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VECTOR BUNDLES. EINSTEIN EQUATIONS

BY

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In the last years a Finslerian theory of relativity was built from various standpoints [2], [4], [7]. Recently, R. M i r o n has completed a more generalized version of this theory, which was called a Lagrangian theory of relativity in [6]. Some physical aspects of this theory were considered by S. I k e d a in [3]. His considerations show that the geometry of the total space of a vector bundle is useful from a physical viewpoint.

In this paper the Einstein equations and the conservation law on the total space of a vector bundle are written. If the vector bundle is just the tangent bundle to the base manifold we recover the Einstein equations established by R. M i r o n in [6] as well as a new kind of Einstein's equations whose physical meaning remains to be found. If the vector bundle has 1-dimensional fibres, we obtain a geometrical framework for an unitary projective theory.

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1. Vector bundles. Let $\xi=(E, p, M)$, $p: E\to M$, be a vector bundle of paracompact base M and finite dimensional type fibre **F**. We set $n=\dim M$ and $m=\dim \mathbf{F}$. Let us denote by (x^i, y^a) the local coordinates on $p^{-1}(U)\subset E$, where $U\subset M$. In what follows we use $i, j, k, h \ldots = 1, 2, \ldots, n$ and $a, b, c, \ldots = 1, 2, \ldots, m$.

The law of transformation of the local coordinates is the following:

$$x^{i'}=x^{i'}(x^1, \ldots, x^n), \quad y^{a'}=S^{a'}_a(x^1, \ldots x^n)\, y^a. \tag{1.1}$$

If the vector bundle is endowed with a nonlinear connection, then, for every $u\in E$, we have $T_uE=H_uE\oplus V_uE$, where V_uE is the vertical part and H_uE is the horizontal part. A basis of T_uE adapted to this decompositin is (δ_i, ∂_a), where $\delta_i=\partial_i-N^a_i(x, y)\,\partial_a$. Here $(N^a_i(x, y))$ are the local coefficients of the nonlinear connection and ∂_i and ∂_a stand for $\partial/\partial x^i$ and $\partial/\partial y^a$, respectively. The basis dual to it is $(dx^i, \delta y^a)$, where $\delta y^a=dx^a+N^a_i dx^i$.

Definition 1.1. *A linear connection D on the manifold E is said to be a d-connection if it preserves by parallel displacement the horizontal distribution $u\to H_uE$ and vertical distribution $u\to V_uE$.*

If we set:

$$\begin{cases} D_{\delta_k}\delta_j=F^i_{jk}(x, y)\,\delta_i, & D_{\delta_k}\partial_b=L^a_{bk}(x, y)\,\partial_a, \\ D_{\partial_a}\delta_j=M^i_{ja}(x, y)\,\delta_i, & D_{\partial_c}\partial_b=C^a_{bc}(x, y)\,\partial_a, \end{cases} \tag{1.2}$$

then $F^i_{jk}(x, y)$ and $L^a_{bk}(x, y)$ change like the local coefficients of a connection on M, respectively on ξ, and $M^i_{ja}(x, y)$, $C^a_{bc}(x, y)$ are tensor fields on E. A d-connection is completely determined by $F\Gamma=(F^i_{jk}, L^a_{bk}, M^i_{ja}, C^a_{bc})$. (See also [5]).

There exist d-connections on E. For instance, if $F^i_{jk}(x)$ are the local coefficients of a linear connection on M (there exists such a connection because M is paracompact), then $(F^i_{jk}(x), \partial_b N^a_j, 0, 0)$ is a d-connection on E.

A pair of linear connections on M and ξ defines a d-connection on E. Indeed, if $L^a_{bk}(x)$ are the local coefficients of a linear connection on ξ, then $N^a_i(x, y)=L^a_{bk}(x)\, y^b$ are the local coefficients of a nonlinear connection on ξ and $(F^i_{jk}(x), L^a_{bk}(x), 0, 0)$ is a d-connection.

A d-connection $F\Gamma$ is called a Berwald connection if

$$L^a_{bk}=\partial_b N^a_k(x, y),\quad M^i_{ja}(x, y)=0.$$

The d-connections showed above are Berwald connections.

We shall denote by , and | the h- and v-covariant derivative, respectively, associated to the d-connection D.

The Ricci identities introduce five torsions :

$$(1.3)\qquad \begin{cases} T^i_{jk}=F^i_{jk}-F^i_{kj},\quad R^a_{jk}=\delta_k N^a_j-\delta_j N^a_k,\quad \overset{1}{P}{}^a_{jb}=\partial_b N^a_j-L^a_{bj}, \\ \overset{2}{P}{}^i_{jb}=M^i_{jb},\quad S^a_{bc}=C^a_{bc}-C^a_{cb}, \end{cases}$$

and six curvatures :

$$(1.4)\qquad \begin{cases} R^i_{jkh}=\delta_h F^i_{jk}+F^l_{jk}F^i_{lh}-k \mid h+M^i_{ja}R^a_{kh}, \\ R^a_{bkh}=\delta_h L^a_{bk}+L^c_{bk}L^a_{ch}-k \mid h+C^a_{be}R^e_{kh}, \\ \overset{1}{P}{}^a_{bkc}=\partial_c L^a_{bk}-C^a_{bc|k}+C^a_{bd}\, \overset{1}{P}{}^d_{kc}, \\ \overset{2}{P}{}^i_{jkc}=\partial_c F^i_{jk}-M^i_{jc|k}+M^i_{jb}\, \overset{1}{P}{}^b_{kc}, \\ M^i_{jbc}=\partial_c M^i_{jb}+M^h_{jb}M^i_{hc}-b \mid c, \\ S^a_{bcd}=\partial_d C^a_{bc}+C^e_{bc}C^a_{ed}-c \mid d, \end{cases}$$

for a d-connection $F\Gamma$. Here and in the following $-k \mid h$ means the substraction of the previous terms after having changed the indices one to another one.

2. Metrical structures on E. Metrical d-connections. A metrical structure on E is a tensor field G on E of type $(0, 2)$, symmetric and nondegenerate. If such a metrical structure G is given, then there exists a canonical nonlinear connection on ξ defined by the orthogonal distribution to the vertical distribution with respect to G. In what follows we shall refer only to this nonlinear connection. It is obvious that, with respect to the adapted frame to this nonlinear connection, G can be written as follows :

$$(2.1)\qquad G=g_{ij}(x, y)\, dx^i\otimes dx^j+h_{ab}(x, y)\, \delta y^a\otimes\delta y^b$$

Definition 2.1. *A d-connection on E is said to be metrical if*

$$(2.2)\qquad g_{ij|k}=0,\quad g_{ij}|_a=0,\quad h_{ab|k}=0,\quad h_{ab}|_c=0,$$

hold.

There exist metrical d-connections. Indeed, if $\overset{\circ}{F\Gamma}=(\overset{\circ}{F}{}^i_{jk}, \overset{\circ}{L}{}^a_{bk}, \overset{\circ}{M}{}^i_{ja}, \overset{\circ}{C}{}^a_{bc})$ is any d-connection on E, then the d-connection whose local coefficients are given below is metrical.

$$(2.3)\qquad \begin{cases} F^i_{jk}=\overset{\circ}{F}{}^i_{jk}+\dfrac{1}{2}g^{ih}g_{hk\overset{\circ}{|}j} \\ L^a_{bj}=\overset{\circ}{L}{}^a_{bj}+\dfrac{1}{2}h^{ac}hg_{cb\overset{\circ}{|}a} \\ M^i_{jb}=\overset{\circ}{M}{}^i_{jb}+\dfrac{1}{2}g^{ih}g_{hi}\overset{\circ}{|}_b \\ C^a_{bc}=\overset{\circ}{C}{}^a_{bc}+\dfrac{1}{2}h^{ad}h_{db}\overset{\circ}{|}_c \end{cases}$$

where $\overset{\circ}{;}$ and $\overset{\circ}{|}$ denotes the h- and v-covariant derivative, respectively, associted to $\overset{\circ}{F\Gamma}$.

The formulas (2.3) can be thought of as a process of metrization of any d-connection. This process will be called Kawaguchi metrization.

We say that a d-connection is h-v-metrical with respect to G given by (2.1), if $g_{ij|k}=0$ and $h_{ab}|_c=0$. We remark that there exist h-v-metrical connections which are not metrical. Indeed, it is easy to check that the following Berwald connection

$$(2.4)\qquad \begin{cases} \tilde{F}^i_{jk}=\dfrac{1}{2}g^{ih}(\delta_k g_{hj}+\delta_j g_{hk}-\delta_h g_{jk}) \\ \tilde{L}^a_{bj}=\partial_b N^a_j \\ \tilde{M}^i_{jb}=0 \\ \tilde{C}^a_{bc}=\dfrac{1}{2}h^{ad}(\partial_b h_{dc}+\partial_c h_{db}-\partial_d h_{bc}) \end{cases}$$

is h-v-metrical but it is not metrical.

Theorem 2.1. *If two skew-symmetrical tensor fields T^i_{jk} and S^a_{bc} are given, then there exists an unique Berwald connection which is h-v metric and has $h(hh)$- and $v(vv)$-torsions the tensor fields T^i_{jk} and S^a_{bc}, respectively. Its local coefficients are as follows :*

$$(2.5)\qquad \begin{cases} \hat{F}^i_{jk}=\tilde{F}^i_{jk}+\dfrac{1}{2}g^{ih}(g_{hr}T^r_{jk}-g_{jr}T^r_{hk}+g_{kr}T^r_{jh}) \\ \hat{L}^a_{bk}=\partial_b N^a_k \\ \hat{M}^i_{ja}=0 \\ \hat{C}^a_{bc}=\tilde{C}^a_{bc}+\dfrac{1}{2}h^{ad}(h_{de}S^e_{bc}-h_{be}S^e_{dc}+h_{ce}S^e_{bd}). \end{cases}$$

Proof. All Berwald connections have the form $(F^i_{jk}+\tau^i_{jk},\ \partial_b N^a_k,\ 0,\ C^a_{bc}+\tilde{\tau}^a_{bc})$, where τ^i_{jk} and $\tilde{\tau}^a_{bc}$ are arbitrary tensor fields. Imposing that such a connection be h-v metrical and its $h(hh)$-and $v(vv)$ — torsions to be just T^i_{jk} and S^a_{bc}, respectively, one obtains that τ^i_{jk} and $\tilde{\tau}^a_{bc}$ are uniquely determined and they have the expressions from (2.5), q.e.d.

Theorem 2.2. *There exists an unique metrical d-connection with $h(hh)$- and $v(vv)$-torsions T^i_{jk} and S^a_{bc} prescribed obtained by Kawaguchi metrization of a h-v metrical Berwald connection. Its local coefficients are as follows:*

$$
(2.6)\qquad \begin{cases} F^i_{jk}=\hat{F}^i_{jk} \\ L^a_{bk}=\partial_b N^a_k+\dfrac{1}{2}h^{ac}[\delta_k g_{bc}-(\partial_b N^d_k)h_{dc}-(\partial_c N^d_k)h_{db}] \\ M^i_{jb}=\dfrac{1}{2}g^{ik}\partial_b g_{jk} \\ C^a_{bc}=\hat{C}^a_{bc}. \end{cases}
$$

Proof. By the Kawaguchi metrization of the unique Berwald h-v metrical connection given by (2.5) one obtains (2.6), q.e.d.

3. Einstein equations on E. Let E be the total space of the vector bundle (E, p, M). Suppose that E is furnished with a metrical structure G and denote by D the metrical d-connection having $h(hh)$-and $v(vv)$-torsions prescribed, given locally by (2.6).

We associate to D the following Einstein equation

$$
(3.1)\qquad \text{Ric }(D)-(1/2)\,\mathbf{R}G=\varkappa\,\mathbf{T},
$$

where Ric (D) and $\mathbf{R}$ denote the Ricci tensor and the scalar curvature of D, respectively, $\varkappa$ is a constant and $\mathbf{T}$ is a tensor field of type $(0,2)$ called the energy-momentum tensor.

Remark 3.1. The tensor field from the left hand of the eq. (3.1) is not symmetric nor free divergence since D has torsion.

To express (3.1) by using the curvature of the d-connection D, let us put $X_\alpha=\{\delta_i, \partial_a\}$.
Then we have:

$$
(3.2)\qquad D_{X_\gamma}X_\beta=\Gamma^\alpha_{\beta\gamma}X_\alpha,\quad \alpha, \beta, \gamma, \delta\ldots=1, 2, \ldots, n+m,
$$

$$
(3.3)\qquad \mathbf{T}^\alpha_{\beta\gamma}=\Gamma^\alpha_{\beta\gamma}-\Gamma^\alpha_{\gamma\beta}+W^\alpha_{\beta\gamma},\ \text{where } [X_\alpha, X_\beta]=W^\gamma_{\alpha\beta}X_\gamma,
$$

$$
(3.4)\qquad \mathbf{R}^\alpha_{\beta\gamma\delta}=X_\delta\Gamma^\alpha_{\beta\gamma}+\Gamma^\varphi_{\beta\gamma}\Gamma^\alpha_{\varphi\delta}-\gamma\,|\,\delta+\Gamma^\alpha_{\beta\varphi}W^\varphi_{\gamma\delta},
$$

$$
(3.5)\qquad Ric\ (D)=\mathbf{R}_{\beta\gamma}=\mathbf{R}^\alpha_{\beta\gamma\alpha},
$$

$$
(3.6)\qquad \mathbf{R}=G^{\alpha\beta}\mathbf{R}_{\alpha\beta},
$$

and the eq. (3.1) becomes:

$$
(3.7)\qquad \mathbf{R}_{\alpha\beta}-\frac{1}{2}\mathbf{R}G_{\alpha\beta}=\varkappa\mathbf{T}_{\alpha\beta}
$$

It results that it is equivalent to the following equations:

$$
(3.8)\qquad \begin{cases} R_{ij}-\dfrac{1}{2}(R+S)\,g_{ij}=\varkappa\mathbf{T}_{ij}, \quad \overset{1}{P}_{ai}=\varkappa\mathbf{T}_{ai}, \quad \overset{2}{P}_{ia}=-\varkappa\mathbf{T}_{ia} \\ S_{ab}-\dfrac{1}{2}(R+S)\,h_{ab}=\varkappa\mathbf{T}_{ab},\ \text{where: } \overset{2}{P}_{ia}=\overset{2}{P}{}^k_{ika}, \end{cases}
$$

$$
R_{ij}=R^h_{ijh},\quad \overset{1}{P}_{ai}=\overset{1}{P}{}^b_{aib},\quad S_{ab}=S^c_{abc},\quad R=g^{ij}R_{ij},\ S=h^{ab}S_{ab}.
$$

All tensor fields from (3.8) are distinguished tensor fields on E i.e. in their laws of transformations to a change of local coordinates y^a does not appear explicitely.

The conservation law $D_{X_\alpha}(\mathbf{R}^\alpha_\beta-(1/2)\,\mathbf{R}\delta^\alpha_\beta)=0$, where $\mathbf{R}^\alpha_\beta=G^{\alpha\gamma}\mathbf{R}_{\gamma\beta}$ can be written as follows:

$$
(3.9)\qquad \begin{cases} \left[R^i_j-\dfrac{1}{2}(R+S)\,\delta^i_j\right]_{|i}+\overset{1}{P}{}^a_j|_a=0, \\ \left[S^a_b-\dfrac{1}{2}(R+S)\,\delta^a_b\right]|_a-\overset{2}{P}{}^i_{b|i}=0, \end{cases}
$$

where $R^i_j=g^{ik}R_{kj}$,

$$
S^a_b=g^{ac}S_{cb},\quad \overset{1}{P}{}^a_j=g^{ab}\overset{1}{P}_{bj},\quad \overset{2}{P}{}^i_b=g^{ij}\overset{2}{P}_{jb}.
$$

Generally, the eqs. (3.9) are not identically satisfied; so it appears that the energy-momentum tensor is now conservatif.

Definition 3.1. *The eqs. (3.8) will be called the Einstein equations on the total space E of the vector bundle ξ.*

4. Some particular cases. a) Let us take $\xi=(TM, \tau, M)$, where M is a generalized Lagrange space i.e. $M=(M^n, g_{ij}(x, y))$ (cf. R. Miron [6]). If some additional conditions on $g_{ij}(x, y)$ are fullfiled (see R. Miron [6]) then $g_{ij}(x, y)$ determines an unique nonlinear connection on (TM, τ, M). Let $(N^i_j(x, y))$ be its local coefficients, $(i, j, k, \ldots=1, 2, \ldots n$, and let (δ_i, ∂_i) be the frame adapted to it. The following Riemannian metric on TM appears as natural:

$$
(4.1)\qquad G=g_{ij}(x, y)\,dx^i\otimes dx^j+g_{ij}(x, y)\,\delta y^i\otimes\delta y^j,\ \text{where}
$$

$$
\delta y^i=dy^i+N^i_j dx^j.
$$

As in the general case, a linear d-connection on TM is completely determined by a set of functions on TM, let say

$$
F\Gamma=(F^i_{jk},\ L^i_{jk},\ M^i_{jk},\ C^i_{jk}).
$$

Let J be the natural almost tangent structure on TM i.e.

$$
J(\delta_i)=\partial_i,\quad J(\partial_i)=0.
$$

Definition 4.1. *A linear d-connection D on TM is said to be normal if $DJ=0$.*

It is easy to see that a normal linear d-connection is characterized by $L^i_{jk}=F^i_{jk}$ and $M^i_{jk}=C^i_{jk}$, so a normal linear d-connection is completely determined by $F\Gamma=(F^i_{jk}, C^i_{jk})$, where F^i_{jk} and C^i_{jk} have the laws of transformation like a linear connection and a tensor on M, respectively, if the local coordinates are changed.

Theorem 4.1. *Given two skew-symmetrical tensor fields T^i_{jk} and S^i_{jk}, there exists an unique metrical normal linear d-connection on TM which has T^i_{jk} and S^i_{jk} as $h(hh)$-and $v(vv)$-torsions, respectively. Its local coefficients are as follows:*

$$(4.2)\qquad \begin{cases} F^i_{jk}=\dfrac{1}{2}g^{ih}(\delta_j g_{hk}+\delta_k g_{hj}-\delta_h g_{jk}+g_{hr}T^r_{jk}-g_{jr}T^r_{hk}+g_{kr}T^r_{jh}),\\[2ex] C^i_{jk}=\dfrac{1}{2}g^{ih}(\partial_j g_{hk}+\partial_k g_{hj}-\partial_h g_{jk}+g_{hr}S^r_{jk}-g_{jr}S^r_{hk}+g_{kr}S^r_{jh}). \end{cases}$$

Proof. Taking any linear d-connection $F\Gamma=(F^i_{jk}, C^i_{jk})$ and imposing the conditions $g_{ij|k}=0$, $g_{ij}|_k=0$, $F^i_{jk}-F^i_{kj}=T^i_{jk}$ and $C^i_{jk}-C^i_{kj}=S^i_{jk}$, one gets that F^i_{jk} and C^i_{jk} are uniquely determined as in (4.2). q.e.d.

Einstein equations associated to the metrical, normal, linear d-connection given by the Theorem 4.1 are just the Einstein equations obtained by R. M i r o n in [6].

b) Preserving the hypothesis from a) we only change the metric G as follows:

$$(4.3)\qquad G=g_{ij}(x, y)\, dx^i\otimes dx^j-g_{ij}(x, y)\, \delta y^i\otimes\delta y^j$$

This G is nondegenerate but it is nondefinite. However, the Theorem 4.1 is still true. The Einstein equations associated to the metrical, normal, linear d-connection stated by it, written with respect to the adapted frame (δ_i, ∂_i) are as follows:

$$(4.4)\qquad \begin{cases} R_{ij}-\dfrac{1}{2}(R-S)\, g_{ij}=\varkappa\mathbf{T}_{ij}, \qquad S_{ij}-\dfrac{1}{2}(R-S)\, g_{ij}=\varkappa\mathbf{T}_{(i)(j)}\\[2ex] \overset{1}{P}_{ij}=\varkappa\mathbf{T}_{(i)j}, \qquad \overset{2}{P}_{ij}=-\varkappa\mathbf{T}_{i(j)} \end{cases}$$

where $R_{ij}=R^k_{ijk}$, $R=g^{ij}R_{ij}$,

$S_{ij}=S^k_{ijk}$, $S=g^{ij}S_{ij}$ and in the right hand appear the components of the energy-momentum tensor with respect to the adapted frame.

Remark 4.1. The eqs. (4.4) could be also interesting for physicists because G is nondefinite. Its signature is allways (n, n).

Remark 4.2. Setting $P(\delta_i)=-\delta_i$, $P(\partial_i)=-\delta_i$ one obtains an almost product structure on TM which satisfies $G(PX, PY)=-G(X, Y)$ for any vector fields X and Y on TM and G given by (4.3). Therefore, (TM, P, G) is an almost hyperbolic manifold.

c) Now, let us take $\xi=(E, p, M)$ with dim $\mathbf{F}=1$. If (U, φ) is a local chart on M, let $(x^1, \ldots, x^n, x^0)$ be the local coordinates of a point $u\in p^{-1}(U)$. These coordinates change as follows (cf. (1.1)):

$$(4.5)\qquad x^{i'}=x^{i'}(x^1, \ldots x^n)\quad x'^0=f(x^1, \ldots, x^n)\cdot x^0,$$

where f is a real function locally defined on M, $f\neq 0$.

The formulas (4.5) show that the manifold E is the most general framework for an unitary projective theory (cf. [8], p. 233).

A nonlinear connection on ξ will be defined by a set of functions (N_i) on E such that $\delta_i=\partial_i-N_i\partial_0$ verify $\delta_{i'}=(\partial_{i'} x^i)\,\delta_i$, where $\partial_0=\partial/\partial x^0$ and $\delta_{i'}=\delta/\delta x^{i'}$.

A linear d-connection will be completely determined by the following set of functions on E, $F\Gamma=(F^i_{jk}, L_k, M^i_j, C)$ where $L_k=L^0_{0k}$, $M^i_j=M^i_{j0}$, $C=C^0_{00}$.
Such a connection has four torsions:

$$(4.6)\qquad T^i_{jk}=F^i_{jk}-F^i_{kj},\quad R_{kh}=\delta_h N_k-\delta_k N_h,\quad \overset{1}{P}_j=\partial_0 N_j-L_j,\quad \overset{2}{P}{}^i_j=M^i_j$$

and four curvatures:

$$(4.7)\qquad \begin{cases} R^i_{jkh}=\delta_h F^i_{jk}+F^l_{jk}F^i_{lh}-h|k+M^i_j R_{kh}\\ \widetilde{R}_{kh}=\delta_h L_k-\delta_k L_h+C R_{kh}\\[1ex] \widehat{P}_k=\partial_0 L_k-\partial_k C+\partial_0(CN_k)\\[1ex] \overset{2}{P}{}^i_{jk}=\partial_0 F^i_{jk}-\partial_k M^i_j+\partial_0(N_k M^i_j)+M^i_h F^h_{jk}-M^h_j F^i_{hk} \end{cases}$$

The h- and v-covariant derivatives are defined as in the general case.

Let be $G=g_{ij}(x^1, \ldots x^n, x^0)\; dx^i\otimes dx^j+g_{00}(x^1, \ldots x^n, x^0)\;(\delta x^0)^2$, where $\delta x^0=dx^0+N_i dx^i$, a Riemannian metric on E.

There exists a metrical d-connection with T^i_{jk} prescribed. Its local coefficients are as follows:

$$(4.8)\qquad \begin{cases} F^i_{jk}=\dfrac{1}{2}g^{ih}(\delta_j g_{hk}+\delta_k g_{hj}-\delta_h g_{jk}+g_{hr}T^r_{jk}-g_{jr}T^r_{hk}+g_{kr}T^r_{jh})\\[2ex] L_k=\dfrac{1}{2}g_{00}^{-1}\delta_k g_{00},\qquad M^i_j=\dfrac{1}{2}g^{ik}\partial_0 g_{kj},\qquad C=\dfrac{1}{2}g_{00}^{-1}\partial_0 g_{00}. \end{cases}$$

Einstein's equations associated to the metrical d-connection given by (4.8), written with respect to the adapted frame, are as follows:

$$(4.9)\qquad \begin{cases} R_{ij}-\dfrac{1}{2}Rg_{ij}=\varkappa\mathbf{T}_{ij}\\[2ex] \overset{1}{P}_i=\varkappa\mathbf{T}_{i0},\qquad \overset{2}{P}{}^k_{jk}=-\varkappa\mathbf{T}_{0i},\qquad Rg_{00}=-2\,\mathbf{T}_{00} \end{cases}$$

where $R=g^{ij}R_{ij}$.

The conservation law looks as follows:

$$\left\{\begin{array}{l} R^{i}_{j|i} - \frac{1}{2} R_{ij} + \partial_0 (g_{00}^{-1} \overset{1}{P}_j) - g_{00}^{-1} M^{i}_{j} \overset{1}{P}_i = 0, \\ \frac{1}{2} \partial_0 R + g^{ij} \overset{2}{P}_{j|i} = 0. \end{array}\right. \tag{4.10}$$

Remark 4.3. If $(M, g_{ij}(x))$ is a Lorentz manifold and we set $G = g_{ij}(x^1, x^2, x^3, x^4)\, dx^i \otimes dx^j + g_{00}(x^1, \ldots, x^4)\, (\delta x^0)^2$, $i, j = 1, \ldots, 4$, then the first group from eqs. (4.9) are just the Einstein equations for $(M, g_{ij}(x))$. The following two groups can be thought of or interpreted as Maxwell equations. Therefore, a way for developping an unitary projective theory has appeared.

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ON STABILITY OF PERTURBED NONLINEAR PULSE SYSTEMS

BY

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1. Introduction. The object of this paper is to establish criteria for the stability and asymptotic stability as well as for the uniqueness of a solution of the nonlinear time-varying differential system with pulse effect of the form

$$x' = f(t, x) + g(t, x), \quad t \neq t_i, \quad \Delta x/_{t=t_i} = I_i(x), \tag{1.1}$$

where x, f, g are n-dimensional vectors, $I_i(x)$ is, for each $i \in N^* = N \setminus \{0\}$, an n-dimensional vector, the meaning of $\Delta x/_{t=t_i} = I_i(x)$ is that $\Delta x/_{t=t_i} \equiv x(t_i+0) - x(t_i) = I_i(x(t_i))$, $i \in N^*$, the moments of pulse actions t_i being such that $t_i \to \infty$ as $i \to \infty$ and $x = x(t)$ is a solution of (1.1).

The stability problem of (1.1) has been investigated by few authors (e.g. see [1], [6], [7], [8], [9]).

The study of pulse differential systems taking into account their connection with the discrete systems and with the theory of distributions in the sense of L. Schwartz has been done in the book [6]. One of the frequently used method in studying stability problems is the Lyapunov direct method but there is no a systematic technique for the construction of Lyapunov functions. B. P. D e m i d o v i c h [1] and later C. V. P a o [5] have considered a different approach, motived by the concept of an equivalent inner product and a dissipative operator, which not only maintains the advantage of the Lyapunov direct method in the study of the stability but also offers a direct technique in estimating the stability region without relying on the construction of the Lyapunov function.

In this paper we will extend the results from [10] to the pulse differential system of the form (1.1) by using the method of B. P. D e m i d o v i c h.

2. Preliminary results. Let R^n be the n-dimensional Euclidean space. For any two column vectors $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T$ in R^n, the inner product is defined by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i. \tag{2.1}$$

The norm induced by the inner product $\langle .,. \rangle$ is given by $\|x\| = (\langle x, x \rangle)^{1/2}$. Throughout this paper, we shall reserve $\langle .,. \rangle$ as the inner product defined by (2.1) and $\|.\|$ as the norm of the Hilbert space $(R^n, \langle .,. \rangle)$. To establish our results, we need the following three lemmas.