Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXII, s. I a, Matematică, 1986, f. 3

A LAGRANGIAN THEORY OF RELATIVITY. II*)

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III. Invariant form of Einstein's Equations. The canonical form of the metric of a generalized Lagrange space M^n and the local pseudo-orthogonal frames with respect to which one realizes the canonical form lead to the invariance group of the metric. Using these frames, we can obtain the invariant components of the geometrical objects of the space M^n. The equations of the motion of the pseudo-orthogonal frames and their integrability conditions give the fundamental equations of M^n and of course the canonical form of Einstein's Equations.

§ 1. The invariance group of the metric of a generalized Lagrange space. In every point $(x, y) \in \pi^{-1}(U) \subset TM$ we consider the quadratic differential form $\tilde{\omega} = g_{ij}(x, y)\, dx^i dx^j$, associated to the generalized Lagrange space $M^n = (M, g_{ij}(x, y))$. Then $\tilde{\omega}$ has a global character on the tangent bundle TM.

Supposing that $\tilde{\omega}$ has the canonical form

$$\tilde{\omega} = (\tilde{\omega}^1)^2 + \dots + (\tilde{\omega}^p)^2 - (\tilde{\omega}^{p+1})^2 - \dots - (\tilde{\omega}^n)^2, \tag{1.1}$$

we denote by $a', b', \dots$ indices with the values $1, \dots, p$; by $a'', b'', \dots$ indices with the values $p+1, \dots, n$ and by $a, b, \dots, 1, \dots, n$.

We use, in the sequel, the Vrănceanu symbols [66] :

$$\varepsilon_{ab} = \begin{cases} \delta_{a'b'}, & \text{for } a=a',\ b=b' \\ 0, & \text{for } a=a',\ b=b'' \text{ or } a=a'',\ b=b' \\ -\delta_{a''b''}, & \text{for } a=a'',\ b=b''. \end{cases} \tag{1.2}$$

If ε^{ab} are given by

$$\varepsilon_{ab}\varepsilon^{bc} = \delta_a^c, \tag{1.3}$$

then the canonical form (1.1) can be written :

$$\tilde{\omega} = \varepsilon_{ab}\tilde{\omega}^a\tilde{\omega}^b \tag{1.1'}$$

Let $\mathcal{R} = ((x, y)\,;\, e_a^i)$ be the local frames in which the canonical form (1.1)' holds. It results :

$$g_{ij}e_a^i e_b^j = \varepsilon_{ab}, \tag{1.4}$$

i.e. $\mathcal{R}$ is a pseudo-orthogonal local frame. Denoting by $\mathcal{R}^* = ((x, y); e_i^a)$ the dual of $\mathcal{R}$ we get :

*) The first part of this paper has been published in the same journal, f. 2 of T. XXXII.

(1.5) $$e^i_a e^b_i = \delta^b_a, \quad (\Rightarrow e^i_a e^a_j = \delta^i_j)$$

and the pseudo-orthogonality of $\mathcal{R}^*$ is expressed by:

(1.4)′ $$\varepsilon^{ab} = g^{ij} e^a_i e^b_j.$$

It follows from (1.4) and (1.4)′:

(1.6) $$g_{ij} = \varepsilon_{ab} e^a_i e^b_j, \quad g^{ij} = \varepsilon^{ab} e^i_a e^j_b, \quad \varepsilon_{ac} e^c_i = g_{ij} e^j_a.$$

The transformations of the local pseudo-orthogonal frames $\mathcal{R} \to \bar{\mathcal{R}}$ are given by

(1.7) $$\bar{e}^i_a = \bar{c}^b_a e^i_b, \quad \det \| \bar{c}^b_a(x, y) \| \neq 0$$

and the transformations $\mathcal{R}^* \to \bar{\mathcal{R}}^*$ are given by

(1.7)′ $$\bar{e}^a_i = c^a_b e^b_i, \quad \det \| c^a_b(x, y) \| \neq 0.$$

Then

(1.8) $$c^b_c \bar{c}^c_a = \delta^b_a, \quad c^c_a \bar{c}^b_c = \delta^b_a.$$

But (1.7) or (1.7)′ are the transformations of the pseudoorthogonal frames. This means:

(1.9) $$\varepsilon_{ab} \bar{c}^a_c \bar{c}^b_d = \varepsilon_{cd}, \quad \varepsilon^{ab} c^c_a c^d_b = \varepsilon^{cd},$$

which imply:

(1.10) $$c^a_b = \varepsilon_{bc} \bar{c}^c_d \varepsilon^{da}.$$

Consequently, we have:

Theorem 1.1. *The transformations of the pseudo-orthogonal frames $\mathcal{R} \to \bar{\mathcal{R}}$, which preserve the canonical form (1.1) of the metric $g_{ij}(x, y)$, together with the product of transformations is a group, isomorphic to the multiplicative group of the pseudo-orthogonal matrix: $\| c^a_b \|$, (1.9).*

We use the pseudo-orthogonal frames $\mathcal{R} = ((x, y); e^i_a)$ in order to represent the geometrical objects of the generalized Lagrange space M^n by their invariant components in $\mathcal{R}$.

A distinguished tensor field $T^{i_1 \dots i_r}_{j_1 \dots j_s}(x, y)$ is represented in $\mathcal{R}$ by its invariant components:

$$T^{a_1 \dots a_r}_{b_1 \dots b_s}(x, y) = T^{i_1 \dots i_r}_{j_1 \dots j_s} e^{a_1}_{i_1} \dots e^{a_r}_{i_r} e^{j_1}_{b_1} \dots e^{j_s}_{b_s}.$$

Indeed, for a transformation (1.1), I, $T^{a_1 \dots a_r}_{b_1 \dots b_s}(x, y)$ are not changed. Conversely, when $T^{a_1 \dots a_r}_{b_1 \dots b_s}(x, y)$ are given, the components $T^{i_1 \dots i_r}_{j_1 \dots j_s}(x, y)$ are uniquely determined by:

$$T^{i_1 \dots i_r}_{j_1 \dots j_s}(x, y) = T^{a_1 \dots a_r}_{b_1 \dots b_s} e^{i_1}_{a_1} \dots e^{i_r}_{a_r} e^{b_1}_{j_1} \dots e^{b_s}_{j_s}.$$

For example, from (1.6) we see that the invariant components of the fundamental tensor $g_{ij}(x, y)$ are ε_{ab} and those of $g^{ij}(x, y)$ are ε^{ab}. A transformation (1.7), $\mathcal{R} \to \bar{\mathcal{R}}$, has as effect the transformation of the invariant components by the law:

$$\bar{T}^{a_1 \dots a_r}_{b_1 \dots b_s} = c^{a_1}_{c_1} \dots c^{a_r}_{c_r} \bar{c}^{d_1}_{b_1} \dots \bar{c}^{d_s}_{b_s} T^{c_1 \dots c_r}_{d_1 \dots d_s}.$$

For example (1.9) gives the transformation law of the invariant components ε_{ab} of the fundamental tensor field $g_{ij}(x, y)$.

Let $\overset{\circ}{N}$ be a fixed non-linear connection on the tangent bundle TM and $\{\delta/\delta x^i, \partial/\partial y^i\}$ be the local adapted basis to the horizontal distribution $\overset{\circ}{N}$ and to the vertical distribution V, respectively. The vector fields $\{\delta/\delta x^i, \partial/\partial y^i\}$ there are represented in $\mathcal{R}$ in the form:

(1.11) $$\frac{\delta}{\delta s^a} = e^i_a \frac{\delta}{\delta x^i}, \quad \frac{\partial}{\partial \sigma^a} = e^i_a \frac{\partial}{\partial y^i}.$$

The dual adapted basis $\{dx^i, \delta y^i\}$ to the distributions $\overset{\circ}{N}$ and V is represented in $\mathcal{R}$ by:

(1.11)′ $$ds^a = e^a_i dx^i, \quad \delta\sigma^a = e^a_i \delta y^i,$$

So that for a transformation (1.7) we have

(1.12) $$\begin{cases} d\bar{s}^a = c^a_b \, ds^b, \quad \delta\bar{\sigma}^a = c^a_b \delta\sigma^b, \\ \dfrac{\delta}{\delta \bar{s}^a} = \bar{c}^b_a \dfrac{\delta}{\delta s^b}, \quad \dfrac{\partial}{\partial \bar{\sigma}^a} = \bar{c}^b_a \dfrac{\partial}{\partial \sigma^b}. \end{cases}$$

The non-holonomy coefficients [66] of the non-holonomic coordinates (s^a, σ^a) in $\pi^{-1}(U) \subset TM$ are defined by

(1.13) $$w^a_{bc} = e^a_i \left(\frac{\delta e^i_b}{\delta s^c} - \frac{\delta e^i_c}{\delta s^b} \right), \quad v^a_{bc} = e^a_i \left(\frac{\partial e^i_b}{\partial \sigma^c} - \frac{\partial e^i_c}{\partial \sigma^b} \right).$$

To a transformation $\mathcal{R} \to \bar{\mathcal{R}}$ (1.7), (1.9) w^a_{bc}, v^a_{bc} obey the transformation laws:

(1.14) $$\begin{aligned} \bar{w}^a_{b_1 c_1} c^{b_1}_b c^{c_1}_c &= w^{a_1}_{bc} c^a_{a_1} - \left(\frac{\delta c^a_b}{\delta s^c} - \frac{\delta c^a_c}{\delta s^b} \right), \\ \bar{v}^a_{b_1 c_1} c^{b_1}_b c^{c_1}_c &= v^{a_1}_{bc} c^a_{a_1} - \left(\frac{\partial c^a_b}{\partial \sigma^c} - \frac{\partial c^a_c}{\partial \sigma^b} \right); \end{aligned}$$

That means that w^a_{bc} and v^a_{bc} are not d-tensor fields on M.

§ 2. The motion equations of the pseudo-orthogonal frames $\mathcal{R}$. Let us consider $L\Gamma(\overset{\circ}{N}) = (L^i_{jk}, C^i_{jk})$, the Lagrange connection which is metrical with respect to $g_{ij}(x, y)$, having h- and v- given torsions, expressed in the Theorem 22, I:

(2.1) $$\begin{aligned} L^i_{jk} &= \overset{\circ}{L}{}^i_{jk} + \frac{1}{2} g^{ir} (g_{rh} T^h{}_{jk} - g_{jh} T^h{}_{rk} + g_{kh} T^h{}_{jr}), \\ C^i_{jk} &= \overset{\circ}{C}{}^i_{jk} + \frac{1}{2} g^{ir} (g_{rh} S^h{}_{jk} - g_{jh} S^h{}_{rk} + g_{kh} S^h{}_{jk}), \end{aligned}$$

where $C\overset{\circ}{\Gamma}(\overset{\circ}{N})$ is the $\overset{\circ}{N}$-canonical connection (cf. Theorem 2.1, I).

The h- and v- covariant derivatives of the vector fields $e^i_a (a = 1, \dots, n)$ of the frames $\mathcal{R}$, with respect to $L\Gamma(\overset{\circ}{N})$, are d-tensor fields of type (1,1)

They are represented in $\mathcal{R}$ by their invariant components:

$$c^i_{a|k}=\overset{*}{L}{}^c_{ab}e^b_k e^i_c,\quad c^i_a|_k=\overset{*}{C}{}^c_{ab}e^b_k e^i_c. \tag{2.2}$$

The last equations give *the motion equations* of the pseudo-orthogonal frames $\mathcal{R}$ on the coordinate neighbourhood $\pi^{-1}(U)$ in TM. For the dual frames $\mathcal{R}^*$ the motion equations are:

$$c^a_{i|k}=-\overset{*}{L}{}^a_{bc}e^b_i e^c_k,\quad c^a_i|_k=-\overset{*}{C}{}^a_{bc}e^b_i e^c_k. \tag{2.2}'$$

Then $\overset{*}{L}{}^a_{bc}(x,y)$, $\overset{*}{C}{}^a_{bc}(x,y)$ will be called *the invariant components* of d-connection $L\Gamma(\overset{\circ}{N})$, (2.1) and $L\Gamma^*=(\overset{*}{L}{}^a_{bc},\overset{*}{C}{}^a_{bc})$ will be called the invariant form of the connection $L\Gamma(\overset{\circ}{N})$.

From (2.2) we get:

$$\overset{*}{L}{}^a_{bc}=\left(\frac{\delta e^i_b}{\delta x^k}+L^i_{jk}e^j_b\right)e^k_c c^a_i,\quad \overset{*}{C}{}^a_{bc}=\left(\frac{\partial e^i_b}{\partial y^k}+C^i_{jk}e^j_b\right)e^k_c c^a_i. \tag{2.3}$$

Proposition 2.1. *With respect to a transformation $\mathcal{R}\to\bar{\mathcal{R}}$ given by (1.7) the invariant components $\overset{*}{L}{}^a_{bc}$, $\overset{*}{C}{}^a_{bc}$ of the d-conection $L\Gamma(\overset{\circ}{N})$ obey the transformation law:*

$$\begin{aligned}\overset{*}{\bar L}{}^a_{b_1c_1}c^{b_1}_b c^{c_1}_c&=c^a_{a_1}\overset{*}{L}{}^{a_1}_{bc}-\frac{\delta c^a_b}{\delta s^c},\\ \overset{*}{\bar C}{}^a_{b_1c_1}c^{b_1}_b c^{c_1}_c&=c^a_{a_1}\overset{*}{C}{}^{a_1}_{bc}-\frac{\partial c^a_b}{\partial\sigma^c}.\end{aligned} \tag{2.4}$$

Proposition 2.2. $\overset{*}{T}{}^a_{bc}$ *and* $\overset{*}{S}{}^a_{bc}$, *given by*

$$\overset{*}{T}{}^a_{bc}=\overset{*}{L}{}^a_{bc}-\overset{*}{L}{}^a_{cb}-w^a_{bc},\quad \overset{*}{S}{}^a_{bc}=\overset{*}{C}{}^a_{bc}-\overset{*}{C}{}^a_{cb}-v^a_{bc} \tag{2.5}$$

are d-invariant tensor fields.

Indeed, (1.14) and (2.4) show that $\overset{*}{T}{}^a_{bc}$ and $\overset{*}{S}{}^a_{bc}$ have a transformation law of the tensorial type. Evidently, $\overset{*}{T}{}^a_{bc}$, $\overset{*}{S}{}^a_{bc}$ are anti-symmetric in the inferior indices. They are called h- and respectively v- invariant torsion tensor fields of the $L\Gamma^*$.

We define *the invariant curvature* tensor fields of $L\Gamma(N)$ by:

$$\begin{aligned}\overset{*}{R}{}_b{}^a{}_{cd}&=\frac{\delta\overset{*}{L}{}^a_{bc}}{\delta s^d}-\frac{\delta\overset{*}{L}{}^a_{bd}}{\delta s^c}+\overset{*}{L}{}^f_{bc}\overset{*}{L}{}^a_{fd}-\overset{*}{L}{}^f_{bd}\overset{*}{L}{}^a_{fc}-\overset{*}{L}{}^a_{bf}w^f_{cd}+\overset{*}{C}{}^a_{bf}R^f{}_{cd},\\ \overset{*}{P}{}_b{}^a{}_{cd}&=\frac{\partial\overset{*}{L}{}^a_{bc}}{\partial\sigma^d}-\frac{\delta\overset{*}{C}{}^a_{bd}}{\delta s^c}+\overset{*}{L}{}^f_{bc}\overset{*}{C}{}^a_{fd}-\overset{*}{C}{}^f_{bd}\overset{*}{L}{}^a_{fc}-\overset{*}{L}{}^a_{bf}\overset{*}{C}{}^f_{cd}+\overset{*}{C}{}^a_{bf}\overset{*}{L}{}^f_{dc}+\overset{*}{C}{}^a_{bf}P^f{}_{cd}+\overset{*}{L}{}^a_{bf}C^f_{cd},\\ \overset{*}{S}{}_b{}^a{}_{cd}&=\frac{\partial\overset{*}{C}{}^a_{bc}}{\partial\sigma^d}-\frac{\partial\overset{*}{C}{}^a_{bd}}{\partial\sigma^c}+\overset{*}{C}{}^f_{bc}\overset{*}{C}{}^a_{fd}-\overset{*}{C}{}^f_{bd}\overset{*}{C}{}^a_{fc}-\overset{*}{C}{}^a_{bf}v^f_{cd},\end{aligned} \tag{2.6}$$

in which $R^a{}_{bc}$, $P^a{}_{bc}$ and C^a_{bc} are the invariant components of the d-tensor fields $R^i{}_{jk}$, $P^i{}_{jk}$ and C^i_{jk}, torsions of the d-connection $L\Gamma(\overset{\circ}{N})$.

The tensorial character of $\overset{*}{R}{}_b{}^a{}_{cd}$, $\overset{*}{P}{}_b{}^a{}_{cd}$ and $\overset{*}{S}{}_b{}^a{}_{cd}$ can be proved by a straightforward computation. In the following paragraph we will prove this character in another way.

Putting $T^a{}_{bc}$ and $S^a{}_{bc}$ as the invariant components of the torsions $T^i{}_{jk}$, $S^i{}_{jk}$ of $L\Gamma(\overset{\circ}{N})$, we have a first important result:

Theorem 2.1. *Between the invariant torsion tensors $\overset{*}{T}{}^a{}_{bc}$, $\overset{*}{S}{}^a{}_{bc}$ of the invariant form $L\Gamma^*$ and the torsion tensors $T^i{}_{jk}$, $S^i{}_{jk}$ of the d-connection $L\Gamma(\overset{\circ}{N})$ there exists the relations:*

$$\overset{*}{T}{}^a{}_{bc}=T^a{}_{bc},\quad \overset{*}{S}{}^a{}_{bc}=S^a{}_{bc}. \tag{2.7}$$

Indeed, (1.13), (2.3) and (2.5) have as consequence (2.7).

Let us denote by $R_b{}^a{}_{cd}$, $P_b{}^a{}_{cd}$ and $S_b{}^a{}_{cd}$ the invariant components of the curvature tensor fields $R_j{}^i{}_{kh}$, $P_j{}^i{}_{kh}$ and $S_j{}^i{}_{kh}$ of the connection $L\Gamma(\overset{\circ}{N})$. In the next paragraph we prove similar equalities to (2.7) for the curvatures.

§ 3. The algorithm of h- and v-invariant derivatives.

The invariant form $L\Gamma^*=(\overset{*}{L}{}^a_{bc},\overset{*}{C}{}^a_{bc})$ of the d-connection $L\Gamma(\overset{\circ}{N})$ allows to define h- and v-invariant derivatives for the tensor fields given by their invariant components in the pseudo-orthogonal frames $\mathcal{R}$. For example, if $K^a_b(x,y)$ is a tensor field, we introduce its h- and v-invariant derivatives, respectively, by:

$$\begin{aligned}K^a_{b,\,c}&=\frac{\delta K^a_b}{\delta s^c}+\overset{*}{L}{}^a_{dc}K^d_b-\overset{*}{L}{}^d_{bc}K^a_d,\\ K^a_{b;\,c}&=\frac{\partial K^a_b}{\partial\sigma^c}+\overset{*}{C}{}^a_{dc}K^d_b-\overset{*}{C}{}^d_{bc}K^a_d.\end{aligned} \tag{3.1}$$

But this rule is general, for an arbitrary tensor field.

This definition is satisfactory, because we have:

Proposition 3.1. $K^a_{b,\,c}$ *and* $K^a_{b;\,c}$ *are the invariant components of the covariant derivatives* $K^i_{j|k}$ *and* $K^i_j|_k$:

$$K^i_{j|k}=K^a_{b,\,c}e^i_a c^b_j c^c_k,\quad K^i_j|_k=K^a_{b;\,c}e^i_a c^b_j c^c_k. \tag{3.2}$$

Indeed, from $K^i_j=K^a_b e^i_a c^b_j$, applying the h- and v- covariant derivatives and taking into account (2.2), (2.2)' and (3.1) we get (3.2).

Proposition 3.2. *The invariant components ε_{ab} of the fundamental tensor $g_{ij}(x,y)$ of the generalized Lagrange space M^n have the properties:*

$$\varepsilon_{ab,\,c}=0,\quad \varepsilon_{ab;\,c}=0. \tag{3.3}$$

Corollary 3.1. *The invariant components $\overset{*}{L}{}^a_{bc}$, $\overset{*}{C}{}^a_{bc}$ of the d-connection $L\Gamma(\overset{\circ}{N})$ verify the equations:*

$$\varepsilon_{ad}\overset{*}{L}{}^d_{bc}+\varepsilon_{bd}\overset{*}{L}{}^d_{ac}=0,\quad \varepsilon_{ad}\overset{*}{C}{}^d_{bc}+\varepsilon_{bd}\overset{*}{C}{}^d_{ac}=0. \tag{3.4}$$

The equations (3.4) show that some $\overset{*}{L}{}^a_{bc}$, $\overset{*}{C}{}^a_{bc}$ vanish.
In the following, we consider the brackets :

$$\left[\frac{\delta}{\delta s^d}, \frac{\delta}{\delta s^c}\right] = \frac{\delta}{\delta s^d}\frac{\delta}{\delta s^c} - \frac{\delta}{\delta s^c}\frac{\delta}{\delta s^d}, \text{ etc.}$$

Proposition 3.3. *These brackets satisfy*

$$\left[\frac{\delta}{\delta s^d}, \frac{\delta}{\delta s^c}\right] = w^a_{cd}\frac{\delta}{\delta s^a} + R^a{}_{dc}\frac{\partial}{\partial \sigma^a},$$

$$(3.5)\qquad \left[\frac{\partial}{\partial \sigma^d}, \frac{\delta}{\delta s^c}\right] = (\overset{*}{C}{}^a_{cd} - C^a_{cd})\frac{\delta}{\delta s^a} - (\overset{*}{L}{}^a_{dc} + P^a{}_{cd})\frac{\partial}{\partial \sigma^a}, \quad \left[\frac{\partial}{\partial \sigma^d}, \frac{\partial}{\partial \sigma^c}\right] = v^a_{cd}\frac{\partial}{\partial \sigma^a}.$$

The proof is a straightforward computation. Also, we have :

Theorem 3.1. *The Ricci formulas or the h- and v- invariant derivatives are given by* :

$$K^a_{b,c,d} - K^a_{b,d,c} = \overset{*}{R}{}^a_{fcd}K^f_b - \overset{*}{R}{}_b{}^f{}_{cd}K^a{}_f - R^f{}_{cd}K^a_{b,f} - \overset{*}{T}{}^f{}_{cd}K^a_{b,f},$$

$$(3.6)\qquad K^a_{b,c;d} - K^a_{b;d,c} = \overset{*}{P}{}_f{}^a{}_{cd}K^f_b - \overset{*}{P}{}_b{}^f{}_{cd}K^a_f - C^f_{cd}K^a_{b,f} - P^f{}_{cd}K^a_{b;f},$$

$$K^a_{b;c;d} - K^a_{b;d;c} = \overset{*}{S}{}_f{}^a{}_{cd}K^f_b - \overset{*}{S}{}_b{}^f{}_{cd}K^a_f - S^f{}_{cd}K^a_{b;f}.$$

These formulas are written for a tensor field $K^a_b(x, y)$, but they have a general character.

Theorem 3.2. *The invariant curvature tensor fields have the properties* :

$$(3.7)\qquad \varepsilon_{af}\overset{*}{R}{}_b{}^f{}_{cd} + \varepsilon_{fb}\overset{*}{R}{}_a{}^f{}_{cd} = 0,$$

$$\varepsilon_{af}\overset{*}{P}{}_b{}^f{}_{cd} + \varepsilon_{fb}\overset{*}{P}{}_a{}^f{}_{cd} = 0, \qquad \varepsilon_{af}\overset{*}{S}{}_b{}^f{}_{cd} + \varepsilon_{fb}\overset{*}{S}{}_a{}^f{}_{cd} = 0.$$

In this theory a very important result is the following :

Theorem 3.3. *Between the tensors* $\overset{*}{R}{}_b{}^a{}_{cd}$, $\overset{*}{P}{}_b{}^a{}_{cd}$, $\overset{*}{S}{}_b{}^a{}_{cd}$ *and* $R_b{}^a{}_{cd}$, $P_b{}^a{}_{cd}$, $S_b{}^a{}_{cd}$ *the following the relations hold* :

$$(3.8)\qquad \overset{*}{R}{}_b{}^a{}_{cd} = R_b{}^a{}_{cd}, \quad \overset{*}{P}{}_b{}^a{}_{cd} = P_b{}^a{}_{cd}, \quad \overset{*}{S}{}_b{}^a{}_{cd} = S_b{}^a{}_{cd}.$$

Proof. By the first integrability conditions of the system (2.2) we have

$$e^r_a R_r{}^i{}_{kh} - T^r{}_{kh}e^i_{a|r} - R^r{}_{kh}e^i_a|_r = \{\overset{*}{R}{}_a{}^f{}_{bd} - \overset{*}{C}{}^f_{ag}R^g{}_{cd} - \overset{*}{L}{}^f{}_{ag}\overset{*}{T}{}^g{}_{bd}\}e^i{}_f e^b_h e^d_h,$$

or

$$e^f_i e^k_b e^h_d e^r_a R_r{}^i{}_{kh} - \overset{*}{L}{}^f_{ag}T^g{}_{bd} - \overset{*}{C}{}^f_{ag}R^g{}_{bd} = \overset{*}{R}{}_a{}^f{}_{bd} - \overset{*}{C}{}^f_{ag}R^g{}_{bd} - \overset{*}{L}{}^f_{ag}\overset{*}{T}{}^g{}_{bd}.$$

Using (2.7) and the invariant components of the curvature tensor $R_f{}^i{}_{kh}$ we obtain the first relation (3.8), etc.

Theorem 3.4. *The integrability conditions of the motion equations (2.2) of the pseudo-orthogonal frames $\mathcal{R}$ are given by* (3.8) *and by the h- and v-invariant derivatives of any order of* (3.8).

§ 4. Invariant form of Einstein's equations. It is our goal to write Einstein's equations in an invariant form, using their invariant components with respect to the pseudo-orthogonal frames $\mathcal{R}$.

Let $L\Gamma(\overset{\circ}{N})$ be the d-connection (2.1) and $L\Gamma^* = (\overset{*}{L}{}^a_{bc}, \overset{*}{C}{}^a_{bc})$ its invariant form with respect to $\mathcal{R}$. From the theorems 2.1 and 3.3 it follows :

$$(4.1)\qquad \overset{*}{T}{}^a{}_{bc} = T^a{}_{bc}, \quad \overset{*}{S}{}^a{}_{bc} = S^a{}_{bc},$$

$$\overset{*}{R}{}_b{}^a{}_{cd} = R_b{}^a{}_{cd}, \quad \overset{*}{P}{}_b{}^a{}_{cd} = P_b{}^a{}_{cd}, \quad \overset{*}{S}{}_b{}^a{}_{cd} = S_b{}^a{}_{cd}.$$

In the frames $\mathcal{R}$ the Ricci tensors R_{ij}, $\overset{1}{P}_{ij}$, $\overset{2}{P}_{ij}$ and S_{ij} are given by their invariant components :

$$(4.2)\qquad R_{bc} = R_b{}^a{}_{ca}, \quad \overset{1}{P}_{bc} = P_b{}^a{}_{ca}, \quad \overset{2}{P}_{bc} = P_b{}^a{}_{ac}, \quad S_{bc} = S_b{}^a{}_{ca}$$

and the scalar curvatures R and S :

$$(4.3)\qquad R = \varepsilon^{ab}R_{ab}, \quad S = \varepsilon^{ab}S_{ab}.$$

We put

$$\overset{*}{R}_{bc} = \overset{*}{R}{}_b{}^a{}_{ca}, \quad \overset{*}{\overset{1}{P}}_{bc} = \overset{*}{P}{}_b{}^a{}_{ca}, \quad \overset{*}{\overset{2}{P}}_{bc} = \overset{*}{P}{}_b{}^a{}_{ac}, \quad \overset{*}{S}_{bc} = \overset{*}{S}{}_b{}^a{}_{ca}.$$

Proposition 4.1. *The invariant components of the Ricci tensors have the properties* :

$$(4.4)\qquad \overset{*}{R}_{bc} = R_{bc}, \quad \overset{*}{\overset{1}{P}}_{bc} = \overset{1}{P}_{bc}, \quad \overset{*}{\overset{2}{P}}_{bc} = \overset{2}{P}_{bc}, \quad \overset{*}{S}_{bc} = S_{bc},$$

$$\overset{*}{R} = R = \varepsilon^{ab}\overset{*}{R}_{ab}, \quad \overset{*}{S} = S = \varepsilon^{ab}\overset{*}{S}_{ab}.$$

Theorem 4.1. *Einstein's equations, with respect to the d-connection (2.1) have in the pseudo-orthogonal frames $\mathcal{R}$ the invariant form* :

$$\overset{*}{R}_{ab} - \frac{1}{2}(\overset{*}{R} + \overset{*}{S})\varepsilon_{ab} = \varkappa \overset{1}{T}_{ab}, \quad \overset{*}{\overset{1}{P}}_{ab} = \varkappa \overset{3}{T}_{ab},$$

$$(4.5)\qquad \overset{*}{S}_{ab} - \frac{1}{2}(\overset{*}{S} + \overset{*}{R})\varepsilon_{ab} = \varkappa \overset{2}{T}_{ab}, \quad \overset{*}{\overset{2}{P}}_{ab} = -\varkappa \overset{4}{T}_{ab},$$

where $\overset{1}{T}_{ab}, \ldots$ *are the invariant form of the components of energy-momentum tensor fields* $T_{ij}, \ldots$, *respectively*.

We say that (4.5) is *the canonical form of Einstein's equations*, with respect to the d-connection $L\Gamma(\overset{\circ}{N})$, (2.1).

Corollary 4.1. *The Einstein equations for the vacuum have the invariant form* :

$$(4.6)\qquad \overset{*}{R}_{ab} = 0, \quad \overset{*}{S}_{ab} = 0, \quad \overset{*}{\overset{1}{P}}_{ab} = 0, \quad \overset{*}{\overset{2}{P}}_{ab} = 0.$$

We are interested in the invariant form of the conservation law. To this purpose we remark:

$$\dot{R}^{a}{}_{b}=\varepsilon^{ac}\dot{R}_{cb},\ \dot{S}^{a}{}_{b}=\varepsilon^{ac}\dot{S}_{cb},\ \overset{1}{\dot{P}}{}^{a}{}_{b}=\varepsilon^{ac}\overset{1}{\dot{P}}_{cb},\ \overset{2}{\dot{P}}{}^{a}{}_{b}=\varepsilon^{ac}\overset{2}{\dot{P}}_{cb}.$$

Using the Theorems 5.2, II and 4.1 we get:

Theorem 4.2. *The invariant form of the conservation law is:*

$$\left[\dot{R}^{a}{}_{b}-\frac{1}{2}(\dot{R}+\dot{S})\delta^{a}_{b}\right]_{,a}+\overset{1}{\dot{P}}{}^{a}{}_{b;a}=0,\quad \left[\dot{S}^{a}{}_{b}-\frac{1}{2}(\dot{R}+\dot{S})\delta^{a}_{b}\right]_{;a}-\overset{2}{\dot{P}}{}^{a}{}_{b,a}=0.$$

Theorem 4.3. *The divergence of the energy, momentum has the invariant form* $\overset{1}{T}{}^{a}_{b,a}+\overset{3}{T}{}^{a}_{b;a}=0,\ \overset{4}{T}{}^{a}_{b,a}+\overset{2}{T}{}^{a}_{b;a}=0.$

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Received 10 IV 1985

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXII, s. I a, Matematică, 1986, f. 3

VECTOR BUNDLES. EINSTEIN EQUATIONS

BY

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In the last years a Finslerian theory of relativity was built from various standpoints [2], [4], [7]. Recently, R. Miron has completed a more generalized version of this theory, which was called a Lagrangian theory of relativity in [6]. Some physical aspects of this theory were considered by S. Ikeda in [3]. His considerations show that the geometry of the total space of a vector bundle is useful from a physical viewpoint.

In this paper the Einstein equations and the conservation law on the total space of a vector bundle are written. If the vector bundle is just the tangent bundle to the base manifold we recover the Einstein equations established by R. Miron in [6] as well as a new kind of Einstein's equations whose physical meaning remains to be found. If the vector bundle has 1-dimensional fibres, we obtain a geometrical framework for an unitary projective theory.

The author is indebted to Prof. Radu Miron who suggested him the subject of this work.

1. Vector bundles. Let $\xi = (E, p, M)$, $p: E \to M$, be a vector bundle of paracompact base M and finite dimensional type fibre **F**. We set $n = \dim M$ and $m = \dim$ **F**. Let us denote by (x^i, y^a) the local coordinates on $p^{-1}(U) \subset E$, where $U \subset M$. In what follows we use $i, j, k, h\ldots = 1, 2, \ldots, n$ and $a, b, c, \ldots = 1, 2, \ldots, m$.

The law of transformation of the local coordinates is the following:

$$x^{i'} = x^{i'}(x^1, \ldots, x^n), \quad y^{a'} = S^{a'}_a(x^1, \ldots x^n)\, y^a. \tag{1.1}$$

If the vector bundle is endowed with a nonlinear connection, then, for every $u \in E$, we have $T_uE = H_uE \oplus V_uE$, where V_uE is the vertical part and H_uE is the horizontal part. A basis of T_uE adapted to this decompositin is (δ_i, ∂_a), where $\delta_i = \partial_i - N^a_i(x, y)\, \partial_a$. Here $(N^a_i(x, y))$ are the local coefficients of the nonlinear connection and ∂_i and ∂_a stand for $\partial/\partial x^i$ and $\partial/\partial y^a$, respectively. The basis dual to it is $(dx^i, \delta y^a)$, where $\delta y^a = dx^a + N^a_i dx^i$.

Definition 1.1. *A linear connection D on the manifold E is said to be a d-connection if it preserves by parallel displacement the horizontal distribution $u \to H_uE$ and vertical distribution $u \to V_uE$.*

If we set:

$$\begin{cases} D_{\delta_k}\delta_j = F^i_{jk}(x, y)\, \delta_i, & D_{\delta_k}\partial_b = L^a_{bk}(x, y)\, \partial_a, \\ D_{\partial_a}\delta_j = M^i_{ja}(x, y)\, \delta_i, & D_{\partial_c}\partial_b = C^a_{bc}(x, y)\partial_a, \end{cases} \tag{1.2}$$