$$
(4.10)\qquad \begin{cases} R^i_{j|i} - \dfrac{1}{2} R_{|j} + \partial_0 (g_{00}^{-1} \overset{1}{P}_j) - g_{00}^{-1} M^i_j \overset{1}{P}_i = 0, \\ \dfrac{1}{2} \partial_0 R + g^{ij} \overset{2}{P}_{j|i} = 0. \end{cases}
$$

Remark 4.3. If $(M, g_{ij}(x))$ is a Lorentz manifold and we set $G = g_{ij}(x^1, x^2, x^3, x^4)\, dx^i \otimes dx^j + g_{00}(x^1, \ldots, x^4)\, (\delta x^0)^2$, $i, j = 1, \ldots, 4$, then the first group from eqs. (4.9) are just the Einstein equations for $(M, g_{ij}(x))$. The following two groups can be thought of or interpreted as Maxwell equations. Therefore, a way for developping an unitary projective theory has appeared.

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ON STABILITY OF PERTURBED NONLINEAR PULSE SYSTEMS

BY

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1. Introduction. The object of this paper is to establish criteria for the stability and asymptotic stability as well as for the uniqueness of a solution of the nonlinear time-varying differential system with pulse effect of the form

$$
(1.1)\qquad x' = f(t, x) + g(t, x), \; t \neq t_i, \; \Delta x/_{t=t_i} = I_i(x),
$$

where x, f, g are n-dimensional vectors, $I_i(x)$ is, for each $i \in N^* = N \setminus \{0\}$, an n-dimensional vector, the meaning of $\Delta x/_{t=t_i} = I_i(x)$ is that $\Delta x/_{t=t_i} \equiv x(t_i + 0) - x(t_i) = I_i(x(t_i))$, $i \in N^*$, the moments of pulse actions t_i being such that $t_i \to \infty$ as $i \to \infty$ and $x = x(t)$ is a solution of (1.1).

The stability problem of (1.1) has been investigated by few authors (e.g. see [1], [6], [7], [8], [9]).

The study of pulse differential systems taking into account their connection with the discrete systems and with the theory of distributions in the sense of L. Schwartz has been done in the book [6]. One of the frequently used method in studying stability problems is the Lyapunov direct method but there is no a systematic technique for the construction of Lyapunov functions. B. P. Demidovich [1] and later C. V. Pao [5] have considered a different approach, motived by the concept of an equivalent inner product and a dissipative operator, which not only maintains the advantage of the Lyapunov direct method in the study of the stability but also offers a direct technique in estimating the stability region without relying on the construction of the Lyapunov function.

In this paper we will extend the results from [10] to the pulse differential system of the form (1.1) by using the method of B. P. Demidovich.

2. Preliminary results. Let R^n be the n-dimensional Euclidean space. For any two column vectors $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T$ in R^n, the inner product is defined by

$$
(2.1)\qquad <x, y> = \sum_{i=1}^{n} x_i y_i.
$$

The norm induced by the inner product $<.,.>$ is given by $\|x\| = (<x, x>)^{1/2}$. Throughout this paper, we shall reserve $<.,.>$ as the inner product defined by (2.1) and $\|.\|$ as the norm of the Hilbert space $(R^n, <.,.>)$. To establish our results, we need the following three lemmas.

Lemma 2.1. ([5]). *Let V be a real symmetric positive definite matrix. Then, the inner product $<.,.>_V$ defined by*

$$<x, y>_V = <x, Vy>, \quad x, y \in R^n, \tag{2.2}$$

is equivalent to $<.,.>$.

Lemma 2.2 ([5]). *Let $<.,.>$ be the inner product defined by (2.2). If $z(t)$ is differentiable in the interval $[t_0, \infty)$, then $\| z(t) \|_V^2$ is differentiable and*

$$\frac{d}{dt} \| z(t) \|_V^2 = 2 < \frac{d}{dt} z(t), z(t) >, \qquad t \in [t_0, \infty), \tag{2.3}$$

where $\| . \|_V$ is the norm induced by $<.,.>_V$.

A more general formulation of Lemma 2.2 has been given by T. K a t o [4].

Lemma 2.3 ([1]). *Assume that $x \in R^n, \Omega \subseteq R^n$ and $f(t, x) = (f_1(t, x), f_2(t, x), \ldots, f_n(t, x)) \in C_x^1([t_0, \infty) \times \Omega)$. Let us denote $\lambda(t, x)$ and $\Lambda(t, x)$ respectively the smallest and the largest eigenvalue of the symmetrised Jacobi's matrix*

$$J_s(t, x) = \frac{1}{2}\left[\frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i}\right]. \tag{2.4}$$

Then, for any two points (t, x) and $(t, x+h)$ such that $\xi = x + \tau h \in \Omega$ with $0 \leq \tau \leq 1$, an inequality

$$\lambda_m(t, x) \| h \|^2 \leq < f(t, x+h) - f(t, x), h > \leq \Lambda_M(t, x) \| h \|^2, \tag{2.5}$$

is fulfiled, where $\lambda_m(t, x) = \min \{\lambda(t, \xi), \tau \in [0,1]\}$ and $\Lambda_M(t, x) = \max \{\Lambda(t, \xi), \tau \in [0,1]\}$.

With respect to the system (1.1), we note that its solutions are, in general, piece-wise continuous, with the possible discontinuity points t_i. For that reason the definitions of various kinds of stability differ from their analogous from the case of the differential equations.

Definition 2.1 ([8]). *A solution $x = x(t)$ of the pulse system (1.1) is said to be stable, if given any $\varepsilon < 0$, $\eta > 0$, there exists $\delta = \delta(\varepsilon, \eta) > 0$ such that for any solution $y = y(t)$ of the system (1.1) for which $\| x(t_0) - y(t_0) \| < \delta$ we have $\| x(t) - y(t) \| < \varepsilon$ for all $t \geq t_0$ such that $| t - t_i | > \eta$.*

Definition 2.2 ([8]). *A solution $x = x(t)$ of the pulse system (1.1) is said to be asymptotically stable if it is stable (as above) and, in addition, if there exists $\delta_0 > 0$ such that for any solution $y = y(t)$ of (1.1) for which $\| x(t_0) - y(t_0) \| < \delta_0$ we have $\lim\limits_{t \to \infty} \| x(t) - y(t) \| = 0$.*

3. Main results. In this section we shall investigate the nonlinear pulse system of differential equations (1.1) under suitable regularity conditions, namely, we suppose that for $t \in [t_0, \infty)$ and $x \in R^n$,

$$f(t, x) \in C_{t;x}^{(0;1)}([t_0, \infty) \times R^n), \; g(t, x) \in C_{t;x}^{(0;0)}([t_0, \infty) \times R^n), \qquad I_i(x) \in C^1(R^n), \quad i \in N^*. \tag{3.1}$$

With respect to the moments of pulse actions t_i, we assume that

$$0 < \theta_1 \leq t_{i+1} - t_i \leq \theta_2. \tag{3.2}$$

Because of its relationship with the stability problem we first state a uniqueness result by weakening the Lipschitz condition.

Theorem 3.1. *Assume that the following conditions hold:*

i) there exists a continuous function $\gamma(t)$ on $[t_0, \infty)$, such that the largest characteristic number $\Lambda(t, x)$ of the symmetric Jacobi matrix $J_s(t, x)$ given by (2.4) satisfies

$$\Lambda(t, x) \leq \gamma(t), \text{ for any } t \in [t_0, \infty) \text{ and } x \in R^n; \tag{3.3}$$

ii) for any $i \in N^*$, the largest characteristic numbers $\Gamma_i(x)$ of the matrices $[E + I_i'(x)]^T [E + I_i'(x)]$ satisfy

$$\Gamma_i(x) \leq \alpha^2, \; \alpha \in (0, \infty); \tag{3.4}$$

iii) there exists a continuous function $k(t)$ on $[t_0, \infty)$ such that for any $x, y \in R^n$, $t \in [t_0, \infty)$

$$< g(t, x) - g(t, y), x - y > \leq k(t) \| x - y \|^2. \tag{3.5}$$

Then, given any initial value x_0, there exists a unique solution $x = x(t)$ of (1.1) with $x(t_0) = x_0$. Moreover, for any two solutions $x = x(t)$ and with $x(t_0) = x_0$, $y(t_0) = y_0$

$$\| x(t) - y(t) \| \leq K \| x_0 - y_0 \| \exp\left[\int_{t_0}^{t} \left(\frac{1}{\theta} \ln \alpha + \gamma(\tau) + k(\tau)\right) d\tau\right] \tag{3.6}$$

where $\theta = \theta_1$ if $\alpha \geq 1$ and $\theta = \theta_2$ if $0 < \alpha < 1$, and $K \geq 1$ a constant.

Proof. Let $x = x(t)$ and $y = y(t)$ be any two solutions of (1.1) with $x(t_0) = x_0$, $y(t_0) = y_0$ and let $z(t) = x(t) - y(t)$. Then, for $t \neq t_i$ we have $z'(t) = f(t, x) - f(t, y) + g(t, x) - g(t, y)$ and $z(t_0) = x_0 - y_0$. By Lemma 2.2 and (3.5) one obtains

$$\frac{d}{dt}(\| z(t) \|^2) = 2 < f(t, x) - f(t, y), z(t) > + 2 < g(t, x) - g(t, y), z(t) >$$

$$\leq 2 < f(t, x) - f(t, y), z(t) > + 2k(t) \| z(t) \|^2.$$

By Lemma 2.3 we have

$$2 < f(t, x) - f(t, y), z(t) > \leq 2\Lambda(t, x) \| z(t) \|^2 \leq 2\gamma(t) \| z(t) \|^2,$$

and therefore,

$$\frac{d}{dt}(\| z(t) \|^2) \leq 2(\gamma(t) + k(t)) \| z(t) \|^2.$$

From here, for $t_i < t < t_{i+1}$, we get

$$\| z(t) \|^2 \leq \| z(t_i + 0) \|^2 \exp\left[2 \int_{t_i}^{t} (\gamma(\tau) + k(\tau))\, d\tau\right]. \tag{3.7}$$

From the jump condition $\Delta x|_{t=t_i} = x(t_i + 0) - x(t_i) = I_i(x(t_i))$ we get

$$z(t_i + 0) = x(t_i + 0) - y(t_i + 0) = z(t_i) + I_i'(\xi)\, z(t_i) = [E + I_i'(\xi)]\, z(t_i).$$

Therefore,

$$\| z(t_i+0) \|^2 = < z(t_i+0),\ z(t_i+0) > = < [E+I_i'(\xi)]\, z(t_i),$$
$$[E+I_i'(\xi)]\, z(t_i) > \leqslant \Gamma_i(\xi) \| z(t_i) \|^2 \leqslant \alpha^2 \| z(t_i) \|^2.$$

The inequality (3.7) together with

$$(3.8) \qquad \| z(t_i+0) \|^2 \leqslant \alpha^2 \| z(t_i) \|^2,$$

yield

$$\| z(t) \|^2 \leqslant \alpha^2 \| z(t_i) \|^2 \exp \Big[2 \int_{t_i}^{t} (\gamma(\tau)+k(\tau))\, d\tau\Big],\ t_i < t < t_{i+1},$$

from which by induction one obtains

$$(3.9) \qquad \| z(t) \| \leqslant \alpha^i \exp \Big[\int_{t_0}^{t} (\gamma(\tau)+k(\tau))\, d\tau\Big] \cdot \| z(t_0) \|,\ t_i < t < t_{i+1}.$$

From $0 < \theta_1 \leqslant t_{i+1} - t_i \leqslant \theta_2$ it follows $(t-t_0)/\theta_2 - 1 \leqslant i \leqslant (t-t_0)/\theta_1$ and therefore

$$\alpha^i \leqslant \alpha^{\frac{t-t_0}{\theta_1}} = \exp [(1/\theta_1) \ln \alpha)\, (t-t_0)] \text{ if } \alpha \geqslant 1,$$

or,

$$\alpha^i \leqslant \alpha^{\frac{t-t_0}{\theta_2}} \cdot \frac{1}{\alpha} = \frac{1}{\alpha} \exp [((1/\theta_2) \ln \alpha) \cdot (t-t_0)] \text{ if } 0 < \alpha < 1.$$

Thus, (3.9) becomes

$$(3.10) \quad \| z(t) \| \leqslant K \| z(t_0) \| \exp \Big[\int_{t_0}^{t} ((1/\theta) \ln \alpha + \gamma(\tau) + k(\tau))\, d\tau\Big],\ t \in [t_0, \infty),$$

where $K \geqslant 1$ is a constant depending on α. This relation is equivalent to (3.6).

To show the uniqueness of a solution, let $x_0 = y_0$. Since both $\gamma(t)$ and $k(t)$ are bounded on any finite interval $[t_0, T]$, it follows from (3.6) that $\| x(t) - y(t) \| = 0$ for all $t \in [t_0, T]$. Since T can be arbitrarily chosen, it follows that $\| x(t) - y(t) \| = 0$ for all $t \in [t_0, \infty)$. Hence the theorem is proved.

Remarks 1. When $I_i(x) \equiv 0$, $i = 1, 2, \ldots$, i.e. (1.1) is an ordinary differential system, then $\Gamma_i(x) \equiv 1$, $i = 1, 2, \ldots$, $\alpha = 1$, $K = 1$, $\theta = \theta_1$ and Theorem 3.1 reduces to Theorem 3.1 [10].

2. When $f(t, x) = A(t)\, x$, where $A(t)$ is a continuous matrix on the interval $[t_0, \infty)$, $\gamma(t)$ from condition (3.3) is the largest characteristic number of $(A(t) + A^T(t))/2$.

3. If $g(t, x)$ is Lipschitz continuous with respect to x with a non-negative Lipschitz function $k(t)$ on $[t_0, \infty)$, then (3.5) is satisfied since

$$< g(t, x) - g(t, y),\ x - y > \leqslant \| g(t, x) - g(t, y) \| \cdot \| x - y \| \leqslant k(t) \| x - y \|^2.$$

4. From the inequality (3.6) it follows that, if

$$\lim_{t \to \infty} \int_{t_0}^{t} [(1/\theta) \ln \alpha + \gamma(\tau) + k(\tau)]\, d\tau < \infty \text{ (or } = -\infty),$$

then any solution of (1.1) is stable (asymptotically stable). That happens if, for example, $(1/\theta) \ln \alpha + \gamma(t) + k(t) < 0$, $t \in [t_0, \infty)$.

In order to give some results concerning the stability of the system (1.1) besides the mentioned regularity conditions on $f(t, x)$ and $g(t, x)$, we shall assume that an equilibrium solution of (1.1) is $x = 0$, what happens if $f(t, 0) = g(t, 0) = 0$ and $I_i(0) = 0$, $i = 1, 2, \ldots$. The stability and asymptotic stability of a solution in the following theorems are in the sense of global stability, i.e., the stability region is the whole space R^n.

Theorem 3.2. *Assume that the following conditions are satisfied:*

i) *there exists a continuos function $k(t)$ on $[t_0, \infty)$ such that*

$$(3.11) \qquad < g(t, x),\ x > \leqslant k(t) \| x \|^2, \text{ for any } x \in R^n \text{ and } t \in [t_0, \infty);$$

ii) *for any $i \in N^*$, the largest characteristic numbers $\Gamma_i(x)$ satisfy (3.4);*

iii) *there exist non-negative continuous functions $\beta(t)$ and $\gamma(t)$ such that for some $T > t_0$*

$$(3.12) \qquad \Lambda(t, x) \leqslant \gamma(t), \text{ for } t \in [t_0, T),\ x \in R^n,$$

$$(3.13) \qquad \Lambda(t, x) \leqslant -\beta(t), \text{ for } t \in [T, \infty),\ x \in R^n,$$

with $\beta(t) \geqslant k(t) + (1/\theta) \ln \alpha$ for $t \in [T, \infty)$.

Then the zero solution of (1.1) is stable. In addition, if we have

$$(3.14) \qquad \lim_{t \to \infty} \int_{T}^{t} (\beta(\tau) - k(\tau) - (1/\theta) \ln \alpha)\, d\tau = \infty,$$

it is asymptotically stable.

Proof. By the regularity conditions, given any initial value x_0, there exists a solution of (1.1), $x = x(t)$, such that $x(t_0) = x_0$. Then, for $t \neq t_i$, we have $x'(t) = f(t, x(t)) + g(t, x(t))$ and therefore

$$(3.15) \qquad \frac{d}{dt} (\| x(t) \|^2) = 2 < f(t, x(t)),\ x(t) > + 2 < g(t, x(t)),\ x(t) >.$$

Since

$$< f(t, x(t)),\ x(t) > = < f(t, x(t)) - f(t, 0),\ x(t) >,$$

according to Lemma 2.3 and (3.13), (3.14), we have

$$< f(t, x(t)),\ x(t) > \leqslant \gamma(t), \text{ for } t \in [t_0, T),$$
$$< f(t, x(t)),\ x(t) > \leqslant -\beta(t), \text{ for } t \in [T, \infty),$$

and by (3.11)

$$< g(t, x(t)),\ x(t) > \leqslant k(t) \| x(t) \|^2.$$

We put $\alpha(t) = \gamma(t)$, for $t \in [t_0, T)$ and $\alpha(t) = -\beta(t)$ for $t \in [T, \infty)$. Then (3.15) becomes

$$\frac{d}{dt} (\| x(t) \|^2) \leqslant 2\, (\alpha(t) + k(t)) \| x(t) \|^2,\ t \in [t_0, \infty),\ t \neq t_i.$$

From here, for $t_i < t < t_{i+1}$, we obtain

$$(3.16) \qquad \| x(t) \|^2 \leq \| x(t_i+0) \|^2 \exp \left[2 \int_{t_i}^{t} (\alpha(\tau) + k(\tau))\, d\tau \right].$$

Inasmuchas

$$x(t_i+0) = x(t_i) + I_i(\xi)\, x(t_i) = [E + I_i'(\xi)]\, x(t_i),$$

we have

$$\| x(t_i+0) \|^2 \leq \Gamma_i(x) \| x(t_i) \|^2 \leq^2 \alpha^2 \| x(t_i) \|^2,$$

and by an argument similar to that from the proof of Theorem 3.1 it follows

$$\| x(t) \| \leq K \| x_0 \| \exp \left[\int_{t_0}^{t} ((1/\theta) \ln \alpha + \alpha(\tau) + k(\tau))\, d\tau \right]$$

or still

$$\| x(t) \| \leq K \| x_0 \| \exp \left[\int_{t_0}^{T} ((1/\theta) \ln \alpha + \gamma(\tau) + k(\tau))\, d\tau \right].$$

$$\exp \left[- \int_{T}^{t} (\beta(\tau) - k(\tau) - (1/\theta) \ln \alpha)\, d\tau \right], \; t \in [t_0, \infty), \quad K \geq 1.$$

Since $\gamma(t) + k(t) + (1/\theta) \ln \alpha$ is bounded on $[t_0, T]$ we have

$\exp \left[\int_{t_0}^{T} ((1/\theta) \ln \alpha + \gamma(\tau) + k(\tau))\, d\tau \right] = M_1$, and therefore

$$(3.17) \qquad \| x(t) \| \leq M \| x_0 \|, \; t \in [t_0, T)$$

and

$$(3.18) \qquad \| x(t) \| \leq M \exp \left[- \int_{T}^{t} (\beta(\tau) - k(\tau) - (1/\theta) \ln \alpha)\, d\tau \right] \| x_0 \|, \; t \in [T, \infty),$$

where $M = KM_1$.

From (3.17) and (3.18) it follows the conclusion of our theorem.

Remark. The inequality (3.13) implies that the matrix $-J_a(t, x)$ is positive definite (or semi-definite) on $[T, \infty)$ with the smallest eigenvalue $\beta(t)$. It is seen that a sufficient condition for (3.14) to hold is that $\inf \{\beta(t) - k(t) \,;\, t \in [T, \infty)\} > (\ln \alpha)/\theta$.

In this case the zero solution is exponentially asymptotically stable.

Corollary. *Assume, under the same regularity conditions on $f(t, x)$ and $g(t, x)$, that (3.12) holds for $t \in [t_0, \infty)$, $x \in R^n$. If (3.11) occurs for any $x \in R^n$, $t \in [t_0, \infty)$ and $k(t) \leq 0$ on $[T, \infty)$ with $\gamma(t) \leq -k(t) - (1/\theta) \ln \alpha$ for $t \in [T, \infty)$, then the zero solution is stable.*

If, in addition, $\lim_{t \to \infty} \int_{T}^{t} (\beta(\tau) + k(\tau) + (1/\theta) \ln \alpha)\, d\tau = -\infty$, *then it is asymptotically stable.*

Indeed, from the same proof as in Theorem 3.2 it follows

$$\| x(t) \| \leq K \exp \left[\int_{t_0}^{t} (\gamma(\tau) + k(\tau) + (1/\theta) \ln \alpha)\, d\tau \right] \| x_0 \|, \; t \in [t_0, \infty)$$

where, from

$$(3.19) \qquad \| x(t) \| \leq M \exp \left[\int_{T}^{t} (\gamma(\tau) + k(\tau) + (1/\theta) \ln \alpha)\, d\tau \right] \| x_0 \|, \; t \in [T, \infty).$$

From (3.19) the conclusion of the theorem immediately follows.

Remark. Condition (3.11) in Theorem 3.1 is a weakeness of condition (3.5) in Theorem 3.1.

If $g(t, x)$ satisfies a Lipschitz condition with respect to x, then we have the following theorem which ensure not only the stability of the null solution but also that of any solution.

Theorem 3.3. *Assume that the following conditions are satisfied :*

i) *there exist non-negative functions $\beta(t)$ and $\gamma(t)$ such that for some $T \geq t_0$ (3.12) and (3.13) hold ;*

ii) *for any $i \in N^*$, the largest characteristic numbers $\Gamma_i(x)$ satisfy (3.4);*

iii) *there exists a continuous function $k(t)$ on $[t_0, \infty)$ such that*

$$(3.20) \qquad \langle g(t, x) - g(t, y),\, x - y \rangle \leq k(t) \| x - y \|^2, \text{ for any } x, y \in R^n, \; t \in [t_0, \infty)$$

with $\beta(t) \geq k(t) + (1/\theta) \ln \alpha$ on $[T, \infty)$.

Then any solution $x = x(t)$ of (1.1) is stable. If, in addition,

$$(3.21) \qquad \lim_{t \to \infty} \int_{T}^{t} (\beta(\tau) - k(\tau) - (1/\theta) \ln \alpha)\, d\tau = \infty,$$

it is asymptotically stable. In particular, if $\inf \{\beta(t) - k(t) \,;\, t \in [T, \infty)\} > (1/\theta) \ln \alpha$, then $x = x(t)$ is exponentially asymptotically stable.

Proof. By the hypothesis of the Lipschitz condition (3.20) we have for any $x, y \in R^n$

$$\langle g(t, x) - g(t, y),\, x - y \rangle \leq k(t) \| x - y \|^2, \; t \in [t_0, \infty).$$

It follows from Theorem 3.1 that for any two solutions $x = x(t)$, $y = y(t)$ with $x(t_0) = x_0$, $y(t_0) = y_0$

$$\| x(t) - y(t) \| \leq K \| x_0 - y_0 \| \exp \left[\int_{t_0}^{t} ((1/\theta) \ln \alpha + \gamma(\tau) + k(\tau))\, d\tau \right], \; t \in [t_0, \infty).$$

Since $\gamma(t)$, $k(t)$ are bounded on $[t_0, T]$ there exists a number $M > 0$ such that

$$\| x(t) - y(t) \| \leq M \exp \left[- \int_{T}^{t} (\beta(\tau) - k(\tau) - (1/\theta) \ln \alpha)\, d\tau \right] \| x_0 - y_0 \|, \; t \in [T, \infty),$$

where $\beta(t) - k(t) - (1/\theta) \ln \alpha > 0$.

It follows by the same arguments as in the proof of Theorem 3.2 the stability and asymptotic stability of any solution $x = x(t)$.

Remark. When $f(t, x) = A(t)\, x$, where $A(t)$ is a continuous matrix on $[t_0, \infty)$, $\gamma(t)$ from (3.12) is the largest characteristic number of $(A(t) + A^T(t))/2$, and (3.13) means that this matrix is negative definite on $[T, \infty)$ with the smallest characteristic number $-\beta(t)$.

We return again to the system (1.1) under the same regularity hypotheses and the assumption that it has the null solution, i.e., $f(t, 0) = g(t, 0) = 0$ and $I_i(0) = 0$, $i \in N^*$.

In the following, the condition (3.11) in Theorem 3.2 will be weakened to some extent, using the concept of an equivalent inner product [5]. We observe that if V is a constant matrix then there exist two positive constants δ, δ' such that

$$(3.22) \qquad \delta' \| x \|^2 \leq \| x \|_v^2 = < x, Vx > \leq \delta \| x \|^2.$$

Theorem 3.4. *Assume that the following conditions hold:*

i) *there exists a continuous function $k(t)$ on $[t_0, \infty)$ such that*

$$(3.23) \qquad < g(t, x), Vx > \leq k(t) \| x \|^2, \quad x \in R^n, \quad t \in [t_0, \infty),$$

where $V = (a_{ij})$, $1 \leq i, j \leq n$, is a positive definite matrix;

ii) *for any $i \in N^*$, the largest characteristic numbers $\Gamma_i(x)$ satisfy (3.4);*

iii) *the largest characteristic number $\tilde{\Lambda}(t, x)$ of the generalized symmetric Jacobi matrix $\tilde{J}_s(t, x) = (a_{ij}\, \partial f_i/\partial x_j + a_{ji}\, \partial f_j/\partial x_i)/2$ is such that*

$$(3.24) \qquad \tilde{\Lambda}(t, x) \leq -\beta, \quad x \in R^n, \quad t \in [t_0, \infty),$$

where $\beta \geq 0$ and $k = \sup \{k(t); t \in [T, \infty)\} < \beta$ for some $T > t_0$. Then, if $(\beta - k)/\delta - (1/\theta) \ln \alpha > 0$, the null solution of (1.1) is asymptotically stable.

Proof. Let $x = x(t)$ be any solution of (1.1) with the initial condition $x(t_0) = x_0$. Let $\| x(t) \|_v^2 = < x(t), Vx(t) >$ be. By Lemma 2.2 for $t \neq t_i$ we have

$$d(\| x(t) \|_v^2)/dt = 2 < f(t, x(t)), Vx(t) > + 2 < g(t, x(t)), Vx(t) >.$$

On the other hand, by Lemma 2.3 and (3.24), one obtains

$$2 < f(t, x(t)), Vx(t) > = 2 < Vf(t, x(t)) - Vf(t, 0), x(t) > \leq$$

$$\leq \tilde{\Lambda}(t, x) \| x(t) \|^2 \leq -\beta \| x(t) \|^2.$$

From here and taking into account (3.23) we get

$$(3.25) \qquad d(\| x(t) \|_v^2)/dt \leq -2(\beta - k(t)) \| x(t) \|^2, \quad t \neq t_i, \quad t \in [t_0, \infty).$$

According to (3.22) and by hypothesis iii) it follows

$$d(\| x(t) \|_v^2)/dt \leq -2\, (\beta - k) \| x(t) \|^2 \leq -2\lambda_1 \| x(t) \|_v^2, \quad t \in [T, \infty), \quad t \neq t_i,$$

where $\lambda_1 = (\beta - k)/\delta$. By the same argument as in the proofs of Theorems 3.1 and 3.2 we get

$$(3.26) \qquad \| x(t) \|_v \leq K \| x(T) \|_v \exp [-(\lambda_1 - (1/\theta) \ln \alpha)(t - T)], \quad t \in [T, \infty).$$

On the other hand, for $t \in [t_0, T]$ from (3.25) it follows

$$(3.27) \qquad \| x(t) \|_v \leq K \| x_0 \|_v \exp \left[\int_{t_0}^{t} ((k(\tau) - \beta)/\delta'' + (1/\theta) \ln \alpha)\, d\tau \right] \leq$$

$\leq M_1 \| x_0 \|_v$, where $M_1 = K \sup \{ \int_{t_0}^{t} ((k(\tau) - \beta)/\delta' + 1/\theta \ln \alpha)\, d\tau;\ t \in [t_0, T] \}$ and

$\delta'' = \delta'$ for those $t \in [t_0, T]$ for which $k(t) \geq \beta$ and $\delta'' = \delta$ for those $t \in [t_0, T]$ for which $k(t) < \beta$. From (3.26) and (3.27) it follows the conclusion of the theorem.

Remark. When $0 < \alpha < 1$ the inequality $(\beta - k)/\delta - (1/\theta) \ln \alpha > 0$ is obviously satisfied.

Corollary. *Under the same assumptions as in Theorem 3.4, except with (3.23) replaced by*

$$(3.28) \qquad \| g(t, x) \| \leq \varphi(t) \| x \|, \quad x \in R^n, \quad t \in [t_0, \infty),$$

where $\varphi(t)$ is a continuous function on $[t_0, \infty)$, if for some $T > t_0$ we have $k_1 = \sup \{\varphi(t);\ t \in [t_0, \infty)\} < \beta/\delta$ and $(\beta - k_1\delta)/\delta - (1/\theta) \ln \alpha > 0$, then the null solution is asymptotically stable.

Indeed, by (3.22) and (3.28) we have

$$< g(t, x), Vx > = < g(t, x), x >_v \leq \| g(t, x) \|_v \cdot \| x \|_v \leq$$

$$\leq \delta \| x \| \cdot \| g(t, x) \| \leq \delta \varphi(t) \| x \|^2,$$

i.e., the inequality (3.23) with $k(t) = \delta \varphi(t)$.

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