Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXII, s. I a, Matematică, 1986, f. 3

ON THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

BY

N. PARHI and S. K. NAYAK*

1. In this paper, we study asymptotic behaviour of solutions of

$$(r(t)\ y')' - p(t)\ y^{\alpha} = f(t), \tag{1}$$

where p, r and f are real-valued continuous functions on $[0, \infty)$ such that $r(t) > 0$ and $\alpha > 0$ is a ratio of odd integers. We also prove a result for solutions of more general equations of the form

$$(r(t)\ y')' + q(t)\ (y')^{\beta} - p(t)\ y^{\alpha} = f(t), \tag{2}$$

where q is a real-valued continuous function on $[0, \infty)$ and $\beta > 0$ is a ratio of odd integers. Every solution $y(t)$ of (1) or (2) that is considered here will be assumed to exist on some ray $[T_y, \infty)$, $T_y \geq 0$. A solution $y(t)$ of (1) or (2) is said to be nonoscillatory if there exists a $T \geq T_y$ such that $y(t) \neq 0$ for $t \geq T$; it is said to be oscillatory if for every $t_1 \geq T_y$ there exist t_2, t_3 such that $t_1 < t_2 < t_3$, $y(t_2) < 0$ and $y(t_3) > 0$; and it is said to be weakly oscillatory if it is not nonoscillatory, that is, $y(t)$ is oscillatory or $y(t) \leq 0$ or $y(t) \geq 0$. In [5, 6], sufficient conditions have been obtained for nonoscillation of solutions of (1). These results can easily be generalized to include (2).

Asymptotic behaviour of solutions of equations of the type (1) with $p(t) < 0$ has been studied by many authors, viz., Hatvani [3], Chang [1], Teufel [10], Spikes [8] and Graef and Spikes [2]. It seems that not much work has been done concerning asymptotic behaviour of solutions of equations of the type (1) or (2) with $p(t) \geq 0$. Marini and Zezza [4] studied the asymptotic behaviour of solutions of (1) with $\alpha = 1$, $f(t) \equiv 0$ and $p(t) \geq 0$. Taliaferro [9] studied the asymptotic behaviour of solutions of (1) with $r(t) \equiv 1$, $f(t) \equiv 0$, $\alpha > 1$ and $p(t) \geq 0$. In [7], the first author studied the asymptotic behaviour of solutions of differential inequalities of the type

$$(r(t)\ y')' - p(t)\ y^{\alpha} \geq 0,$$

where p, r and α are the same as in (1) with $p(t) \geq 0$. Clearly, these results hold good for (1) with $p(t) \geq 0$ and $f(t) \geq 0$.

2. Here we consider (1) or (2) with $p(t) \geq 0$, and $f(t) \geq 0$. In some of the results, we allow $f(t)$ to change sign. For $\alpha = 1$, (1) reduces to

* Supported by U.G.C., India.

$$(r(t)\,y')'-p(t)\,y=f(t). \tag{3}$$

The corresponding homogeneous equation is given by

$$(r(t)\,x')'-p(t)\,x=0. \tag{4}$$

Remark. From a result due to Marini and Zezza [4, Theorem 5, p. 10] it follows that for every solution $\varphi(t)$ of (4) $r(t)\,\varphi'(t)$ is unbounded as $t\to\infty$ if and only if

$$\lim_{t\to\infty}\int_\sigma^t p(s)\left(\int_\sigma^s \frac{d\theta}{r(\theta)}\right)ds=\infty,\ \sigma\geqslant 0. \tag{5}$$

If $\varphi(t)>0$ for large t, then $\lim\limits_{t\to\infty} r(t)\varphi'(t)=\infty$, iff (5) holds.

Let $\varphi(t)$ be a solution of (4) with initial conditions $\varphi(a)>0$ and $\varphi'(a)\geqslant 0$, $a\geqslant 0$. So, $\varphi(t)>0$ for $t\geqslant a$. Clearly,

$$\psi(t)=\varphi(t)\int_a^t \frac{ds}{r(s)\,\varphi^2(s)},\quad t\geqslant a,$$

is a solution of (4) and $\{\varphi(t),\ \psi(t)\}$ forms a basis for the linear space of solutions of (4). It is easy to verify that

$$y_0(t)=\psi(t)\int_a^t f(s)\,\varphi(s)ds-\varphi(t)\int_a^t f(s)\,\psi(s)\,ds,\ t\geqslant a,$$

is a particular solution of (3). So any solution $y(t)$ of (3) is given by $y(t)=y_0(t)+\lambda\psi(t)+\mu\varphi(t)$, where λ and μ are reals.

Theorem 2.1. *Let $p(t)>0$ and (5) hold. If $\varphi(t)$ is unbounded such that*

$$\lim_{t\to\infty}\int_\sigma^t f(s)\,\varphi(s)\,ds=\infty,\ \sigma\geqslant a,$$

then every bounded solution of (3) tends to zero as $t\to\infty$ if and only if $\lim\limits_{t\to\infty}[f(t)/p(t)]=0$.

Proof. Let $y(t)$ be a bounded solution of (3). So it can be written as

$$y(t)=y_0(t)+\lambda\psi(t)+\mu\varphi(t)=$$

$$=\varphi(t)\left[\left(\lambda+\int_a^t f(s)\,\varphi(s)\,ds\right)\left(\int_a^t \frac{ds}{r(s)\,\varphi^2(s)}\right)+\left(\mu-\int_a^t f(s)\,\psi(s)\,ds\right)\right].$$

Since $y(t)$ is bounded and $\lim\limits_{t\to\infty}\varphi(t)=\infty$, we conclude that

$$\lim_{t\to\infty}\left[\left(\lambda+\int_a^t f(s)\,\varphi(s)\,ds\right)\left(\int_a^t \frac{ds}{r(s)\,\varphi^2(s)}\right)+\left(\mu-\int_a^t f(s)\,\psi(s)\,ds\right)\right]=0.$$

Thus,

$$\lim_{t\to\infty} y(t)=\lim_{t\to\infty}\frac{\left(\lambda+\int_a^t f(s)\,\varphi(s)\,ds\right)\left(\int_a^t \frac{ds}{r(s)\,\varphi^2(s)}\right)+\left(\mu-\int_a^t f(s)\,\psi(s)\,ds\right)}{1/\varphi(t)}=$$

$$=-\lim_{t\to\infty}\frac{\lambda+\int_a^t f(s)\,\varphi(s)\,ds}{r(t)\,\varphi'(t)}.$$

From the preceding remark and the choice of $\varphi(t)$, it follows that $r(t)\,\varphi'(t)\to\infty$ as $t\to\infty$. So, using l'Hospital's rule, we have

$$\lim_{t\to\infty} y(t)=-\lim_{t\to\infty}\frac{f(t)\,\varphi(t)}{(r(t)\,\varphi'(t))'}=-\lim_{t\to\infty}\frac{f(t)}{p(t)}.$$

This completes the proof of the theorem.

Remark. From Theorem 3 due to Marini and Zezza [4, p. 5] it follows that a solution $\varphi(t)$ of (4) is unbounded iff

$$\lim_{t\to\infty}\int_\sigma^t \frac{1}{r(s)}\left(\int_\sigma^s p(\theta)\,d\theta\right)ds=\infty,\qquad \sigma\geqslant 0. \tag{6}$$

Also a positive solution $\varphi(t)$ of (4) tends to infinity as $t\to\infty$ if (5) holds and $\lim\limits_{t\to\infty}\int_\sigma^t \frac{ds}{r(s)}=\infty,\quad \sigma\geqslant 0.$

The following example illustrates the above theorem.

Example: Consider

$$(2/3)\,(t^3\,y')'-2t\,y=-(1/3)\,(2\cos t-1)+2\,(\sin t)/t,\qquad t>0, \tag{7}$$

and

$$(2/3)\,(t^3\,x')'-2\,tx=0,\ t>0,$$

We take $\varphi(t)=t$. Clearly, all the conditions of Theorem 2.1 are satisfied. Hence all bounded solutions of (7) tend to zero as $t\to\infty$. In particular, $y(t)=-1/(8t^3)(t^2-8\cos t)$ is a bounded nonoscillatory solution of (7) which tends to zero as $t\to\infty$.

Theorem 2.2. *Let $p(t)>0$ and (5) hold. If $\varphi(t)$ is unbounded such that*

$$\lim_{t\to\infty}\int_\sigma^t f(s)\,\varphi(s)\,ds<\infty,\qquad \sigma\geqslant a,$$

then every bounded solution of (3) tends to zero as $t\to\infty$.

Proof. We proceed as in Theorem 2.1 to obtain

$$\lim_{t\to\infty} y(t)=-\lim_{t\to\infty}\frac{\lambda+\int_a^t f(s)\,\varphi(s)\,ds}{r(t)\,\varphi'(t)}=0.$$

Example. Consider

$$(t^3y')'-3t\,y=-(1/t^4)\,(t^2\sin t+7t\,\cos t-12\sin t),\ t\geqslant\sigma>0, \tag{8}$$

and

$$(t^3x')'-3tx=0,\ t\geqslant\sigma>0.$$

We take $\varphi(t)=t$. Clearly, all the conditions of Theorem 2.2 are satisfied. Hence all bounded solutions of (8) tend to zero as $t\to\infty$. In particular, $y(t)=t^{-3}\sin t$ is a bounded oscillatory solution of (8) which tends to zero as $t\to\infty$.

Corollary 2.3. *If $p(t)>0$ and $f(t)\geqslant 0$ such that $\lim\limits_{t\to\infty}[f(t)/p(t)]=0$ and (5) and (6) hold, then all bounded solutions of (3) tend to zero as $t\to\infty$.*

Proof. Let $\varphi(t)$ be a solution of (4) with initial conditions $\varphi(a)>0$ and $\varphi'(a)\geq 0$, $a\geq 0$. So $\varphi(t)>0$ for $t\geq a$.

Since $f(t)\geq 0$, then

$$\lim_{t\to\infty}\int_{\sigma}^{t} f(s)\,\varphi(s)\,ds=\infty \quad\text{or}\quad \lim_{t\to\infty}\int_{\sigma}^{t} f(s)\,\varphi(s)\,ds<\infty, \quad \sigma\geq a.$$

In the former case the conclusion is obtained from Theorem 2.1 and in the latter case the result follows from Theorem 2.2.

Example. $((1/t)\,y')'-y=1/t-3/t^4$, $t\geq 2$.

$y(t)=-1/t$ is a bounded solution of the equation which tends to zero as $t\to\infty$.

Theorem 2.4. *Let* $\int_0^\infty \frac{dt}{r(t)}=\infty$, $\int_0^\infty p(t)\,dt=\infty$ *and* $\int_0^\infty |f(t)|\,dt<\infty$. *If* $y(t)$ *is a bounded nonoscillatory solution of (1), then* $y(t)\to 0$ *as* $t\to\infty$.

Proof. Since $y(t)$ is nonoscillatory, from the assumption on $f(t)$ it follows that there is no loss of generality if we take $y(t)$ ultimately positive. Let $y(t)>0$ for $t\geq t_0>0$. To complete the proof of the Theorem it is enough to show that $\overline{\lim\limits_{t\to\infty}}\, y(t)=0$. If possible, let $\overline{\lim\limits_{t\to\infty}}\, y(t)=k$, $k>0$. We consider three cases, viz., (i) $y'(t)$ is ultimately positive, (ii) $y'(t)$ is weakly oscillatory and (iii) $y'(t)$ is ultimately negative and derive a contradiction in each of these cases.

Let $y'(t)>0$ for $t\geq t_1\geq t_0$. Integrating (1) from t_1 to t, we get

$$r(t)\,y'(t)\geq \int_{t_1}^{t} p(s)\,y^{\alpha}(s)\,ds+\int_{t_1}^{t} f(s)\,ds\geq y^{\alpha}(t_1)\int_{t_1}^{t} p(s)\,ds-\int_{t_1}^{t} |f(s)|\,ds.$$

So, $r(t)\,y'(t)>1$ for large t. This in turn implies that $y(t)\to\infty$ as $t\to\infty$, a contradiction.

If possible, let $y'(t)$ be weakly oscillatory. So there exists a sequence $<t_n>$ of zeros of $y'(t)$ such that $t_n\to\infty$ as $n\to\infty$. Let $\sigma>t_0$. Integrating (1) from σ to t_n, we obtain

$$\int_{\sigma}^{t_n} p(t)\,y^{\alpha}(t)\,dt=-r(\sigma)\,y'(\sigma)-\int_{\sigma}^{t_n} f(t)\,dt.$$

From the first mean value theorem of integral calculus it follows that

$$\int_{\sigma}^{t_n} p(t)\,y^{\alpha}(t)\,dt=y^{\alpha}(\xi_n)\int_{\sigma}^{t_n} p(t)\,dt,$$

where

$$\sigma\leq\xi_n\leq t_n. \quad\text{So,}\quad \sigma\leq\lim_{n\to\infty}\xi_n\leq\infty \quad\text{and}$$

$$\overline{\lim_{n\to\infty}}\, y^{\alpha}(\xi_n)=-\lim_{n\to\infty}\Big[\int_{\sigma}^{t_n} p(t)\,dt\Big]^{-1}\Big[r(\sigma)\,y'(\sigma)+\int_{\sigma}^{t_n} f(t)\,dt\Big]=0.$$

But $\overline{\lim\limits_{t\to\infty}}\, y(t)=k>0$ implies that $0<\overline{\lim\limits_{n\to\infty}}\, y^{\alpha}(\xi_n)<\infty$. Hence a contradiction.

Finally, let $y'(t)$ be ultimately negative. So there exists $t_1\geq t_0$ such that $y'(t)<0$ for $t\geq t_1$. Now integrating (1) from t_1 to t yields

$$\int_{t_1}^{t} p(s)\,y^{\alpha}(s)\,ds\leq -r(t_1)\,y'(t_1)-\int_{t_1}^{t} f(s)\,ds,$$

that is,

$$y^{\alpha}(t)\leq\Big[\int_{t_1}^{t} p(s)\,ds\Big]^{-1}\Big[-r(t_1)\,y'(t_1)+\int_{t_1}^{t} |f(s)|\,ds\Big].$$

So $k^{\alpha}=\overline{\lim\limits_{t\to\infty}}\, y^{\alpha}(t)\leq 0$, a contradiction.

Hence the theorem is proved.

Example. $y''-y=-\frac{2(1+\sin t)}{t^2}-\frac{4\cos t}{t^3}+\frac{6(2+\sin t)}{t^4}$, $t>0$, $y(t)=\frac{2+\sin t}{t^2}$ is a bounded nonoscillatory solution which goes to zero as $t\to\infty$.

Theorem 2.5. *Let* $f(t)\geq 0$ *and* $0<\alpha\leq 1$. *If*

$$\int_0^\infty \frac{1}{r(t)}\Big(\int_0^t p(s)\,ds\Big)\,dt<\infty \quad\text{and}\quad \int_0^\infty \frac{1}{r(t)}\Big(\int_0^t f(s)\,ds\Big)<\infty,$$

then all ultimately positive solutions of (2) with $q(t)\geq 0$ *are bounded.*

Proof. Let $y(t)$ be an ultimately positive solution of (2) such that $y(t)>0$ for $t\geq t_0>0$. Let $y'(t)$ be oscillatory with consecutive zeros at a and $b\,(t_0<a<b)$ such that $y'(t)<0$ for $t\in(a,b)$. Integrating (2) from a to b, we obtain

$$0=[r(t)\,y'(t)]_a^b=\int_a^b [f(t)+p(t)\,y^{\alpha}(t)-q(t)\,(y'(t))^{\beta}]\,dt>0,$$

a contradiction. So $y'(t)\geq 0$ or $y'(t)\leq 0$ for large t. If $y'(t)\leq 0$, then $y(t)$ is bounded. If $y'(t)\geq 0$ for $t\geq t_1\geq t_0$, then integrating (2) from t_1 to t, we obtain

$$r(t)\,y'(t)\leq r(t_1)\,y'(t_1)+\int_{t_1}^{t} f(s)\,ds+y^{\alpha}(t)\int_{t_1}^{t} p(s)\,ds,$$

that is,

$$y'(t)\leq R(t)+P(t)\,y^{\alpha}(t),$$

where

$$R(t)=\frac{r(t_1)\,y'(t_1)}{r(t)}+\frac{1}{r(t)}\int_{t_1}^{t} f(s)\,ds \quad\text{and}\quad P(t)=\frac{1}{r(t)}\int_{t_1}^{t} p(s)\,ds. \quad\text{So}$$

$$y^{-\alpha}(t)\,y'(t)\leq y^{\alpha}(t)\,R(t)+P(t)\leq y^{-\alpha}(t_1)\,R(t)+P(t).$$

If $0<\alpha<1$, then

$$\frac{y^{1-\alpha}(t)}{1-\alpha}\leq\frac{y^{1-\alpha}(t_1)}{1-\alpha}+y^{-\alpha}(t_1)\int_{t_1}^{t} R(s)\,ds+\int_{t_1}^{t} P(s)\,ds.$$

If $\alpha=1$, then

$$\log y(t)\leq\log y(t_1)+y^{-1}(t_1)\int_{t_1}^{t} R(s)\,ds+\int_{t_1}^{t} P(s)\,ds.$$

Let $c\in(t_1,t)$ be fixed. Then

$$\int_{t_1}^{t}\frac{1}{r(s)}\left(\int_{t_1}^{t}p(\theta)\,d\theta\right)ds=L+\int_{c}^{t}\frac{1}{r(s)}\left(\int_{t_1}^{s}p(\theta)\,d\theta\right)ds\geqslant$$
$$\geqslant L+\left(\int_{c}^{t}\frac{ds}{r(s)}\right)\left(\int_{t_1}^{c}p(\theta)\,d\theta\right),$$

where $L=\int_{t_1}^{c}\frac{1}{r(s)}\left(\int_{t_1}^{s}p(\theta)\,d\theta\right)ds$. Hence $\int_{0}^{\infty}\frac{1}{r(t)}\left(\int_{0}^{t}p(s)\,ds\right)dt<\infty$ implies that $\int_{0}^{\infty}\frac{dt}{r(t)}<\infty$. This in turn implies that $\int_{0}^{\infty}R(t)\,dt<\infty$. As a consequence, $\overline{\lim_{t\to\infty}}\ y(t)<\infty$.

Hence the theorem is proved.

Theorem 2.6. *If $f(t)\geqslant 0$, $0<\alpha\leqslant 1$ and $\int_{0}^{\infty}\frac{1}{r(t)}\left(\int_{0}^{t}p(s)\,ds\right)dt<\infty$, then all ultimately negative solutions of (1) are bounded.*

For the proof see [7].

Corollary 2.7. *Suppose that the conditions of Theorem 2.5 are satisfied. Then all non-oscillatory solutions of (1) with $0<\alpha\leqslant 1$ are bounded.*

Example. $(t^3y')'-1/t^2\ y=1/t^2-1/t^4$, $t>1$.
$y(t)=1/t^2-1$, $t>1$, is a bounded nonoscillatory solution of the equation.

Remark. Corollary 2.7 does not hold good for $\alpha>1$. This is clear from the equation

$$(t^3y')'-(3/t^3-1/t^7)\ y^3=1/t^3,\quad t\geqslant 1,$$

which admits an unbounded nonoscillatory solution $y(t)=t$ even though the conditions of Theorem 2.5 are satisfied.

REFERENCES

1. Chang, S. H. — *Boundedness theorems for certain second order nonlinear differential equations*, J. Math. Anal. Appl., 31 (1970), 509–516.
2. Graef, J. R. and Spikes P. W. — *Asymptotic behaviour of solutions of a second order nonlinear differential equation*, J. Diff. Eqs., 17 (1975), 461–476.
3. Hatvani, L. — *On the asymptotic behaviour of the solutions of $(p(t)\,x')'+q(t)\,f(x)=0$*, Pub. Math. Debrecen, 19 (1972), 225–237.
4. Marini, M. and Zezza P. — *On the asymptotic behaviour of the solutions of a class of second order linear differential equations*, J. Diff. Eqs., 28 (1978), 1–17.
5. Parhi, N. and Nayak, S. K. — *On nonoscillatory behaviour of solutions of nonlinear differential equations*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 65 (1978), 58–62.
6. Parhi, N. and Nayak S. K. — *Nonoscillation of second order nonhomogeneous differential equations*, J. Math. Anal. Appl., 102 (1984), 62–74.
7. Parhi, N. — *On nonoscillatory solutions of second order differential inequalities*, Acta Math. Hung., 42 (1983), 101–109.
8. Spikes, P. W. — *Behaviour of solutions of the differential equation $y''+q(t)\,y^p=r(t)$* Appl. Anal., 4 (1974), 253–264.
9. Taliaferro, S. D. — *Asymptotic behaviour of solutions of $y''=\varphi(t)\,y^{\lambda}$*, J. Math. Anal. Appl., 66 (1978), 95–134.
10. Teufel, H. (Jr.) — *On the behaviour of solutions of sublinear second order differential equations*, Proc. Amer. Math. Soc., 32 (1972), 445–451.

Received 10.VIII. 1985

Department of Mathematics
Berhampur University
Berhampur — 760007
Orissa, India

Analele științifice ale Universității „Al. I. Cuza" din Iași
Tomul XXXII, s. I a, Matematică, 1986, f. 3

SUR LE PROBLEME DE DARBOUX POUR L'ÉQUATION $\frac{\partial^2 z}{\partial x\partial y}\in F(x,y,z)$

PAR

GEORGETA TEODORU

Ayant pour point de départ le cas classique ([1], p. 149, [2], p. 209) dans les travaux [3], [4] nous avons défini le problème de Darboux pour l'équation

(1) $$\frac{\partial^2 z}{\partial x\partial y}\in F(x,y,z),\quad (x,y)\in D=\{(x,y)\mid 0\leqslant x\leqslant a,\ 0\leqslant y\leqslant b\},$$

comme le problème de la détermination d'une solution de cette équation, qui satisfasse aux conditions

(2) $$z(x,0)=\sigma(x),\ z(0,y)=\tau(y),\ \sigma(0)=\tau(0),\ (x,y)\in D,\ z\in\Omega,$$

en admettant les hypothèses suivantes :

(H_1) $F:D\times\Omega\to 2^{R^n}$ est une application multivoque à valeurs compactes convexes et non vides de $\mathbf{R}^n$, où Ω est un ensemble ouvert dans $\mathbf{R}^n$,

(H_2) Pour tout $(x,y)\in D$, l'application $z\to F(x,y,z)$ est semi-continue supérieurement sur Ω,

(H_3) Pour tout $z\in\Omega$, l'application $(x,y)\to F(x,y,z)$ est mesurable par rapport à la mesure de Lebesgue sur D,

(H_4) Il existe une fonction $g:D\to\mathbf{R}_+$, $g\in\mathcal{L}^1(D;\mathbf{R}_+)$, telle que $\|u\|\leqslant g(x,y)$, $\forall u\in F(x,y,z)$, $\forall(x,y)\in D$, $\forall z\in\Omega$,

(H_5) Il existe un ensemble convexe, compact, $M\subset\Omega$ et un point $(x_0,y_0)\in\,]0,a]\times]0,b]$, tels que

$$\int_0^{x_0}\int_0^{y_0}g(\xi,\eta)\,d\xi d\eta\leqslant d(M,C_\Omega),$$

où $d(M,C_\Omega)$ est la distance de M à $C_\Omega=\mathbf{R}^n-\Omega$,

(H_6) Les fonctions $\sigma\in AC([0,a];\mathbf{R}^n)$, $\tau\in AC([0,b];\mathbf{R}^n)$ satisfont à la condition $\sigma(0)=\tau(0)$,

(H_7) Les valeurs de la fonction $\alpha:D\to\mathbf{R}^n$, définie par $\alpha(x,y)=\sigma(x)+\tau(y)-\sigma(0)$, $(x,y)\in D$, appartiennent à l'ensemble M, lorsque $(x,y)\in\Delta=[0,x_0]\times[0,y_0]\subseteq D$.