$\{a, b\} \in N(G)$ and $T^p = T^p(a, b, G)$, $p = p(a, b, G)$ defined by (6)–(8). Let us denote G_1 and G_2 the subgraphs of G generated by $\{a\} \cup (X \setminus T^p)$ and T^p, respectively and by $I(x, G)$, $I(x, G_1)$, $I(x, G_2)$ the sets of the vertices adjacent to x in G, G_1 and G_2, respectively. If G_1 and G_2 are complementary graphs of some interval graphs, by Theorem B, we can consider transitive orientations for G_1 and G_2 such that

$$(20)\quad \begin{aligned} &\text{a) } I^+(x, G_1) \supset I^+(y, G_1) \text{ or } I^+(x, G_1) \subset I^+(y, G_1)\,;\ \forall x, y \in (X \setminus T^p) \cup \{a\}, \\ &\text{b) } I^+(x, G_2) \supset I^+(y, G_2) \text{ or } I^+(x, G_2) \subset I^+(y, G_2)\,;\ \forall x, y \in T^p . \end{aligned}$$

Since $G \in \mathcal{G}$ we have $I(x, G) \supset T^p$ or $I(x, G) \cap T^p = \varnothing$, for all $x \in X \setminus T^p$. Let us consider

$$(21)\quad \begin{aligned} X_1 &= \{x \in X \setminus T^p \mid I(x, G) \cap T^p = \varnothing\} \cup \{a\}, \\ X_2 &= \{x \in X \setminus T^p \mid I(x, G) \supset T^p\} . \end{aligned}$$

By using Lemma 1 we conclude that for all $x, y \in X_1$, $x \notin I(y, G)$; because of $G \in \mathcal{G}$ it is clear that $X_1 \cap X_2 = \varnothing$, $X_1 \cup X_2 = \{a\} \cup (X \setminus T^p)$. For the given transitive orientation of G_1 one defines

$$(22)\quad \begin{aligned} X_2^1 &= \{x \in X_2 \mid a \in I^+(x, G_1)\} , \\ X_2^2 &= \{x \in X_2 \mid x \in I^+(a, G_1)\} . \end{aligned}$$

Since $I(a, G_1) = X_2$ and G_1 is transitively oriented, it is easy to prove that $X_2^1 \cap X_2^2 = \varnothing$, $X_2^1 \cup X_2^2 = X_2$ and

$$(23)\quad I^+(x, G_1) \supset X_2^2, \text{ for all } x \in X_2^1.$$

Now, G_1 and G_2 being transitively oriented, let us define the following orientation of the edges of G:

$$(24)\quad \begin{aligned} I^+(x, G) &= I^+(x, G_1), \text{ for all } x \in X_2^2 \cup (X_1 \setminus \{a\}), \\ I^+(x, G) &= T^p \cup I^+(x, G_1), \text{ for all } x \in X_2^1, \\ I^+(x, G) &= I^+(x, G_2) \cup X_2^2, \text{ for all } x \in T^p, \\ I^-(x, G) &= I(x, G) \setminus I^+(x, G), \text{ for all } x \in X. \end{aligned}$$

Since G_1, G_2 are transitively oriented and $I(x, G) \supset T^p$ or $I(x, G) \cap T^p = \varnothing$ for all $x \in X \setminus T^p$, it is easy to prove that (24) give a transitive orientation, of G. Further, as a consequence of (20) and (24) one obtains

$$I^+(x, G) \supset I^+(y, G) \text{ or } I^+(x, G) \subset I^+(y, G), \text{ for all } x, y \in X.$$

By using Theorem B it follows that G is the complementary graph of an interval graph. Theorem 4 is proved.

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A „GREEDY" $0(|V|^3)$-ALGORITHM FOR THE MAXIMUM CARDINALITY MATCHING PROBLEM

BY

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1. Introduction. Let $G = (V, E)$ be a finite graph. If $A \subseteq E$ and $v \in V$, then *the degree of v with respect to A*, denoted $d_A(v)$, is the number of the edges of A incident with v (clearly, $d_E(v)$ is the degree $d(v)$ of the vertex v in G).

A *matching* of G is a set $M \subseteq E$ such that $d_M(v) \leqslant 1$ for each vertex $v \in V$. Let us denote $\mathcal{M}_G$ the family of all matchings of the graph G. If $M \in \mathcal{M}_G$, then $V = \mathrm{SAT}_M \cup \mathrm{EXP}_M$, where

$$\mathrm{SAT}_M = \{v\,;\ v \in V,\ d_M(v) = 1\}$$

is the set of *saturated* vertices with respect to M, and

$$\mathrm{EXP}_M = \{v\,;\ v \in V,\ d_M(v) = 0\}$$

is the set of *exposed* vertices with respect to M.

A *maximum matching* of G is a matching M^* of maximum cardinality among all the matchings of $\mathcal{M}_G$. A characterization of a maximum matching was given by Berge [1] in terms of alternating paths. Using this characterization, Edmonds [4] has designed an $O(|V|^3)$ algorithm for determining a maximum matching of a given graph $G = (V, E)$.

An *alternating path* with respect to the matching M in the graph G is a path $P = v_1, e_1, v_2, e_2, \ldots, v_n, e_n, v_{n+1}$ from v_1 to v_{n+1}, of length n, such that there are no two consecutive edges on P belonging to M or $E \setminus M$ (note that all the vertices and edges are distinct). If $v \in \mathrm{EXP}_M$, then $\mathcal{E}_M(v)$ ($O_M(v)$) denote the set of all vertices accesible from v by an alternating path of *even* (*odd*) length. With these notations the Berge's characterization of a maximum matching can be stated: „M^* is a maximum matching if and only if

$$(1)\qquad O_M(v) \subseteq \mathrm{SAT}_M,$$

for each vertex $v \in \mathrm{EXP}_{M^*}$".

The Edmonds' „blossom" algorithm, starting with an arbitrary matching M_0, try to find, at each stage, a vertex $v \in \mathrm{EXP}_M$ (M is the current matching) and an alternating path P of odd length from v_0 to a vertex $v_1 \in \mathrm{EXP}_M$. If such a vertex does not exist then the current matching is a maximum matching, else $M := M \Delta E(P)$ is the new current matching, having an edge more than its predecessor. One of the difficulties of the algorithm is the contraction of the „blossoms" (closed odd alternating paths). In this paper we give an algorithm for finding a maximum matching without „contractions" in a general graph. This algorithm with a greedy flavour, despite its simplicity, has also an $O(|V|^3)$ time complexity in the worst-case. Furthermore, it can be used to solve some „ordered" versions of the maximum cardinality matching problem.

Let us remark that our approach is different from that in [3].

2. A greedy maximum matching algorithm Let $G=(V,E)$ be a graph of order n.

Since the family

$$\mathcal{J}=\{A\ ;\ A\subseteq V,\ \exists M\in\mathcal{M}_G:\ A\subseteq \mathrm{SAT}_M\}$$

is a matroid [5], we can outline the following greedy procedure in order to obtain a base for this matroid:

1. Let $v_1, v_2, \ldots, v_n$ be an arbitrary ordering of V; $S=\emptyset$
2. **For** $i=1$ **to** n **do**
3.   **if** $S\cup\{v_i\}\in\mathcal{J}$ **then** $S:=S\cup\{v_i\}$ **endif**
4. **end**

The output S is exactly SAT_M, where M is a maximum cardinality matching. If we want to obtain M, and on the other hand, in order to implement efficiently the test in line 3, we modify the above procedure obtaining:

1. Let $v_1, v_2, \ldots, v_n$ be an arbitrary ordering of V
2. $M:=\emptyset$, $\mathrm{SAT}:=\emptyset$, $\mathrm{EXP}:=\emptyset$, $S:=\emptyset$
3. **For** $i=1$ **to** n **do**
4.   **if** $v_i\in SAT$
5.   **then** $S:=S\cup\{v_i\}$
6.   **else**
7.   **if** $A=O_M(v_i)\cap(\{v_{i+1},\ldots,v_n\}\setminus SAT)\neq\emptyset$
8.   **then** let v_j the first vertex in A and P the odd alternating path from v_i to v_j
9. $\mathrm{SAT}:=\mathrm{SAT}\cup\{v_i,v_j\}$, $M:=M\Delta E(P)$, $S:=S\cup\{v_i\}$
10.   **else** $EXP:=EXP\cup\{v_i\}$
11.   **endif**
12.   **endif**
13. **end**

At the end of this procedure the sets SAT and EXP are exactly the sets of saturated vertices, respectively exposed vertices with respect to the final matching M, and since $\mathrm{SAT}=S$, M is of maximum cardinality. These facts can be easily deduced from the following two lemmas.

Lemma 1. *If $M_0\in\mathcal{M}_G$, then there exists a maximum matching $M^*\in\mathcal{M}_G$ such that $S(M_0)\subseteq S(M^*)$.*

Proof. If $M\in\mathcal{M}_G$ and $vw\in M$, then let us define $\mathrm{match}_M(v)=w$ and $\mathrm{match}_M(w)=v$.

Let $M_0\in\mathcal{M}_G$ and M^* be a maximum matching. Then, $M_1=M^*\cup\{v\ \mathrm{match}_{M_0}(v);\ v\in\mathrm{SAT}_{M_0}\cap\mathrm{EXP}_{M^*}\}\setminus\{\mathrm{match}_{M_0}(v)\mathrm{match}_{M^*}(\mathrm{match}_{M_0}(v));\ v\in\mathrm{SAT}_{M_0}\cap\mathrm{EXP}_{M^*}\}$ is a matching in G which satisfies $|M_1|=|M^*|$ (hence M_1 is a maximum matching) and $S(M_0)\subseteq S(M_1)$.

Lemma 2. *Let $M_0\in\mathcal{M}_G$ and $v_0\in\mathrm{EXP}_{M_0}$. If $O_{M_0}(v_0)\subseteq\mathrm{SAT}_{M_0}$, then for any $M\in\mathcal{M}_G$ such that $S(M_0)\subseteq S(M)$ we have:*

$$v_0\in\mathrm{EXP}_M \text{ and } O_M(v_0)\subseteq\mathrm{SAT}_M.$$

Proof. Let M_0, M and v_0 satisfying the hypothesis of the lemma. Let us suppose that $v_0\in\mathrm{SAT}_M$. Then, the connected component which contains v_0 of the subgraph spanned by $M_0\Delta M$ in G is an odd alternating path P with respect to M_0, starting from v_0 and finishing in an exposed vertex v_f with respect to M_0, contradicting the hypothesis (Since $v_0\in\mathrm{SAT}_M$ then there exists $v_1=\mathrm{match}_{M\setminus M_0}(v_0)$ and hence v_0, v_0v_1, v_1 is an odd alternating path with respect to M_0. If $v_1\in\mathrm{EXP}_{M_0}$, this is the odd alternating path P; otherwise $v_1\in\mathrm{SAT}_{M_0}$, hence there exists $v_2=\mathrm{match}_{M_0\setminus M}(v_1)$. Clearly $v_2\in\mathrm{SAT}_{M_0}\subseteq\mathrm{SAT}_M$ and hence there exists $v_3=\mathrm{match}_{M\setminus M_0}(v_2)$. Since M_0 and M are matchings, it follows that $v_0, v_0v_1, v_1, v_1v_2, v_2, v_2v_3, v_3$ is an odd alternating path with respect to M_0. If $v_3\in\mathrm{EXP}_{M_0}$, this is the odd alternating path P, otherwise we can continue the above construction and since the vertices set V is finite, and all the vertices of the constructed paths are distinct, we must finish with the odd alternating path P desired).

Now, if $O_M(v_0)\cap\mathrm{EXP}_M\neq\emptyset$, let P' an odd alternating path from v_0 to $v\in\mathrm{EXP}_M$. But then, the matching $M_1=M\Delta E(P')$ satisfies $S(M_0)\subseteq S(M_1)$ and $v_0\in\mathrm{SAT}_{M_1}$, contradicting the first part of the lemma.

The only difficulty of the procedure described is the test in line 7, whose implementation must give the vertex v_j and the path P (if they exist). We shall do this by performing a depth first search from v_i, the current vertex scanned by the algorithm. If M is the current matching, then EXP_M contains EXP, (the set of definitively exposed vertices), and $V\setminus(\mathrm{EXP}\cup\mathrm{SAT}_M)$, (the set of yet-unmatched vertices). Let *first* (v_k) be the first vertex in $V\setminus(\mathrm{EXP}\cup\mathrm{SAT}_M)$ adjacent with the vertex $v_k\in V$. We note that the time spent by the algorithm to maintain *first* (v_k) is proportional to $d(v_k)$, using for this two stacks for each v_k:

(i) *adj-sat* (v_k), containing an unordered list of the vertices adjacent with v_k, which are saturated by the current matching (initially *adj-sat* (v_k) is empty for each v_k; when, in the algorithm, a vertex v becomes saturated, then v is added in *adj-sat* (v_k) for each vertex v_k adjacent with v);

(ii) *adj-exp*(v_k), containing initially the ordered list of the vertices adjacent with $v_k(\mathrm{EXP}_M=\emptyset)$. Using this stack, *first* (v_k) can be obtained by popping *adj-exp*(v_k) until a vertex from $V\setminus(\mathrm{EXP}\cup\mathrm{SAT}_M)$ is determined or the stack becomes empty. In this last case, *first* (v_k) is setted on $n+1$.

The depth first search from a vertex v can be now described:

procedure $DFS(v)$

1. stack v in P' // P' is an empty stack controlling the depth first search // *best* $:=$ *first* (v); $P:=\{v, best\}$
2. **while** P' is nonempty **do**
3.   **while** *adj-sat* (top of P') is nonempty **do**
4. w_1: top of *adj-sat* (top of P')
5.   **if** w_1 is unvisited and w_1 is not in P'
6.   **then**
7.   mark w_1 visited

```
          w : match_M(w_1)
          stack w in P'
          if best > first (w)
            then
            best : =first (w)
            P : P' ∪ best
          endif
        else
            pop adj-sat (top of P')
        endif
      end
    pop P'
  end
  end DFS(v).
```

Let us remark that in P' are introduced, during the DFS(v), all the vertices of $\mathscr{E}_M(v)$. Each vertex of $SAT_M \cap O_M(v)$ is used once for the updating of *best* (when it is declared visited) and several times for the construction of the alternating paths. Since each stack *adj-sat*(v_k) have at most $2|M|$ vertices and each vertex visited by DFS(v) is in SAT_M, it follows that the time spent by DFS(v) is of $O(|M|^2)$.

If the value of *best* determined by DFS(v) is not $n+1$, then P gives the information needed to modify the current matching; indeed if $P = v_i, w_1, w_2, \ldots, w_k$, *best* then the augmented matching is $M \cup \{v_i \mathrm{match}_M(w_1), w_1 \mathrm{match}_M(w_2), \ldots, w_{k-1} \mathrm{match}(w_k), w_k best\} \setminus \bigcup_{i=1}^{k} \{w_i \mathrm{match}_M(w_i)\}$.

We shall denote this (in the algorithm given bellow) by *augment M*. The greedy maximum matching algorithm can be now definitively described.

The greedy maximum matching algorithm.

Input: A graph $G=(V, E)$ given by $|V|$ stacks *adj-exp*(v) $(v \in V)$, containing an ordered list of the vertices adjacent with v; the order $v_1, v_2, \ldots, v_n$ of V is arbitrary but the same for each stack.

Output: $M \subseteq E$ a maximum matching of G.

```
 M : =∅, SAT : =∅, EXP =∅
 for i =1 to n do
    adj-sat (v_i) : =∅
 end
 for i =1 to n do
   if v_i ∉ SAT
     then
      for each v in SAT do
        store the level of adj-sat (v)
      end
      call DFS(v_i)
      for each v in SAT do
        restore the level of adj-sat (v)
      end
      if best < n+1
```



```
        then
          SAT : =SAT ∪ {v_i, best}
          augment M
          for each v in adj-exp (v_i) do
            stack v_i in adj-sat (v)
      end
      for each v in adj-exp (best) do
          stack best in adj-sat (v)
        end
      else
            EXP : =EXP ∪ {v_i}
        endif
      endif
 end
 output M
 end
```

The time complexity of the algorithm is determined by the time spent by the repeated callings of DFS (v_i), $i=\overline{1, n}$, which is

$$0(1^2)+0(2^2)+\ldots+0(\nu(G)^2)=0(\nu(G)^3)$$

where $\nu(G)$ is the cardinality of the maximum matching of G.

3. Some applications. Let $\leqslant$ a preorder on the vertices set V of the graph $G=(V, E)$.

An $\leqslant$ — *optimal matching* in G is a matching $M^* \in \mathscr{M}_G$ such that:

$$\forall v \in EXP_{M^*}\ O_{M^*}(v) \subseteq SAT_{M^*} \tag{1}$$

$$\forall v \in EXP_{M^*},\ \forall v_1 \in \mathscr{E}_{M^*}(v)\ v \leqslant v_1 \tag{2}$$

Clearly an $\leqslant$ — optimal matching M^* is a maximum matching (by condition (1)). Furthermore, if we choose an ordering $v_1, v_2, \ldots, v_n$ of V such that $v_i \leqslant v_j \Rightarrow i < j$, then the matching obtained with our greedy algorithm is $\leqslant$ — optimal (If $v_i \in EXP_{M^*}$ each vertex v_j accesible from v_i on an even alternating path has $j \geqslant i$, otherwise lemma 2 would be contradicted).

Now, if we choose the preorder $v_i \leqslant v_j \Leftrightarrow d_G(v_i) \geqslant d_G(v_j)$, then an $\leqslant$ — optimal matching is a maximum degree matching [2], which can be used in edge-graph colouring.

If we choose $v_i \leqslant v_j \Leftrightarrow d_G(v_i) \leqslant d_G(v_j)$, then an $\leqslant$ — optimal matching, given by our algorithm is a maximum matching with a minimum sum of the degrees of the saturated vertices. It seems that this ordering is more appropiate in order to increase the efficiency of our algorithm.

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