0) = (u_2^0, T_2^0)$ in (3.10), we have proved:

Theorem 3.2. *The Problem 2.1 has a unique solution.*

Finally, the relation (3.10) permit us to establish also a stability result, that is

Theorem 3.3. *Any perturbation of the initial data (1.7)–(1.8) decreases exponentially in time if*

$$\max\{\|\tau\|_{L^\infty(0,\theta;L^\infty(\partial\Omega))}, \min\{|T_1^0|_\infty, |T_2^0|_\infty\}\} \leqslant a^{-1}C_0^{-1}. \tag{3.11}$$

Proof. If condition (3.11) is satisfied, then $\mathcal{M}$ has distinct eigenvalues which are also strictly negative and the proof is completed using classical results on spectral decomposition.

Remark 3.1. The relation (3.11) is equivalent to the unicity criterion (see [2]), in the steady case.

The present article was preceded by the note [6] which was presented on the Int. Spring School on Evolution Equations, Dobrichovice by Prague, May 21–25, 1984. After that it appeared the note [7] in which the same problem is considered in a particular case. We were surprised to learn from this note that the Galerkin method does not work in the tridimensional case.

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Received 22.XI. 1985

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXII, s. I a, Matematică, 1986, f. 3

A NOTE ON INTERVAL GRAPHS

BY

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1. Let $G=(X, E)$ be a simple nonoriented graph, where X is the set of vertices and E is the set of the edges for G. G is an interval graph if there exists a set $\mathcal{I}$ of open intervals on real line such that G and the intersection graph associated to $\mathcal{I}$ are isomorphic. Necessary and sufficient conditions for interval graphs are given by C. C. Lekkerkerker and J. Ch. Boland [2], P. C. Gilmore and J. J. Hoffman [1], B. G. Mirkin [3]. An algorithm for solving the problem whether a given graph is an interval graph is described by B. G. Mirkin and S. N. Rodin [4,5].

Theorem A. (Gilmore-Hoffman). *A graph G is an interval graph if and only if its complementary graph is transitively oriented and every cycle of length four in G has at least one triangulator.*

For an oriented graph $G=(X, A)$, where X is the set of the vertices and A is the set of the arcs of G, for $x\in X$ we denote by $I^+(x, G)$, $I^-(x, G)$, or simply $I^+(x)$, $I^-(x)$, the sets defined by

$$I^+(x, G) = \{y \mid y\in X,\ (x, y)\in A\},$$
$$I^-(x, G) = \{y \mid y\in X,\ (y, x)\in A\}.$$

Theorem B (Mirkin). *A graph $G=(X, E)$ is an interval graph if and only if its complementary graph, $\bar{G}$, is transitively oriented and for any transitive orientation of $\bar{G}$ and any x, $y\in X$ we have*

$$I^+(x, \bar{G}) \subset I^+(y, \bar{G}) \text{ or } I^+(x, \bar{G}) \supset I^+(y, \bar{G}).$$

2. In this note we give some new structural properties of interval graphs. If $G=(X, E)$ is a simple nonoriented graph and $x\in X$, then we denote by $I(x, G)$, or simply $I(x)$, the set of the vertices of G which are adjacent to x, i.e.

$$I(x, G) = \{y \mid y\in X,\ [x, y]\in E\}.$$

Theorem 1. *Let $G=(X, E)$ be the complementary graph of an interval graph. If a, $b\in X$, $a\notin I(b)$ then*

$$I(x) \supset I(b)\setminus I(a), \text{ for all } x\in I(a)\setminus I(b). \tag{1}$$

Proof: Suppose a, $b\in X$, $a\notin I(b)$; if $x\in I(a)\setminus I(b)$ and $I(b)\setminus I(a)=\varnothing$ it is clear that (1) holds. Now consider $y\in I(b)\setminus I(a)$; if $y\notin I(x)$ then in the complementary graph $\bar{G}$, of G, there exists the cycle of length four $[ayxba]$ which has no triangulator and therefore G should not be an interval graph, (Theorem A). Hence we have to accept $y\in I(x)$ and the Theorem is proved. For a graph G any connected component having at least two vertices will be named nontrivial connected component. The next corollaries can be obtained directly from Theorem 1 and Theorems A, B.

Corollary 1. *If G is the complementary graph of an interval graph then G has at most one nontrivial connected component.*

Corollary 2. *If $G=(X, E)$ is the complementary graph of an interval graph and $a, b\in X$, $a\notin I(b)$ then*

$$I(x)\supset I(b)\setminus I(a), \text{ for all } x\in I(a)\setminus I(b),$$
$$I(y)\supset I(a)\setminus I(b), \text{ for all } y\in I(b)\setminus I(a). \tag{2}$$

Corollary 3. *A graph G is the complementary graph of an interval graph if and only if G is transitively oriented and for all $a, b\in X$, $a\notin I(b)$ the relations (2) hold.*

Corollary 4. *If $G=(X, E)$ is a graph and (1) holds for all $a, b\in X$, $a\notin I(b)$ then for each transitive orientation of G we have*

$$I^+(x)\supset I^+(y) \text{ or } I^+(x)\subset I^+(y), \text{ for all } x, y\in X.$$

Theorem 2. *If $G=(X, E)$ is a simple nonoriented graph and $I(a)\supset I(b)$ or $I(b)\supset I(a)$ for all $a, b\in X$, $a\notin I(b)$ then G is the complementary graph of an interval graph.*

Proof. We shall prove this theorem by induction on the number of vertices for the given graph. It is clear that we can suppose that G is connected. Tacking account on Theorem B and Corollary 4 it is sufficient to prove that G is transitively oriented. Suppose that Theorem is true for all the graphs having at most $n-1$ vertices, $n\geq 2$, and consider a graph $G=(X, E)$ having n vertices. Let a_1 be a vertex of G such that

$$|I(a_1)|\geq |I(x)|, \text{ for all } x\in X,$$

where $|I(x)|$ denotes the number of the elements of the set $I(x)$.

a) If $x, y\in X\setminus I(a_1)$ then $x\notin I(y)$, since $|I(a_1)|\geq |I(y)|$ and therefore $I(a_1)\supset I(y)$. Hence we can suppose $X\setminus I(a_1)=\{a_1, \ldots, a_s\}$ such that

$$I(a_1)\supset I(a_2)\supset\ldots\supset I(a_s), \tag{3}$$

and $a_i\notin I(a_j)$, for all $i, j\in\{1, \ldots, s\}$.
We remark that if $s=1$ a transitive orientation of G can be obtained from a transitive orientation of the subgraph of G generated by $X\setminus\{a_1\}$ and considering $I^+(a_1)=I(a_1)$, for example. Consequently we can suppose $s\geq 2$.

b) We shall prove that for each $i\in\{1, \ldots, s-1\}$ and each $x\in I(a_i)\setminus I(a_{i+1})$ we have

$$I(x)\supset I(a_k), \text{ for all } k\geq i+1. \tag{4}$$

If $x\in I(a_i)\setminus I(a_{i+1})$ then $x\notin I(a_{i+1})$; hence $I(x)\supset I(a_{i+1})$ or $I(x)\subset I(a_{i+1})$. On the other hand $a_1\in I(x)\setminus I(a_{i+1})$ and consequently $I(x)\supset I(a_{i+1})$ holds. By using (3) it follows that (4) is true.

c) Let G_i, $i=1, \ldots, s-1$ be the subgraphs of G having the sets of vertices $X_i=I(a_i)\setminus I(a_{i+1})$, $i=1, \ldots, s-1$, respectively.
It is clear that

$$I(x, G_i)=I(x, G)\cap X_i,$$

hence for all the subgraphs G_i, $i=1, \ldots, s-1$ the conditions of the theorem are satisfied. So G_i, $i=1, \ldots, s-1$ are transitively oriented.

d) Consider any transitive orientations for the subgraphs G_i, $i=1, \ldots, s-1$ and define an orientation of the edges of G as follows:

$$I^+(a_i, G)=I(a_i, G)\,;\ i=1, \ldots, s$$
$$I^+(x, G)=I^+(x, G_i)\cup I(a_{i+1}, G), \text{ for all } x\in I(a_i)\setminus I(a_{i+1}), \tag{5}$$
$$I^-(x, G)=I(x, G)\setminus I^+(x, G), \text{ for all } x\in X,$$

where $I^+(x, G_i)$, $x\in X_i$, $i=1, \ldots, s-1$ are defined according to the given transitive orientation of the subgraphs G_i, $i=1, \ldots, s-1$.
Taking account on (3) and (4) it is easy to prove that (5) give a transitive orientation of G. The Theorem is proved.

3. Let $G=(X, E)$ be a simple nonoriented graph and $a, b\in X$ such that $a\neq b$, $[a, b]\notin E$; then we define the sets S_a^k, S_b^k, $\bar{S}_a^k$, $\bar{S}_b^k$, $k=0, 1, \ldots$ and T^k, $k=-1, 0, 1, \ldots$ by

$$S_a^0=I(a)\setminus I(b),\ S_b^0=I(b)\setminus I(a),$$
$$\bar{S}_a^0=\{x\in X \mid x\notin I(a)\cup I(b),\ I(x)\cap S_a^0\neq\varnothing\},$$
$$\bar{S}_b^0=\{x\in X \mid x\notin I(a)\cup I(b),\ I(x)\cap S_b^0\neq\varnothing\}, \tag{6}$$
$$T^{-1}=\varnothing,\ T^0=\{a, b\}\cup S_a^0\cup S_b^0\cup\bar{S}_a^0\cup\bar{S}_b^0,$$

and for every $k\geq 0$

$$S_a^{k+1}=\{x\in X \mid x\notin T^k,\ I(x)\supset T^{k-1}\cup S_a^k\cup\bar{S}_b^k,\ I(x)\not\supset\bar{S}_a^k\cup S_b^k\},$$
$$S_b^{k+1}=\{x\in X \mid x\notin T^k,\ I(x)\supset T^{k-1}\cup\bar{S}_a^k\cup S_b^k,\ I(x)\not\supset S_a^k\cup\bar{S}_b^k\},$$
$$\bar{S}_a^{k+1}=\{x\in X \mid x\notin T^k\cup(I(a)\cap I(b)),\ I(x)\cap S_a^{k+1}\neq\varnothing\}, \tag{7}$$
$$\bar{S}_b^{k+1}=\{x\in X \mid x\notin T^k\cup(I(a)\cap I(b)),\ I(x)\cap S_b^{k+1}\neq\varnothing\},$$
$$T^{k+1}=T^k\cup S_a^{k+1}\cup S_b^{k+1}\cup\bar{S}_a^{k+1}\cup\bar{S}_b^{k+1}.$$

It is clear that the sets defined by the recurrence formulae (6), (7) are dependent on G and $[a, b]\notin E$; sometimes we shall denote these sets by $S_a^k(b, G)$, $S_b^k(a, G)$, $\bar{S}_a^k(b, G)$, $\bar{S}_b^k(a, G)$, $T^k(a, b, G)$, respectively.
Tacking account on the fact that G is a finite graph it follows that for any $a, b\in X$, $[a, b]\notin E$, there exists $k\geq 0$ such that $T^k(a, b, G)=T^{k+1}(a, b, G)$ and consequently we can define the number $p(a, b, G)$ by

$$p(a, b, G)=\min\{k \mid T^k(a, b, G)=T^{k+1}(a, b, G)\}. \tag{8}$$

If $G=(X, E)$ is a simple nonoriented graph let $N(G)$ be the set defined by

(9) $$N(G)=\{\{a, b\} \mid a, b\in X,\ [a, b]\notin E,\ I(a)\setminus I(b)\neq\emptyset,\ I(b)\setminus I(a)\neq\emptyset\}.$$

We shall denote by $\mathcal{G}$ the family of those graphs $G=(X, E)$ having the property that for all $\{a, b\}\in N(G)$ the corresponding sets S_a^k, S_b^k, $\bar{S}_a^k$, T^k, $\bar{S}_b^k$ defined by (6) (7) satisfy the conditions

(10)
a) $I(x)\supset S_a^k$, for all $x\in S_b^k$, $k=0, \ldots, p(a, b, G)=p$,

b) $I(x)\supset T^p$ or $I(x)\cap T^p=\emptyset$, for all $x\in X\setminus T^p$,

c) If $A^k=S_a^k\cup S_b^k\cup\bar{S}_a^{k+1}$ and $x, y\in S_a^{k+1}$ then

$$I(x)\cap A^k\supset I(y)\cap A^k \text{ or } I(x)\cap A^k\subset I(y)\cap A^k;\ k=0, \ldots, p.$$

We remark that from $\{a, b\}\in N(G)$ it follows $\{b, a\}\in N(G)$ and consequently when $G\in\mathcal{G}$ the relations (10) remain valid even if a and b are changed between them. The following Lemmas will be usefull to prove that $\mathcal{G}$ contains all the complementary graphs of the interval graphs.

Lemma 1. *Let G be the complementary graph of an interval graph and $\{a, b\}\in N(G)$. If $x\notin I(a)\cap I(b)$ then $I(x)\supset I(a)\setminus I(b)$ or $I(x)\supset I(b)\setminus I(a)$ or $I(x)\subset I(a)\cup I(b)$.*

Proof. Suppose $x\notin I(a)\cap I(b)$, $I(x)\not\supset I(a)\setminus I(b)$, $I(x)\not\supset I(b)\setminus I(a)$ and consider $y\in I(x)$, $z\in I(a)\setminus I(b)$, $u\in I(b)\setminus I(a)$ such that $u, z\notin I(x)$. By using Theorem 1 one obtains $u, z\in I(y)$ and $z\in I(u)$. Consequently, if $y\notin I(a)\cup I(b)$ it follows that G contains the cyclic route $[azubuyxyza]$ of odd length without triangulators and this means that G has no transitive orientations: contradiction; so we cannot accept $y\notin I(a)\cup I(b)$ and the Lemma is proved.

Lemma 2. *Let G be the complementary graph of an interval graph and $\{a, b\}\in N(G)$. If $x\in I(a)\cap I(b)$ then $I(x)\supset I(a)\setminus I(b)$ or $I(x)\supset I(b)\setminus I(a)$.*

Proof: Suppose $x\in I(a)\cap I(b)$ and $I(x)\not\supset I(a)\setminus I(b)$. Let us consider $y\in I(a)\setminus I(b)$, $y\notin I(x)$ and $z\in I(b)\setminus I(a)$; by using Theorem 1 one obtains $y\in I(z)$. If $z\notin I(x)$ then G contains the cyclic route $[ayzbxa]$ of odd length without triangulator, this is impossible because of G has transitive orientation. Consequently we have to accept $z\in I(x)$ and the Lemma is proved.

Lemma 3. *If G is the complementary graph of an interval graph and $\{a, b\}\in N(G)$, then for any transitive orientation of G one and only one of the following two sentences is true:*

(A_1): $I^+(a)\supset S_a^0$, $I^-(b)\supset S_b^0$, $I^+(x)\supset S_a^0$ for all $x\in S_b^0$ and

$I^+(x)\supset S_a^0\cap I(x)$, $I^-(y)\supset I(y)\cap S_b^0$ for all $x\in\bar{S}_a^0$, $y\in\bar{S}_b^0$.

(A_2): $I^+(b)\supset S_b^0$, $I^-(a)\supset S_a^0$, $I^+(x)\supset S_b^0$ for all $x\in S_a^0$ and

$I^+(x)\supset S_b^0\cap I(x)$, $I^-(y)\supset I(y)\cap S_a^0$ for all $x\in\bar{S}_b^0$, $y\in\bar{S}_a^0$,

where S_a^0, S_b^0, $\bar{S}_a^0$, $\bar{S}_b^0$ are defined by (6).

Proof. It is clear that (A_1) and (A_2) cannot be simultaneous true. Let us consider a transitive orientation of G and $\{a, b\}\in N(G)$: two cases are possible: 1. $I^+(a)\cap S_a^0\neq\emptyset$ and 2. $I^+(a)\cap S_a^0=\emptyset$.

Case 1. Consider $z\in I^+(a)\cap S_a^0$; we shall show that (A_1) is true. Since $I(a)\cap S_b^0=\emptyset$ and $I(z)\supset S_b^0$ it follows that $S_b^0\subset I^-(z)$ and from $z\notin I(b)$ one obtains $S_b^0\subset I^-(b)$. On the other hand $S_a^0\subset I(y)$ for all $y\in S_b^0$ and consequently $S_a^0\subset I^+(y)$ for all $y\in S_b^0$. Now consider $x\in\bar{S}_a^0$ and $y\in I(x)\cap S_a^0$; since $y\in S_a^0$ it follows $y\in I^+(a)$ and from $x\in\bar{S}_a^0$ it is clear that $x\notin I(a)$. Hence we must have $y\in I^+(x)$ and therefore $I^+(x)\supset S_a^0\cap I(x)$ for all $x\in\bar{S}_a^0$. Similarly one proves that $I^-(y)\supset S_b^0\cap I(y)$ for all $y\in\bar{S}_b^0$. We have shown that in this case (A_1) is true.

Case 2. If $I^+(a)\cap S_a^0=\emptyset$ then $S_a^0\subset I^-(a)$ and tacking account on Theorem 1 and on the fact that G is transitively oriented it follows $I^+(b)\cap\cap S_b^0\neq\emptyset$. Since $\{b, a\}\in N(G)$ too, the first case just proved shows that in the second case, by changing a and b in (A_1) one obtains a true sentence. But if we change a and b in (A_1) one obtains (A_2) and the Lemma is proved.

Lemma 4. *If G is the complementary graph of an interval graph and $\{a, b\}\in N(G)$ then for any transitive orientation of G and all $x\in S_a^1$, $y\in S_b^1$ either*

(B_1): $I^+(x)\supset T^0\cap I(x)$ and $I^-(y)\supset T^0\cap I(y)$ or

(B_2): $I^-(x)\supset T^0\cap I(x)$ and $I^+(y)\supset T^0\cap I(y)$.

Proof. By Lemma 3, for any transitive orientation of G either (A_1) or (A_2) is true. Tacking account on (6), (7) it is easy to prove that when (A_1) is true then (B_1) is true and when (A_2) is true then (B_2) is also true. On the other hand it is clear that (B_1) and (B_2) cannot be simultaneously true and the Lemma is proved.

Lemma 5. *If G is the complementary graph of an interval graph and $\{a, b\}\in N(G)$ then for all $x\in I(a)\cap I(b)$ we have*

(11) $$I(x)\supset S_a^0\cup S_b^0 \text{ or } I(x)\supset S_b^0\cup\bar{S}_a^0.$$

Proof. Consider $x\in I(a)\cap I(b)$; from Lemma 2, $I(x)\supset S_a^0$ or $I(x)\supset S_b^0$.

a) Suppose $I(x)\supset S_a^0$, $I(x)\not\supset S_b^0$ and consider $y\in S_b^0$, $z\in S_b^0\cap I(y)$. If then $\{a, z\}\in N(G)$ because of $z\notin I(a)$, $y\in I(z)\setminus I(a)$ and $x\in I(a)\setminus I(z)$: hence $y\in I(x)$ and (11) is true. If $z\in I(x)$ then by using Lemma 3 and considering a transitive orientation of G, it follows $y\in I^+(x)$ or $y\in I^-(x)$, and again (11) is true. Conseqaently, when $I(x)\not\supset S_b^0$ it follows $I(x)\supset S_a^0\cup\bar{S}_b^0$. Since $\{b, a\}\in N(G)$ too, we can say that in the case $I(x)\supset S_b^0$ then $I(x)\supset S_b^0\cup\bar{S}_a^0$.

b) It remains to prove the Lemma in the case $I(x)\supset S_a^0\cup S_b^0$. Consider a transitive orientation of G; by Lemma 3 we can suppose that (A_1) is true and because of $\{a, b\}\in N(G)$ it results $I^+(x)\supset S_a^0\cup S_b^0$ or $I^-(x)\supset S_a^0\cup S_b^0$. If $z\in\bar{S}_a^0$ then there exists $y\in S_a^0$ such that $y\in I^+(z)$ and consequently from $z\notin I(x)$ one obtains $I^+(x)\supset S_a^0\cup S_b^0$. Since (A_1) is true, $I^-(u)\supset S_b^0\cap I(u)$ for all $u\in\bar{S}_b^0$ and because of G is transitively oriented one obtains $u\in I^+(x)$, i.e. $u\in I(x)$ and the Lemma is proved.

Lemma 6. *Let G be the complementary graph of an interval graph and $\{a, b\} \in N(G)$. If $k \geq 1$ then for all $x \in S_a^k$, $y \in S_b^k$ and for every transitive orientation of G either $a \in I^+(x) \cap I^-(y)$ or $a \in I^-(x) \cap I^+(y)$.*

Proof. Using Lemma 4 one observes that for $k=1$ either $a \in I^+(x) \cap I^-(y)$ or $a \in I^-(x) \cap I^+(y)$. Suppose that the conclusion of the Lemma 6 is true for $k=1, \ldots, n-1$, $n \geq 2$, and let $x \in S_a^n$, $y \in S_b^n$. Consider a transitive orientation of G such that $a \in I^+(z) \cap I^-(t)$ for all $z \in S_a^{n-1}$ and all $t \in S_b^{n-1}$; since $x \in S_a^n$ there exists $x^1 \in S_a^{n-1} \cup S_b^{n-1}$ such that $x^1 \notin I(x)$. If $x^1 \in S_b^{n-1}$ then we must have $x \in I^-(a)$, because of $x^1 \in I^-(a)$ and $x^1 \notin I(x)$. If $x^1 \in S_a^{n-1}$ then there exists $y^{n-1} \in S_a^{n-1}$ such that $y^{n-1} \in I(x^1) \cap I(x)$; but $x^1 \notin I(a)$ and $a \in I^+(y^{n-1})$, hence $x^1 \notin I^+(y^{n-1})$ and consequently $x \in I^+(y^{n-1})$. Since $x^1 \notin T^{n-2}$ and $x \in S_a^n$ it results

$$I^-(y^{n-1}) \supset T^{n-2} \cap I(y^{n-1}), \quad I(x) \supset T^{n-2}.$$

Taking account on (7) we can consider a sequence y^i, $i=n-2, \ldots, 1, 0$ such that

$$y^i \in T^i - T^{i-1}; \quad i=0, 1, \ldots, n-2,$$

$$y^{i-1} \notin I(y^i); \quad i=1, \ldots, n-1.$$

Since $x \in I^+(y^{n-1})$ and $y^i \in I(x)$, $i=0, \ldots, n-1$, we must have $x \in I^+(y^i)$, $i=0, \ldots, n-2$, and because of $y^0 \in T^0 \supset \{a, b\}$ and G is transitively oriented one obtains $x \in I^+(a)$. Similarly one proves that $y \in I^-(a)$ and the Lemma is proved.

Corollary 5. *If G is the complementary graph of an interval graph and $\{a, b\} \in N(G)$ then for all $k \geq 1$ we have*

$$I(x) \supset S_a^k, \text{ for all } x \in S_b^k. \tag{12}$$

Proof. Consider $k \geq 1$ and $x \in S_a^k$, $y \in S_b^k$. By using Lemma 6 it follows that for any transitive orientation of G we have $a \in I^+(x) \cap I^-(y)$ or $a \in I^-(x) \cap I^+(y)$ and consequently $y \in I^+(x)$ or $x \in I^+(y)$, i.e. $y \in I(x)$.

Corollary 6. *Let G be the complementary graph of an interval graph and $\{a, b\} \in N(G)$. If the set T^k, $k \geq 0$, is defined according to (6), (8) and $x \in X \setminus T^k$, $x \in I(a) \cap I(b)$ then*

$$I(x) \supset T^{k-1} \cup S_a^k \cup S_b^k \text{ or } I(x) \supset T^{k-1} \cup \bar{S}_a^k \cup S_b^k. \tag{13}$$

Proof. Taking account on Lemma 5 it follows that (13) is true for $k=0$. Suppose that (13) holds for every $k \leq n-1$, $n \geq 1$, and consider T^n, S_a^n, S_b^n, $\bar{S}_a^n$, $\bar{S}_b^n$ defined according to (7), $x \in X \setminus T^n$, $x \in I(a) \cap I(b)$. Since (13) 8 is true for $k \leq n-1$ it results

$$I(x) \supset T^{n-2} \cup S_a^{n-1} \cup S_b^{n-1} \text{ or } I(x) \supset T^{n-2} \cup \bar{S}_a^{n-1} \cup S_b^{n-1}. \tag{14}$$

If $x \in X \setminus T^n$ then $x \notin T^{n-1}$ because of $T^{n-1} \subset T^n$ and therefore (14) holds for all $x \in X \setminus T^n$. Hence

$$I(x) \supset T^{n-1}. \tag{15}$$

By using Lemma 6, $x \in I(a) \cap I(b)$ and a transitive orientation of G, as in the proof of the Lemma 5 one obtains

$$I(x) \supset S_a^n \cup \ddot{S}_b^n \text{ or } I(x) \supset S_b^n \cup \dot{S}_a^n. \tag{16}$$

The relations (15), (16) show that (13) is true and for $k=n$ and the Corollary is proved.

Corollary 7. *Let G be the complementary graph of an interval graph and $\{a, b\} \in N(G)$. If $p=p(a, b, G)$ is defined by (8) and $x \in X \setminus T^p$ then*

$$\begin{aligned} &\text{a). } I(x) \supset T^p \text{ if } x \in I(a) \cap I(b), \\ &\text{b) } I(x) \cap T^p = \varnothing \text{ if } x \notin I(a) \cup I(b). \end{aligned} \tag{17}$$

Proof. Suppose $x \in X \setminus T^p$. By using (8) and Corollary 6 it follows that (17a) holds. Since $p \geq 0$, from $x \in X \setminus T^p$, $x \notin I(a) \cup I(b)$ we obtain $x \notin I(a) \cap I(b)$ and $x \notin T^p$; consequently, by Lemma 1, $I(x) \subset I(a) \cup I(b)$. If $I(x) \cap T^p \neq \varnothing$ then $x \in T^p$, impossible; so we have to accept (17b) and the Corollary is proved.

Lemma 7. *Let G be the complementary graph of an interval graph and $\{a, b\} \in N(G)$. If S_a^k, $k \geq 1$, is defined by (6), (7) then for all $x, y \in S_a^k$ we have*

$$I(x) \cap A^{k-1} \supset I(y) \cap A^{k-1} \text{ or } I(x) \cap A^{k-1} \subset I(y) \cap A^{k-1}, \tag{18}$$

where $A^{k-1} = S_b^{k-1} \cup S_a^{k-1} \cup \bar{S}_a^k$.

Proof. Suppose $x, y \in S_a^k$ and a transitive orientation of G. From Lemma 6 it follows that we can consider $a \in I^+(z) \cap I^-(u)$ for all $z \in S_b^{k-1}$ and all $u \in S_a^{k-1}$. Taking account on (7) and on the fact that G is transitively oriented one obtains

$$I^+(t) \supset (S_b^{k-1} \cup \bar{\bar{S}}_a^{k-1} \cup \bar{S}_a^k) \cap I(t), \text{ for all } t \in S_a^k. \tag{19}$$

On the other hand, Theorem B shows that we must have $I^+(x) \subset I^+(y)$ or $I^+(y) \subset I^+(x)$ and consequently (19) and Theorem B implies (18). The Lemma is proved.

Theorem 3. *If $G=(X, E)$ is the complementary graph of an interval graph then $G \in \mathcal{G}$.*

Proof. Consider $\{a, b\} \in N(G)$ and the sets S_a^k, S_b^k, $\bar{S}_a^k$, $\bar{S}_b^k$, T^k, $k = 0, 1, \ldots, p = p(a, b, G)$, defined by (6), (7). The Corollaries 5,7 and the Lemma 7 show that the relations (10) are true. Hence $G \in \mathcal{G}$ and the Theorem is proved.

Theorem 4. *A graph $G=(X, E)$ is the complementary graph of an interval graph if and only if $G \in \mathcal{G}$ and for every $\{a, b\} \in N(G)$ the subgraphs of G generated by T^p and $\{a\} \cup (X \setminus T^p)$ are complementary graphs of some interval graphs, where $p=p(a, b, G)$ and $T^p = T^p(a, b, G)$ are defined according to (6)—(8).*

Proof. Let $G=(X, E)$ be the complementary graph of an interval graph; Theorem 3 and Lemma 1 show that $G \in \mathcal{G}$. Since any subgraph of an interval graph is an interval graph too, it follows that the subgraphs of G having the sets of vertices $\{a\} \cup (X \setminus T^p)$ and T^p respectively are complementary graphs of some interval graphs.

We note that when $N(G) = \varnothing$ then G is the complementary graph of an interval graph as it is shown by Theorem 2. Now suppose $G=(X, E) \in \mathcal{G}$,

$\{a, b\} \in N(G)$ and $T^p = T^p(a, b, G)$, $p = p(a, b, G)$ defined by (6)–(8). Let us denote G_1 and G_2 the subgraphs of G generated by $\{a\} \cup (X \setminus T^p)$ and T^p, respectively and by $I(x, G)$, $I(x, G_1)$, $I(x, G_2)$ the sets of the vertices adjacent to x in G, G_1 and G_2, respectively. If G_1 and G_2 are complementary graphs of some interval graphs, by Theorem B, we can consider transitive orientations for G_1 and G_2 such that

$$(20)\quad \begin{aligned} &\text{a) } I^+(x, G_1) \supset I^+(y, G_1) \text{ or } I^+(x, G_1) \subset I^+(y, G_1)\ ;\ \forall x,\ y \in (X \setminus T^p) \cup \{a\}, \\ &\text{b) } I^+(x, G_2) \supset I^+(y, G_2) \text{ or } I^+(x, G_2) \subset I^+(y, G_2)\ ;\ \forall x,\ y \in T^p . \end{aligned}$$

Since $G \in \mathcal{G}$ we have $I(x, G) \supset T^p$ or $I(x, G) \cap T^p = \varnothing$, for all $x \in X \setminus T^p$. Let us consider

$$(21)\quad \begin{aligned} X_1 &= \{x \in X \setminus T^p \mid I(x, G) \cap T^p = \varnothing\} \cup \{a\}, \\ X_2 &= \{x \in X \setminus T^p \mid I(x, G) \supset T^p\} . \end{aligned}$$

By using Lemma 1 we conclude that for all $x,\ y \in X_1$, $x \notin I(y, G)$; because of $G \in \mathcal{G}$ it is clear that $X_1 \cap X_2 = \varnothing$, $X_1 \cup X_2 = \{a\} \cup (X \setminus T^p)$.
For the given transitive orientation of G_1 one defines

$$(22)\quad \begin{aligned} X_2^1 &= \{x \in X_2 \mid a \in I^+(x,\ G_1)\} , \\ X_2^2 &= \{x \in X_2 \mid x \in I^+(a,\ G_1)\} . \end{aligned}$$

Since $I(a, G_1) = X_2$ and G_1 is transitively oriented, it is easy to prove that $X_2^1 \cap X_2^2 = \varnothing$, $X_2^1 \cup X_2^2 = X_2$ and

$$(23)\quad I^+(x, G_1) \supset X_2^2, \text{ for all } x \in X_1^2.$$

Now, G_1 and G_2 being transitively oriented, let us define the following orientation of the edges of G :

$$(24)\quad \begin{aligned} I^+(x, G) &= I^+(x, G_1), \text{ for all } x \in X_2^2 \cup (X_1 \setminus \{a\}), \\ I^+(x, G) &= T^p \cup I^+(x, G_1), \text{ for all } x \in X_2^1, \\ I^+(x, G) &= I^+(x, G_2) \cup X_2^2, \text{ for all } x \in T^p, \\ I^-(x, G) &= I(x, G) \setminus I^+(x, G), \text{ for all } x \in X. \end{aligned}$$

Since G_1, G_2 are transitively oriented and $I(x, G) \supset T^p$ or $I(x, G) \cap T^p = \varnothing$ for all $x \in X \setminus T^p$, it is easy to prove that (24) give a transitive orientation, of G. Further, as a consequence of (20) and (24) one obtains

$$I^+(x, G) \supset I^+(y, G) \text{ or } I^+(x, G) \subset I^+(y, G), \text{ for all } x,\ y \in X.$$

By using Theorem B it follows that G is the complementary graph of an interval graph. Theorem 4 is proved.

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Received 28.XII. 1985

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Tomul XXXII, s. I a, Matematică, 1986, f. 3

A „GREEDY" $0(|V|^3)$-ALGORITHM FOR THE MAXIMUM CARDINALITY MATCHING PROBLEM

BY

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1. **Introduction.** Let $G = (V, E)$ be a finite graph. If $A \subseteq E$ and $v \in V$, then *the degree of v with respect to A*, denoted $d_A(v)$, is the number of the edges of A incident with v (clearly, $d_E(v)$ is the degree $d(v)$ of the vertex v in G).

A *matching* of G is a set $M \subseteq E$ such that $d_M(v) \leqslant 1$ for each vertex $v \in V$. Let us denote $\mathcal{M}_G$ the family of all matchings of the graph G. If $M \in \mathcal{M}_G$, then $V = \mathrm{SAT}_M \cup \mathrm{EXP}_M$, where

$$\mathrm{SAT}_M = \{v\ ;\ v \in V,\ d_M(v) = 1\}$$

is the set of *saturated* vertices with respect to M, and

$$\mathrm{EXP}_M = \{v\ ;\ v \in V,\ d_M(v) = 0\}$$

is the set of *exposed* vertices with respect to M.

A *maximum matching* of G is a matching M^* of maximum cardinality among all the matchings of $\mathcal{M}_G$. A characterization of a maximum matching was given by Berge [1] in terms of alternating paths. Using this characterization, Edmonds [4] has designed an $O(|V|^3)$ algorithm for determining a maximum matching of a given graph $G = (V, E)$.

An *alternating path* with respect to the matching M in the graph G is a path $P = v_1, e_1, v_2, e_2, \ldots, v_n, e_n, v_{n+1}$ from v_1 to v_{n+1}, of length n, such that there are no two consecutive edges on P belonging to M or $E \setminus M$ (note that all the vertices and edges are distinct). If $v \in \mathrm{EXP}_M$, then $\mathcal{E}_M(v)$ ($O_M(v)$) denote the set of all vertices accesible from v by an alternating path of *even* (*odd*) length. With these notations the Berge's characterization of a maximum matching can be stated : „M^* is a maximum matching if and only if

$$(1)\quad O_M(v) \subseteq \mathrm{SAT}_M,$$

for each vertex $v \in \mathrm{EXP}_M$".

The Edmonds' „blossom" algorithm, starting with an arbitrary matching M_0, try to find, at each stage, a vertex $v \in \mathrm{EXP}_M$ (M is the current matching) and an alternating path P of odd length from v_0 to a vertex $v_1 \in \mathrm{EXP}_M$. If such a vertex does not exist then the current matching is a maximum matching, else $M := M \Delta E(P)$ is the new current matching, having an edge more than its predecessor. One of the difficulties of the