Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXII, s. I a, Matematică, 1986, f. 3

A DISCRETE INTEGRAL INEQUALITY OF CONVOLUTION TYPE IN TWO INDEPENDENT VARIABLES

BY

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In this paper, we extend the author's results of the paper [1] to the case of two independent variables. We shall use the notion of Laurent transform for the double sequences, which is the natural generalization of the Laurent transform (called also z-transform) from the case of the common sequences. It is well known that from the inequality

$$u(x, y)\leqslant f(x, y)+\int_0^x\int_0^y a(x-s, y-t)u(s, t)ds\, dt, \quad x, y\geqslant 0, \tag{1}$$

where all the functions are continuous for $x, y\geqslant 0$ and $a(x, y)\geqslant 0$, it follows

$$u(x, y)\leqslant f(x, y)+\int_0^x\int_0^y r(x-s, y-t)f(s, t)ds\, dt, \quad x, y\geqslant 0 \tag{2}$$

where $r=r(x, y)\geqslant 0$ is the resolvent kernel corresponding to $a=a(x, y)$. The discrete analogue of (1) is

$$u(m, n)\leqslant f(m, n)+\sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{n-1} a(m-1-\alpha, n-1-\beta)u(\alpha, \beta) \tag{3}$$

where $a(m, n)\geqslant 0$ for all integers $m, n\geqslant 0$. Throughout this paper it is assumed that

$$\sum_{\alpha=p}^{k} c(\alpha)=0 \quad \text{and} \quad \prod_{\alpha=p}^{k} c(\alpha)=1 \quad \text{if} \quad k<p \tag{4}$$

For $m=0$ or $n=0$, we have from (3) $u(0, n)\leqslant f(0, n)$ or $u(m, 0)\leqslant f(m, 0)$. We remark that if the sequence $v=\{v(m, n)\}$ is the solution of the system

$$v(m, n)=f(m, n)+\sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{n-1} a(m-1-\alpha, n-1-\beta)v(\alpha, \beta), \quad m, n\geqslant 0 \tag{5}$$

then we have $u(m, n)\leqslant v(m, n)$ for all $m, n\geqslant 0$. This system may be solved using the Laurent transform for sequences.

Definition 1. *We say that a given sequence $b=\{b(m, n)\}$ has the property* (L), *if there exist $\rho_1>0$, $\rho_2>0$ such that*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |b(m, n)|\rho_1^{-m}\rho_2^{-n}<\infty \tag{6}$$

Of course, ρ_1 and ρ_2 depend on $b=\{b(m,n)\}$.

Definition 2. *For the sequence b having the property (L), the Laurent transform is the function $B(w,z)$, given by*

$$B(w,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b(m,n)w^{-m}z^{-n}, \quad |w|>\rho_1, \ |z|>\rho_2 \tag{7}$$

Assuming that $a=\{a,(m,n)\}$, $f=\{f(m,n)\}$ have the property (L) and denoting by $A(w,z)$, $F(w,z)$their Laurent transforms, we obtain from (5)

$$V(w,z)=F(w,z)+(1/wz)A(w,z)V(w,z), \tag{8}$$

where $V(w,z)$ is the Laurent transform of $v=\{v(m,n)\}$, defined for $|w|$ and $|z|$—sufficiently large. From (8) we have

$$\begin{cases} V(w,z)=F(w,z)+(1/wz)R(w,z)F(w,z) \\ R(w,z)=(wzA(w,z)/(wz-A(w,z))). \end{cases} \tag{9}$$

The function $R(w,z)$ may be written as

$$R(m,n)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r(m,n)w^{-m}z^{-n}, \tag{10}$$

where the coefficients $r(m,n)$ are given by the formula

$$r(m,n)=(2\pi i)^{-2}\iint R(w,z)w^{m-1}z^{n-1}\,dw\,dz, \tag{11}$$

the integral being calculated on the product of the circumferences $|w|=\delta_1$, $|z|=\delta_2$ with δ_1, δ_2 = sufficiently large such that this product lies in the domain $|w|>\rho_1$, $|z|>\rho_2$ in which $R(w,z)$ is holomorphic. Using the expansion

$$R(w,z)=A(w,z)\sum_{p=0}^{\infty}(A(w,z)/wz), \tag{12}$$

true for $|w|$, $|z|$ sufficiently large, we conclude that $r(m,n)\geq 0$ for all $m,n\geq 0$, if we remember that $A(w,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a(m,n)w^{-m}z^{-n}$, where $a(m,n)\geq 0$ for $m,n\geq 0$. From (9) it follows that the solution of the system (5) is given by

$$v(m,n)=f(m,n)+\sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{n-1} r(m-1-\alpha,n-1-\beta)f(\alpha,\beta), \tag{13}$$

and thus we can state

Theorem 1. *Assume that a and f have the property (L) and that $a(m,n)\geq 0$ for every $m,n\geq 0$. Then we have from (3) the bound*

$$u(m,n)\leq f(m,n)+\sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{n-1} r(m-1-\alpha,n-1-\beta)f(\alpha,\beta) \tag{14}$$

where $r(m,n)$ are given by (11). If f is nondecreasing, then

$$u(m,n)\leq f(m,n)\Big(1+\sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{n-1} r(\alpha,\beta)\Big), \qquad m,n\geq 0 \tag{15}$$

Corollary 1.1. *Suppose that in (3) $u(m,n)\geq 0$, $a(m,x)\geq 0$, $f(m,n)>0$ for all $m,n\geq 0$, f is nondecreasing and a has the property (L). Then the bound (15) remains true.*

Proof. In this case, from (3) we obtain

$$u(m,n)/f(m,n)\leq 1+\sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{n-1} a(m-1-\alpha,n-1-\beta)(u(\alpha,\beta)/f(\alpha,\beta)) \tag{16}$$

and we can consider the sequence u/f instead of u. From Theorem 1, we have

$$u(m,n)/f(m,n)\leq 1+\sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{n-1} r(\alpha,\beta), \qquad m,n\geq 0 \tag{17}$$

and thus (15) follows.

Corollary 1.2. *Assume that*

(i) the sequences a and f have the property (L),

(ii) a is nonnegative and f is nondecreasing,

(iii) $A(w,z)$ is defined and $A(w,z)-wz\neq 0$ for $|w|>\rho_1$, $|z|>\rho_2$ with $\rho_1,\rho_2\in(0,1)$.

Then we obtain from (3) the bound

$$u(m,n)\leq Mf(m,n), \qquad \text{where} \quad M=\Big(1-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a(m,n)\Big)^{-1}. \tag{18}$$

Proof. Under the cited above hypotheses, $R(w,z)$ defined by (9) is holomorphic for $|w|>\rho_1$, $|z|>\rho_2$ and we can put $w=1$, $z=1$ in (10) to conclude that the series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a(m,n)$ converges and has the sum $R(1,1)$. We obtain (18) directly from (15), because

$$1+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}r(m,n)=1+R(1,1)=(1-A(1,1))^{-1}=\Big(1-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a(m,n)\Big)^{-1}. \tag{19}$$

We observe that our hypotheses imply $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a(m,n)<1$. An example of the sequence $\{a(m,n)\}$ which satisfies the conditions imposed in the Corollary 1.2 is given by $a(m,n)=mna^m b^n$ with $a,b\in(0,(3-\sqrt{5})/2)$.

Corollary 1.3. *Suppose that in (3)*

(i)′ $u(m,n)\geq 0$, $f(m,n)>0$ for all $m,n\geq 0$

and the conditions (ii), (iii) of the Corollary 1.2 are fulfiled. Then the bound (18) remains true.

The following lemma is a generalization of S.M. Singare and B.G. Pachpatte Theorem 1 of the paper [2]:

Lemma. *Assume that*

$$u(m,n)\leq f(m,n)+g(m,n)\sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{n-1}h(\alpha,\beta)u(\alpha,\beta), \qquad m,n\geq 0 \tag{20}$$

where the sequences u, g, h are nonnegative, f is positive, f and g are nondecreasing. Then for all $m, n \geqslant 0$ it follows

$$u(m,n) \leqslant f(m,n) \prod_{\alpha=0}^{m-1} \left(1 + g(m,n) \sum_{\beta=0}^{n-1} h(\alpha,\beta)\right) \tag{21}$$

Proof. Denoting $v(m,n) = u(m,n)/f(m,n)$, we obtain

$$v(m,n) \leqslant 1 + g(m,n) \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} h(\alpha,\beta) v(\alpha,\beta), \tag{22}$$

We put $\varphi(m,n) = v(m,n)$ for $m \leqslant m_0$, $n \leqslant n_0$ and $\varphi(m,n) = 0$ otherwise. Obviously we have

$$\varphi(m,n) \leqslant 1 + \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} g(m_0,n_0) h(\alpha,\beta) \varphi(\alpha,\beta), \tag{23}$$

for all $m, n \geqslant 0$ and consequently, accordingly to the Theorem 1 [2], we conclude that

$$\varphi(m,n) \leqslant \prod_{\alpha=0}^{m-1} \left(1 + g(m_0,n_0) \sum_{\beta=0}^{n-1} h(\alpha,\beta)\right), \tag{24}$$

also for all $m, n \geqslant 0$. From (24) we obtain that (21) is true for $m = m_0$ and $n = n_0$. Because these integers were taken arbitrary, we conclude that (21) is true for all $m, n \geqslant 0$.

Remark 1. If suppose f = nonnegative, the other hypotheses being fulfiled, the conclusion of the Lemma 1 also holds.

Remark 2. Following the same way as in the preceding Lemma 1, supposing that f and g are nondecreasing, we can give bounds for u, starting from the inequalities

$$u(m,n) \leqslant f(m,n) + g(m,n) \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} h(\alpha,\beta) W(u(\alpha,\beta)), \tag{25}$$

or

$$u(m,n) \leqslant f(m,n) + g(m,n) \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} h(\alpha,\beta) \left[u(\alpha,\beta) + \sum_{\mu=0}^{\alpha-1} \sum_{\nu=0}^{\beta-1} c(\mu,\nu) . u(\mu,\nu)\right], \tag{26}$$

which in the case $g(m,n) = 1$ were considered in the papers [2], [3]. Then, inequalities more general than (25) and (26), obtained by the addition in the right side of the linear term

$$\sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} a(m-1-\alpha, n-1-\beta) u(\alpha,\beta) \tag{27}$$

may be also investigated by the method which will be used in what follows. Now, we consider the discrete inequality

$$\begin{aligned} u(m,n) \leqslant f(m,n) + g(m,n) \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} h(\alpha,\beta) u(\alpha,\beta) + \\ + \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} a(m-1-\alpha, n-1-\beta) u(\alpha,\beta), \end{aligned} \tag{28}$$

which corresponds to the linear integral inequality

$$u(x,y) \leqslant f(x,y) + g(x,y) \int_0^x \int_0^y h(s,t) u(s,t) ds\, dt + \int_0^x \int_0^y a(x-s, y-t) u(s,t)\, ds dt, \tag{29}$$

where all the involved functions are continuous and nonnegative for x, $y \geqslant 0$. Denote

$$f_1(m,n) = f(m,n)\left(1 + \sum_{\alpha=0}^{m-1} \sum_{\alpha=0}^{n-1} r(\alpha,\beta)\right), \; g_1(m,n) = g(m,n)\left(1 + \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} r(\alpha,\beta)\right) \tag{30}$$

where $r = \{r(m,n)\}$ is given by (10)+(11).

Theorem 2. *Assume that in* (28) :

(a) *the sequence* $\mathfrak{A}$ *has the property* (L),

(b) u, f, g, h, a *are nonegative sequences*,

(c) f *and* g *are nondecreasing*,

Then for all $m, n \geqslant 0$ *we have*

$$u(m,n) \leqslant f_1(m,n) \prod_{\alpha=0}^{m-1} \left(1 + g_1(m,n) \sum_{\beta=0}^{n-1} h(\alpha,\beta)\right) \tag{31}$$

Proof. Denoting

$$F(m,n) = f(m,n) + g(m,n) \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} h(\alpha,\beta) u(\alpha,\beta), \tag{32}$$

we can write

$$u(m,n) \leqslant F(m,n) + \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} a(m-1-\alpha, n-1-\beta) u(\alpha,\beta) \tag{33}$$

for all $m, n \geqslant 0$. Taking into account that F is nondecreasing and making use of the Corollary 1.1, we obtain

$$u(m,n) \leqslant F(m,n)\left(1 + \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} r(\alpha,\beta)\right) \tag{34}$$

and from (32), (30) it follows

$$u(m,n) \leqslant f_1(m,n) + g_1(m,n) \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} h(\alpha,\beta) u(\alpha,\beta). \tag{35}$$

The Lemma 1, whose hypotheses are satisfied, implies that the bound (31) is true.

Corollary 2.1. *Under the hypotheses of Theorem* 2, *we obtain from* (28) *the bound*

$$(m,n) \leqslant f_1(m,n) \exp\left(g_1(m,n) \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} h(\alpha,\beta)\right), \qquad m, n \geqslant 0 \tag{36}$$

The relation (36) easily follows from (31), using the elementary inequality

$$\prod_{\alpha=0}^{m} \left(1 + \sum_{\beta=0}^{n} c(\alpha,\beta)\right) \leqslant \exp\left(\sum_{\alpha=0}^{m} \sum_{\beta=0}^{n} c(\alpha,\beta)\right) \tag{37}$$

where all $c(\alpha, \beta)$ are nonnegative.

Corollary 2.2. *Suppose that in* (28) *the hypotheses* (a), (b), (c) *of Theorem* 2 *and* (iii) *of Corollary* 1.2 *are fulfiled. Suppose also that the sequences* $\{f(m,n)\}$ *and*

$$\{g_2(m, n)\} = \{g(m, n) \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} h(\alpha, \beta)\} \tag{38}$$

are bounded. Then $\{u(m, n)\}$ *is also bounded.*

Proof. With M defined by (18), using (30), (36) and (38), we have

$$u(m, n) \leqslant Mf(m, n)\exp(Mg_2(m, n)), \tag{39}$$

which implies that u is bounded.

The established inequalities may be applied to obtain bounds for the solution of the system

$$u(m, n) = f(m, n) + g(m, n) \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} h(\alpha, \beta) u(\alpha, \beta) + \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} a(m-1-\alpha, n-1-\beta) u(\alpha, \beta). \tag{40}$$

If we denote $|u(m, n)| = U(m, n)$, $|f(m, n)| = F(m, n)$ etc. then we obtain from (40)

$$U(m, n) \leqslant F(m, n) + G(m, n) \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} H(\alpha, \beta) U(\alpha, \beta) + \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} A(m-1-\alpha, n-1-\beta) U(\alpha, \beta), \tag{41}$$

and we can study the behaviour of the sequence $\{|u(m, n)|\}$. If denote by u_i $(i=1, 2)$ the solutions of (40) corresponding to the free terms f_i $(i=1, 2)$, we obtain bounds for $|u_1 - u_2| = V$, writting

$$V(m, n) \leqslant K + G(m, n) \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} H(\alpha, \beta) V(\alpha, \beta) + \sum_{\beta=0}^{m-1} \sum_{\beta=0}^{n-1} A(m-1-\alpha, n-1-\beta)\, V(\alpha, \beta) \tag{42}$$

$K > 0$ being a constant such that $|f_1(m, n) - f_2(m, n)| \leqslant K$ for all $m, n \geqslant 0$.

Remark 3. In the majorations here obtained appear the nonnegative coefficients $r(m, n)$ given by (10) and (11). Of course, it is difficult to write down the expansion (10), but at least in the case when a is „sufficiently small", more precisely when the hypothesis (iii) of the Corollary 1.2 is fulfiled, we obtain a good information on the behaviour of the sequence $\{u(m, n)\}$ without making use of the concrete values $r(m, n)$ (see the Corollary 2.2).

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Received 21 II 1986

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXII, s. I a, Matematică, 1986, f. 3

QUELQUES METHODES POUR LE CONTROLE OPTIMAL DE PROBLEMES COMPORTANT DES CONTRAINTES SUR L'ETAT

PAR

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I. Introduction. Nous illustrons quelques méthodes pour l'analyse de problèmes de contrôle comportant des contraintes sur l'état, en étudiant un système parabolique soumis à une contrainte locale sur l'état final. Si on impose à l'état d'être essentiellement borné, la méthode des espaces adaptés permet d'exprimer les conditions d'optimalité. Si on impose seulement la positivité de l'état final, un problème auxiliaire permet d'obtenir certaines conditions d'optimalité. Dans les deux cas, les conditions d'optimalité permettent de déduire un résultat de régularité sur le contrôle optimal. Si l'équation d'état est non linéaire, les conditions d'optimalité sont en général obtenues sous une forme non qualifiée. Dans un cas particulier, nous montrons que le problème est qualifié.

II. Position du problème. Soit un ouvert borné de $\mathbf{R}^n$, de frontière Ω régulière. Soit $T>0$; notons $Q = \Omega \times]0, T[$ et $\Sigma = \Gamma \times]0, T[$. Considérons le système:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = u \text{ p.p. dans } Q, \\ \dfrac{\partial y}{\partial n} = 0 \text{ sur } \Sigma, \\ y(\cdot, 0) \equiv 0 \text{ dans } \Omega. \end{cases} \tag{1}$$

Pour u donné dans $L^2(Q)$, nous savons que (1) admet une solution unique y_u dans

$$H^{2,1}(Q) = \{y \in H^1(Q)\,;\ y \in L^2(0, T, H^2(\Omega))\}\,.$$

Soit z_d dans $L^2(\Omega)$ et $N>0$. Le problème est

$$\text{(P)} \quad \begin{cases} \min J(u) = \dfrac{N}{2} \displaystyle\int_Q u^2 dx dt + \dfrac{1}{2} \int_\Omega (y_u(x, T) - z_d(x))^2\, dx, \\ u \geqslant 0 \text{ p.p. sur } Q,\ y_u(\cdot, T) \in K_Y, \end{cases}$$

où K_Y est défini par l'une des deux relations suivantes:

$$\text{(H1)} \quad K_Y = \{y \in H^{2,1}(Q)\,;\ |y(x, T)| \leqslant 1,\ \text{p.p. } x \in \Omega\}\,,$$