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QUASIRIGID BODIES

BY

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1. Let $\mathscr{E}$ be a three-dimensional Euclidian point space, $\mathscr{V}$ the associated vector space and $\mathscr{L}(\mathscr{V})$ the space of second-order tensors over $\mathscr{V}$. A motion (deformation) of a continuous body $\mathscr{B}$ is said to be homogeneous if every straight line is deformed into a straight line. In order that a deformation $\mathbf{x}(\mathbf{X}, t)$ relative to a reference configuration $\varkappa(\mathscr{B})$ should be homogeneous, it is necessary and sufficient that

$$\text{(1)} \qquad \mathbf{x} = \mathbf{x}_0(t) + \mathbf{F}(t)(\mathbf{X} - \mathbf{X}_0), \quad \det\mathbf{F} \neq 0,$$

where $\mathbf{X}_0$ is a fixed place in $\varkappa(\mathscr{B})$, $\mathbf{x}_0$ is a function of time with values in $\mathscr{E}$ and $\mathbf{F}$ is a function of time with values in $\mathscr{L}(\mathscr{V})$. The equations for the velocity field and the acceleration field of the homogeneous motion (1) are

$$\text{(2)} \qquad \dot{\mathbf{x}} = \dot{\mathbf{x}}_0 + \dot{\mathbf{F}}\mathbf{F}^{-1}(\mathbf{x} - \mathbf{x}_0), \qquad \ddot{\mathbf{x}} = \ddot{\mathbf{x}}_0 + \ddot{\mathbf{F}}\mathbf{F}^{-1}(\mathbf{x} - \mathbf{x}_0).$$

The class of homogeneous motions has an important role in mechanics since, on one side, homogeneous motions (1) generalise the rigid motions corresponding to orthogonal $\mathbf{F}$ and, on the other side, homogeneous deformations represent first order local approximations of smooth deformations of continuous bodies.

Principal geometric and kinematics properties of homogeneous motions, with some historical aspects and literature references, can be found in the authoritative works [1] and [2]. From the point of view of the continuum dynamics, homogeneous motions are possible only for suitable contact forces and applied body forces, but this represents a very important restriction of the class of possible homogeneous motions [2, 182—185].

In this note, we propose another dynamical interpretation of homogeneous motions. On the analogy of the definition of the rigid body, which is a continuous body admitting only rigid motions, we call *quasirigid body* a continuous body which may accomplish only homogeneous motions. If this mechanical model is free of any constraint (unconstrained), then it has 12 degrees of freedom (a general rigid body has six degrees of freedom) and therefore will be included in analytical dynamics.

The position vector of the center of mass $\mathbf{x}^*$ and the Euler tensor $\mathbf{E}$ of a quasirigid body $\mathscr{B}$, both at the time t and with respect to $\mathbf{x}_0$, are given by

(3) $$\mathbf{x}^* - \mathbf{x}_0 = M^{-1}\mathbf{F}(t)\int_{\mathcal{B}} (\mathbf{X} - \mathbf{X}_0)\,dm, \quad \mathbf{E} = \mathbf{F}(t)\Big(\int_{\mathcal{B}} (\mathbf{X} - \mathbf{X}_0)\otimes(\mathbf{X} - \mathbf{X}_0)\,dm\Big)\mathbf{F}(t)^T,$$

where M is the mass of $\mathcal{B}$, $\otimes$ denotes the tensor product and $\mathbf{F}^T$ is the transpose of $\mathbf{F}$. The rotational momentum (moment of momentum) of a quasirigid body with respect to $\mathbf{x}_0$ is given by

(4) $$\mathbf{K} = \mathbf{E}(\dot{\mathbf{F}}\mathbf{F}^{-1})^T - \dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{E} + M(\mathbf{x}^* - \mathbf{x}_0)\wedge\dot{\mathbf{x}}_0,$$

where $\wedge$ denotes the exterior product. Because $\dot{\mathbf{F}}\mathbf{F}^{-1}$ represents the sum of the stretching (rate-of-strain) tensor $\mathbf{D}$ and the spin tensor $\mathbf{W}$, we can rewrite the formula (4) as

(5) $$\mathbf{K} = \mathbf{ED} - \mathbf{DE} - \mathbf{EW} - \mathbf{WE} + M(\mathbf{x}^* - \mathbf{x}_0)\wedge\dot{\mathbf{x}}_0.$$

The tensors $\mathbf{K}$ and $\mathbf{W}$ are skew-symmetric, while $\mathbf{D}$ and $\mathbf{E}$ are symmetric.

If $\mathbf{x}_0(t)$ is taken as the place $\mathbf{x}^*$ occupied by the center of mass of $\mathcal{B}$ at the time t, then (5) reduces to

(6) $$\mathbf{K}^* = \mathbf{E}^*\mathbf{D} - \mathbf{D}\mathbf{E}^* - \mathbf{E}^*\mathbf{W} - \mathbf{W}\mathbf{E}^*,$$

$\mathbf{K}^*$ and $\mathbf{E}^*$ being the rotational momentum and the Euler tensor of $\mathcal{B}$ relative to center of mass, respectively.

Theorem 1. *For a quasirigid body, any three of the following four properties of a line imply the fourth:*

(i) *It is a principal axis of inertia at* $\mathbf{x}^*$ *corresponding to a simple proper number.*

(ii) *It is a principal axis of stretching associated to an extreme proper number.*

(iii) *It is the axis of spin.*

(iv) *It is the axis of rotational momentum relativ to* $\mathbf{x}^*$.

Proof. Every principal axis of inertia at $\mathbf{x}^*$ is a principal axis of the tensor $\mathbf{E}^*$ and conversely.

Let $\mathbf{p}$ be a proper vector of the tensors $\mathbf{D}$ and $\mathbf{E}^*$. Then $\mathbf{Dp} = \lambda\mathbf{p}$, $\mathbf{E}^*\mathbf{p} = \mu\mathbf{p}$ and from (6) it follows that $\mathbf{K}^*\mathbf{p} = -(\mathbf{E}^* + \mu\mathbf{1})\mathbf{Wp}$, where $\mathbf{1}$ is the unit tensor. Since the tensor $\mathbf{E}^* + \mu\mathbf{1}$ is invertible (being positive-definite), the equality $(\mathbf{E}^* + \mu\mathbf{1})\mathbf{Wp} = \mathbf{0}$ holds if and only if $\mathbf{Wp} = \mathbf{0}$.

Let us suppose now that $\mathbf{p}_1$ is a proper vector of the tensors $\mathbf{K}^*$ and $\mathbf{W}$, i.e. we have $\mathbf{K}^*\mathbf{p}_1 = \mathbf{0}$, $\mathbf{Wp}_1 = \mathbf{0}$. In this case, the relation (6) yields

(a) $$\mathbf{E}^*(\mathbf{Dp}_1) - \mathbf{D}(\mathbf{E}^*\mathbf{p}_1) - \mathbf{W}(\mathbf{Ep}_1) = \mathbf{0}.$$

If $\mathbf{p}_1$ is a unit proper vector of the tensor $\mathbf{E}^*$, then there is at least one orthonormal basis $\{\mathbf{p}_i\}$, each member of which is a proper vector of $\mathbf{E}^*$, such that

$$\mathbf{E}^* = \sum_{i=1}^{3} \mu_i\mathbf{p}_i\otimes\mathbf{p}_i, \quad \mu_1 \neq \mu_2, \mu_3.$$

The representation of $\mathbf{D}$ relative to the basis $\{\mathbf{p}_i\}$ is of the form $\mathbf{D} = D_{ij}\mathbf{p}_i\otimes\mathbf{p}_j$, $D_{ij} = D_{ji}$. Hence, and by (a), we obtain $\mathbf{Dp}_1 = D_{11}\mathbf{p}_1$.

In a similar way, if $\mathbf{p}_1$ is a unit proper vector of $\mathbf{D}$, there exists an orthonormal basis of proper vectors of $\mathbf{D}$, denoted $\{\mathbf{p}_i\}$, such that

(b) $$\mathbf{D} = \sum_{i=1}^{3} \lambda_i\mathbf{p}_i\otimes\mathbf{p}_i, \quad \mathbf{W} = \frac{\sqrt{2}}{2}|\mathbf{W}|\mathbf{p}_2\wedge\mathbf{p}_3, \quad \mathbf{E}^* = E^*_{ij}\mathbf{p}_i\otimes\mathbf{p}_j, \quad E^*_{ij} = E^*_{ji},$$

holds, where λ_1 is the least or greatest proper number of $\mathbf{D}$ and $|\mathbf{W}|$ is the norm of the tensor $\mathbf{W}$. The relations (a) and (b) yield the system

$$\sqrt{2}(\lambda_1 - \lambda_2)E^*_{12} - |\mathbf{W}|E^*_{13} = 0, \quad |\mathbf{W}|E^*_{12} + \sqrt{2}(\lambda_1 - \lambda_3)E^*_{13} = 0,$$

having the determinant $2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) + |\mathbf{W}|^2$. It follows that $E^*_{12} = E^*_{13} = 0$, hence $\mathbf{E}^*\mathbf{p}_1 = E^*_{11}\mathbf{p}_1$, except in the nonsignificant case when $\mathbf{W} = \mathbf{0}$ and λ_1 is a multiple proper number of $\mathbf{D}$.

This theorem represents an extension for the quasirigid bodies of a result known for the rigid bodies [2, 45–46]. In the case of a rigid body, $\mathbf{D}$ vanishes and the restrictive condition $\mu_1 \neq \mu_2, \mu_3$ is no more necessary.

2. As it is well known, a rigid body in general may be acted on by a force system consisting in a resultant force $\mathcal{R} \in \mathcal{V}$ and a resultant moment $\mathcal{M} \in \mathcal{V}$ relative to $\mathbf{x}_0$ at each t. Analogously, we suppose that a system of forces acting on the quasirigid body $\mathcal{B}$ consists of two parts: a resultant force $\mathcal{R} \in \mathcal{V}$ and a generalized resultant moment $\mathcal{M} \in \mathcal{L}(\mathcal{V})$ with respect to $\mathbf{x}_0$, at any time of the motion. If there is a body force density $\mathbf{b}$, we have

(7) $$\mathcal{R} = \int_{\mathcal{B}} \mathbf{b}\,dm, \quad \mathcal{M} = \int_{\mathcal{B}} (\mathbf{x} - \mathbf{x}_0)\otimes\mathbf{b}\,dm.$$

In order that $\mathcal{R}$ and $\mathcal{M}$ may be related to certain kinematical quantities, i.e. in order to deduce the equations of motion of a quasirigid body, we shall use D'Alembert's principle. Let us consider a smooth one-parameter family of mappings of the reference configuration $\varkappa(\mathcal{B})$ onto regions of $\mathcal{E}$:

(8) $$\mathbf{x} = \mathbf{x}_0(s) + \mathbf{F}(s)(\mathbf{X} - \mathbf{X}_0), \quad \det\mathbf{F} \neq 0, \quad s \in [s_1, s_2].$$

Since s need not be the time, such a family is called a homogeneous virtual motion. The expression

(9) $$\mathbf{x}' = \mathbf{x}_0'(s) + \mathbf{F}'(s)\mathbf{F}(s)^{-1}(\mathbf{x} - \mathbf{x}_0),$$

is called *virtual velocity* of a particle of the quasirigid body.

Theorem 2. *An unconstrained quasirigid body obeys D'Alembert's principle if and only if relative to an inertial frame it satisfies the following equations of motion*

(10) $$M\ddot{\mathbf{x}}_0 + M\ddot{\mathbf{F}}\mathbf{F}^{-1}(\mathbf{x}^* - \mathbf{x}_0) = \mathcal{R}, \quad M(\mathbf{x}^* - \mathbf{x}_0)\otimes\ddot{\mathbf{x}}_0 + \mathbf{E}(\mathbf{F}^T)^{-1}\ddot{\mathbf{F}}^T = \mathcal{M}.$$

Proof. D'Alembert's principle asserts that the virtual power of the applied forces and of the inertial forces is zero, which is equivalent to the verification of the condition

(11) $$\int_{\mathcal{B}} \mathbf{x}'\cdot(\mathbf{b} - \ddot{\mathbf{x}})\,dm = 0$$

for any virtual velocity $\mathbf{x}'$.

By substituting $\mathbf{x}'$ from (9) into (11), we obtain

$$\mathbf{x}_0' \cdot \int_B (\mathbf{b} - \ddot{\mathbf{x}})dm + tr[\mathbf{F}'\mathbf{F}^{-1} \int_B (\mathbf{x} - \mathbf{x}_0) \otimes (\mathbf{b} - \ddot{\mathbf{x}})dm] = 0,$$

where tr denotes the trace of the corresponding tensor. In view of $(2)_2$, (3) and (7), the preceding relation is equivalent to

$$\mathbf{x}_0' \cdot [M\ddot{\mathbf{x}}_0 + M\ddot{\mathbf{F}}\mathbf{F}^{-1}(\mathbf{x}^* - \mathbf{x}_0) - \mathbf{R}] + (\mathbf{F}'\mathbf{F}^{-1})^T \cdot [M(\mathbf{x}^* - \mathbf{x}_0) \otimes \ddot{\mathbf{x}}_0 + \mathbf{E}(\mathbf{F}^T)^{-1}\mathbf{F}^T - \mathcal{M}] = 0 \tag{12}$$

and since $\mathbf{x}_0'(s)$, $\mathbf{F}'(s)\mathbf{F}(s)^{-1}$ are arbitrary, it follows that the equality (12) is equivalent to equations (10).

Remarks. 1° If we chose $\mathbf{x}_0(t)$ to coincide to $\mathbf{x}^*$, then the motions of a quasirigid body are governed by the system of differential equations

$$M\ddot{\mathbf{x}}^* = \mathcal{R}, \quad \mathbf{E}^*(\mathbf{F}^T)^{-1}\ddot{\mathbf{F}}^T = \mathcal{M}^*, \tag{10'}$$

$\mathcal{M}^*$ being the generalized resultant moment with respect to $\mathbf{x}^*$.

2° In the case of a uniform field $\mathbf{b} = \text{const.}$ the equations of motion (10) and (10') come to $\ddot{\mathbf{x}}_0 = \mathbf{b}$, $\ddot{\mathbf{F}} = \mathbf{0}$ and $\ddot{\mathbf{x}}^* = \mathbf{b}$, $\ddot{\mathbf{F}} = \mathbf{0}$, respectively. One obtains, in this way, the same result as for a homogeneous simple body subject to a homogeneous deformation history from a homogeneous reference [2, 182–183].

3° In order to determine the motion of a quasirigid body to the equations of motion (10), we must add the initial conditions:

$$\mathbf{x}(t_0) = \mathbf{x}_0, \quad \dot{\mathbf{x}}(t_0) = \mathbf{v}_0, \quad \mathbf{F}(t_0) = \mathbf{F}_0, \quad \dot{\mathbf{F}}(t_0) = \dot{\mathbf{F}}_0, \quad \det \mathbf{F}_0 \neq 0. \tag{I}$$

4° Also, the equations of motions of a constrained quasirigid body may be deduced from D'Alembert's principle in a standard way.

5° The idea of considering the quasirigid body model has been suggested by Harley Cohen's paper [3]. We had not access to this paper, but only to a short author's summary (MR 83 c : 70 003), where it is pointed out that he generalizes the equations of motion of a rigid body with a fixed point to bodies undergoing homogeneous deformation with a fixed point. This is obtained by a generalization of Euler's second law (the principle of balance of rotational momentum).

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RECENZII

A. BOREL: *Oeuvres — Collected Papers*, Volume 1: 1948—1958, Volume 2: 1959—1968, Volume 3: 1969—1982. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983, XVI+2226 p., $ 135.10

Aceste trei volume conţin articolele publicate de A. Borel singur sau în colaborare între anii 1948 şi 1982. Nu sînt incluse articolele expozitive de mai mare întindere publicate în Seminarul Bourbaki sau în Lecture Notes in Mathematics. Se găseşte astfel puse la un loc rezultatele cercetărilor unuia din cei mai mari matematicieni, cercetări care au impulsionat şi jalonat dezvoltarea geometriei diferenţiale, topologiei şi geometriei algebrice peste 30 de ani. Majoritatea articolelor sînt reproduse prin fotocopiere. Corecturile ce au fost necesare s-au dat într-un paragraf final. Volumul I se deschide cu un „curriculum vitae" al autorului. În finalul volumului III se dă o listă completă a lucrărilor lui A. Borel. Publicarea acestor volume constituie o contribuţie remarcabilă la dezvoltarea matematicii.

M. Anastasiei

A. S. KECHRIS, D. A. MARTIN and Y. N. MISKEVAKIS (Editors): *Proceedings, Caltech-UCLA Logic Seminar 1979—1981*, Lecture Notes in Mathematics, vol. 1019, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983, V+284 p., $ 13

Volumul conţine lucrări prezentate, în cadrul Seminarului de logică de la UCLA, în perioada 1979—1981. Dintre problemele cercetate menţionăm: Măsuri supercompacte pe $P_{\omega_1}(\lambda)$, rezultate privind teoria Wadge asupra mulţimilor Borel, introducere în Q-teorie şi problemele axate pe diferite aspecte ale Scale-teoriei. În partea finală, sînt prezentate unele probleme deschise, probleme interesante sugerate de studiul efectuat în cadrul menţionat mai sus.

Olga Costinescu

STEPHEN MC ADAM: *Asymptotic Prime Divisors*, Lecture Notes in Mathematics, vol. 1023, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983; IX+118 p., US $ 7.70

În această monografie, autorul prezintă în detaliu comportarea lui $\mathrm{Ass}(R/I^n)$, pentru n mare, în cazul unui inel comutativ noetherian R şi a unui ideal I al lui R. Dacă I^a notează închiderea întreagă a lui I, şirul $\mathrm{Ass}(R/I^n)$, $n = 1, 2, 3, \ldots$ uneori se stabilizează. Autorul dă unele condiţii pentru stabilizarea acestui şir, ca şi condiţii de stabilitate asimptotică pentru şiruri de elemente. Lectura monografiei prezentate necesită numai cunoştinţe clasice de teoria inelelor comutative, astfel încît poate fi făcută de un absolvent al cursurilor de algebră din primii doi ani de studiu

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