7. Ekeland, I. et Temam, R. — *Analyse convexe et problèmes variationnels*, Dunod Gauthier-Villars, Paris, 1974.
8. Lions, J. L. — *Contrôle optimal des systèmes gouvernés par des équations aux dérivées partielles*, Dunod Gauthier-Villars, Paris, 1968.
9. Lions, J. L. — *Function spaces and optimal control of distributed systems*, Cours à l'Université Fédérale de Rio de Janeiro, 1980.
10. Lions, J. L. — *Contrôle de systèmes distribués singuliers*, Gauthier-Villars, Paris, 1983.
11. Lions, J. L. Magenes, E. — *Problèmes aux limites non homogènes et applications* Dunod, Paris, 1968.
12. Mackenroth, U. — *Convex parabolic boundary control problems with pointwise state contraints*, J. Math. An. Appl., 87 (1982), 256—277.
13. Stampacchia, G. — *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier Grenoble, 15 (1965), 185—258.

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INRIA,
Rocquencourt, BP.105
78153 Le Chesnay
France

Dep. Ecuaciones Funcionales
Focultud de Ciencias
35005 Santander
Espagne

Analele științifice ale Universității „Al. I. Cuza" din Iași
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MAXIMUM PRINCIPLE FOR CONTROL PROBLEMS GOVERNED BY A CLASS OF FINITE DIMENSIONAL VARIATIONAL INEQUALITIES

BY

CĂTĂLINA DAVIDEANU

Optimal control problems governed by parabolic and elliptic variational inequalities have been studied in [1]. Here we shall study with the same method some specific cases not covered by the results established there.

This paper is concerned with necessary optimality conditions (maximum principle) for control problems governed by finite dimensional inequalities. Problems of this type appear by the discretization of parabolic variational inequalities (see Example 2.2 below) and in the modelisation of a large class of physical problems.

1. **Necessary optimality conditions**. In this section, we consider the control problem of the following type:

Problem (P_K) : Minimize

$$\int_0^T [g(t, y(t))+h(u(t))]\,dt+\varphi_0(y(T)), \quad 0<T<\infty, \tag{1.1}$$

over the set of all functions $y\in W^{1,2}([0, T] ; R^n)$, $u\in L^2(0, T ; R^m)$ subject to:

$$y'(t)+Fy(t)+\partial I_K(y(t))\ni Bu(t)+f(t) \quad \text{a.e. } t\in]0, T[\tag{1.2}$$

$$y(0)=y_0 \tag{1.3}$$

where $K=\{z\in R^n ; 0\leqslant z_i\leqslant a_i ; i=1, 2, \dots, n\}$.

Using the definition of the operator ∂I_K, the relations (1.2) and (1.3) can be equivalently written as:

$$y_i'(t)+(Fy(t))_i=(Bu(t))_i+f_i(t), \qquad \text{a.e. } \{t ; 0<y_i(t)<a_i\}, \tag{1.4}$$

$$y_i'(t)+(Fy(t))_i\geqslant (Bu(t))_i+f_i(t), \qquad \text{a.e. } \{t ; y_i(t)=0\}, \tag{1.5}$$

$$y_i'(t)+(Fy(t))_i\leqslant (Bu(t))_i+f_i(t), \qquad \text{a.e. } \{t ; y_i(t)=a_i\}, \tag{1.6}$$

$$0\leqslant y_i(t)\leqslant a_i \qquad \text{a.e. } t\in]0, T[, \tag{1.7}$$

$$y_i(0)=y_{0,i}\in K, \tag{1.8}$$

where $i=1, 2, \dots, n$.

Further on, we shall assume that:

(a) $F : R^n\to R^n$ is a local Lipschitzian function and

$(Fy, y) \geq -D_1\|y\|_n^2 - D_2$ for all $y \in R^n$, where $D_1, D_2 \in R_+^*$.

(b) $B : R^m \to R^n$ is a linear continuous operator.

(c) $f \in W^{1,1}([0, T] ; R^n)$, $\varphi_0 : R^n \to R_+$.

(d) $h : R^m \to \overline{R}$ is a convex, lower semicontinuous function and $h(u) \geq b_1\|u\|_m^2 - b_2$ for all $u \in R^m$, where $b_1 \in R_+^*$, $b_2 \in R$.

(e) the function $g: [0, T] \times R^n \to R_+$ is measurable in t, $g(t, 0) \in L^\infty(0, T)$ and for every $\delta > 0$ there exists $L_\delta > 0$, independent of t, so that $|g(t, y) - g(t, z)| + |\varphi_0(y) - \varphi_0(z)| \leq L_\delta\|y - z\|_n$ for all $t \in [0, T]$ and $\|y\|_n + \|z\|_n \leq \delta$.

If $u \in L^2(0, T ; R^m)$ then the finite dimensional variational inequality (1.2) with the initial condition (1.3) has a unique solution $y \in W^{1,2}([0, T] ; R^n)$ and the control problem (P_K) has, at least, one optimal pair.

The maximum principle is presented by the following result :

Theorem 1.1. *Let $(y^*, u^*) \in W^{1,2}([0, T] ; R^n) \times L^2(0, T ; R^m)$ be an arbitrary optimal pair of the problem (P_K).*

Then there exist $p \in BV([0, T] ; R^n)$, the dual extremal arc, and $\xi \in L^\infty(0, T ; R^n)$ so that :

(1.9) $\quad p_i'(t) - ((\partial F(y^*(t)))^* p(t))_i = \xi_i(t), \quad$ a.e. $\{t ; 0 < y_i^*(t) < a_i\}$,

(1.10) $\quad \xi(t) \in \partial g(t, y^*(t)) \quad$ a.e. $t \in]0, T[$

(1.11) $\quad p_i(t) = 0 \quad$ a.e. $\{t ; y_i^*(t) = 0 ;\ (Bu^*(t))_i + f_i(t) \neq F(0))_i\}$,

(1.12) $\quad p_i(t) = 0 \quad$ a.e. $\{t ; y_i^*(t) = a_i ;\ (Bu^*(t))_i + f_i(t) \neq F(a))_i\}$,

(1.13) $\quad p(T) + \partial\varphi_0(y^*(T)) \ni 0$,

(1.14) $\quad B^* p(t) \in \partial h(u^*(t)), \quad$ a.e. $t \in]0, T[$.

Proof. The idea of proof consists of approximating the control problem (P_K) by a family of smooth problems (P_K^ε) for which the necessary optimality conditions are immediate, and then passing to limit in the approximating equations (see [1]).

We consider the sequence of the approximative problems $P_K^\varepsilon(\varepsilon > 0)$.

Problem (P_K^ε) : Minimize

$$\int_0^T [g^\varepsilon(t, y(t)) + h(u(t)) + 1/2\|u(t) - u^*(t)\|_m^2]dt + \varphi_0^\varepsilon(y(T))$$

on all (y, u) subject to :

$$\begin{cases} y'(t) + F^\varepsilon y(t) + \gamma^\varepsilon(y(t)) = Bu(t) + f(t) \text{ a.e. } t \in]0, T[, \\ y(0) = y_0, \end{cases}$$

where $\gamma^\varepsilon : R^n \to R^n$ is defined by $\gamma^\varepsilon(y) = (\beta_1^\varepsilon(y_1), \ldots, \beta_n^\varepsilon(y_n))$ and $\beta_i^\varepsilon(y_i) = \int_R [\beta_{\varepsilon i}(y_i - \varepsilon^2\theta) - \beta_{\varepsilon i}(-\varepsilon^2\theta)]\rho(\theta)d\theta + \beta_{\varepsilon i}(0)$, $i = 1, 2, \ldots, n$.

Here ρ is a C_0^∞-mollifier function on R and

$$\beta_{\varepsilon i}(y_i) = \frac{1}{\varepsilon}(y_i - P_i(y_i)) = \begin{cases} \varepsilon^{-1}y_i & \text{if } y_i \leq 0 \\ 0 & \text{if } 0 < y_i \leq a_i \\ \varepsilon^{-1}(y_i - a_i) & \text{if } y_i > a_i \end{cases}$$

Knowing the expression of β_ε, we calculate the two integrals from the formula of β^ε and we have :

$$I_1 = \int_R \beta_{\varepsilon i}(y_i - \varepsilon^2\theta)\rho(\theta)d\theta = \int_{-\infty}^{\varepsilon^{-2}(y_i - a_i)} \beta_{\varepsilon i}(y_i - \varepsilon^2\theta)\rho(\theta)d\theta + \int_{\varepsilon^{-2}(y_i - a_i)}^{\varepsilon^{-2}y_i} \beta_{\varepsilon i}(y_i - \varepsilon^2\theta)\rho(\theta)d\theta +$$

$$+ \int_{\varepsilon^{-2}y_i}^{\infty} \beta_{\varepsilon i}(y_i - \varepsilon^2\theta)\rho(\theta)d\theta = \frac{1}{\varepsilon}\int_{-\infty}^{\varepsilon^{-2}(y_i - a_i)} (y_i - \varepsilon^2 - a_i)\rho(\theta)d\theta + \frac{1}{\varepsilon}\int_{\varepsilon^{-2}y_i}^{\infty} (y_i - \varepsilon^2\theta)\rho(\theta)d\theta =$$

$$= \frac{y_i - a_i}{\varepsilon}\int_{-\infty}^{\varepsilon^{-2}(y_i - a_i)} \rho(\theta)d\theta + \frac{y_i}{\varepsilon}\int_{\varepsilon^{-2}y_i}^{\infty} \rho(\theta)d\theta - \varepsilon\int_{-\infty}^{\varepsilon^{-2}(y_i - a_i)} \theta\rho(\theta)d\theta - \varepsilon\int_{\varepsilon^{-2}y_i}^{\infty} \theta\rho(\theta)d\theta.$$

$$I_2 = \int_R \beta_{\varepsilon i}(-\varepsilon^2\theta)\rho(\theta)d\theta = \int_{\varepsilon^{-2}a_i}^{-\varepsilon^{-2}a_i} \beta_{\varepsilon i}(-\varepsilon^2\theta)\rho(\theta)d\theta + \int_{-\varepsilon^{-2}a_i}^{0} \beta_{\varepsilon i}(-\varepsilon^2\theta)\rho(\theta)d\theta +$$

$$+ \int_0^\infty \beta_{\varepsilon i}(-\varepsilon^2\theta)\rho(\theta)d\theta = \frac{1}{\varepsilon}\int_{-\infty}^{\varepsilon^{-2}a_i} (-\varepsilon^2\theta - a_i)\rho(\theta)d\theta + \frac{1}{\varepsilon}\int_0^\infty (-\varepsilon^2\theta)\rho(\theta)d\theta =$$

$$= -\varepsilon\int_0^1 \theta\rho(\theta)d\theta.$$

In the calculation of the two given above integrals we have used the definition of the function β_ε and the properties of the function ρ. Moreover, we have made the non-restricting supplementary hypothesis that $a_i \geq \varepsilon^2$ for $i = 1, 2, \ldots, n$. Thus :

$$\beta_i^\varepsilon(y_i) = \frac{y_i - a_i}{\varepsilon}\int_{-\infty}^{\varepsilon^{-2}(y_i - a_i)} \rho(\theta)d\theta + \frac{y_i}{\varepsilon}\int_{\varepsilon^{-2}y_i}^{\infty} \rho(\theta)d\theta + \varepsilon\left(\int_0^1 \theta\rho(\theta)d\theta - \int_{-\infty}^{\varepsilon^{-2}(y_i - a_i)} \theta\rho(\theta)d\theta - \int_{\varepsilon^{-2}y_i}^{\infty} \theta\rho(\theta)d\theta\right).$$

By derivation we have :

$$\dot{\beta}_i^\varepsilon(y_i) = \frac{1}{\varepsilon}\int_{-\infty}^{\varepsilon^{-2}(y_i - a_i)} \rho(\theta)d\theta + \frac{1}{\varepsilon}\int_{\varepsilon^{-2}y_i}^{\infty} \rho(\theta)d\theta.$$

The function g^ε and φ_0^ε are defined by the relations:

$$g^\varepsilon(t, y) = \int_{R_n} g(t, y - \varepsilon^2\theta)\rho_n(\theta)d\theta,$$

$$\varphi_0^\varepsilon(y) = \int_{R_n} \varphi_0(y - \varepsilon^2\theta)\rho_n(\theta)d\theta,$$

where ρ is a C_0^∞-mollifier function on R.

Let $(y^\varepsilon, u^\varepsilon)$ be an optimal pair for (P_K^ε) and $p^\varepsilon \in BV([0, T]; R^n)$ the corresponding dual extremal arc, which satisfies the relations:

(1.15) $$p^{\varepsilon\prime}(t) - \left(\frac{\partial F^\varepsilon}{\partial y}(y^\varepsilon(t))\right)^* p^\varepsilon(t) - \nabla\gamma^\varepsilon(y^\varepsilon(t))p^\varepsilon(t) = \nabla g^\varepsilon(t, y^\varepsilon(t)) \text{ a.e. } t \in]0, T[,$$

(1.16) $$p^\varepsilon(T) + \nabla\varphi_0^\varepsilon(y^\varepsilon(T)) = 0,$$

(1.17) $$B^* p^\varepsilon(t) \in \partial h(u^\varepsilon(t)) + u^\varepsilon(t) - u^*(t) \text{ a.e. } t \in]0, T[.$$

If $u^\varepsilon \to u^*$ strongly in $L^2(0, T; R^m)$, we can show that, for $\varepsilon \to 0$, we get the convergences:

(1.18) $$y^\varepsilon \to y^* \text{ strongly } C([0, T]; R^n) \text{ weakly in } W^{1,2}([0, T]; R^n),$$

(1.19) $$\gamma^\varepsilon(y^\varepsilon) \to Bu^* + f - y^{*\prime} - Fy^* \text{ weakly in } L^2(0, T; R^n),$$

(1.20) $$\nabla g^\varepsilon(t, y^\varepsilon(t)) \to \xi(t) \in \partial g(t, y^*(t)),$$

weak* in $L^\infty(0, T; R^n)$ a.e. $t \in]0, T[$ (see [1], p. 182),

(1.21) $$\nabla\varphi_0^\varepsilon(y^\varepsilon(T)) \to \xi^0 \in \partial\varphi_0(y^*(T)) \text{ in } R^n,$$

(1.22) $$\frac{\partial F^\varepsilon}{\partial y}(y^\varepsilon(t)) \to \xi^1(t) \in \partial F(y^*(t)),$$

weakly in $L^2(0, T; R^{n^2})$ a.e. $t \in]0, T[$.

The last convergence is obtained as a particular case of

Lemma 1.1. *Let $F : R^p \to R^q$ be a locally Lipschitzian map. We define $F^\varepsilon : R^p \to R^q$ by $F^\varepsilon(z) = (F_1^\varepsilon(z), F_2^\varepsilon(z), \ldots, F_q^\varepsilon(z))$ where $F_i^\varepsilon(z) = \int_{R^p} F_i(z - \varepsilon^2\theta)\rho_p(\theta)d\theta$ for $i = 1, 2, \ldots, q$.*

If $\{y^\varepsilon\} \subset L^1(0, T; R^p)$ is a sequence of real functions which satisfies the relations:

(i) $y^\varepsilon \to y$ strongly in $L^1(0, T; R^p)$,

(ii) $\nabla F^\varepsilon(y^\varepsilon(.)) \to \xi$ *weakly in* $L^1(0, T; R^{p\times q})$,

then $\xi(t) \in \partial F(y(t))$ a.e. $t \in]0, T[$.

(*We note the Clarke's generalized derivative of F at y by $\partial F(y)$.*)

Proof. By substituting $u = z - \varepsilon^2\theta$ in the F^ε formula and by using the properties of ρ_p we obtain:

$$F^\varepsilon(z) = \frac{(-1)^p}{\varepsilon^{2p}} \int_{R^p} F(u)\rho_p\left(\frac{z-u}{\varepsilon^2}\right)du.$$

By derivation we get:

$$\nabla F^\varepsilon(z)(w) = \frac{1}{\varepsilon^2}\frac{(-1)^p}{\varepsilon^{2p}} \int_{R^p} F(u)\nabla\rho_p\left(\frac{z-u}{\varepsilon^2}\right)(w)du \text{ for every } w \in R^p.$$

Substituting once again $\varepsilon^{-2}(z-u) = \theta$, the above-mentioned relation becomes:

$$\nabla F^\varepsilon(z)(w) = \frac{1}{\varepsilon^2} \int_{R^p} F(z - \varepsilon^2\theta)\rho_p(\theta)(w)d\theta \text{ for every } w \in R^p.$$

Integrating by parts we get the formula:

$$\nabla F^\varepsilon(z) = \frac{1}{\varepsilon^2}[\rho_p(\theta)F(z - \varepsilon^2\theta)]_{\partial R^p} - \frac{1}{\varepsilon^2}\int_{R^p} \nabla_\theta F(z - \varepsilon^2\theta)\rho_p(\theta)d\theta =$$

$$= \int_{R^p} \nabla F(z - \varepsilon^2\theta)\rho_p(\theta)d\theta.$$

Thus:

$$\nabla F^\varepsilon(z) \in \overline{\text{co}}\{\nabla F(z - \varepsilon^2 \operatorname{supp}\rho_p)\} \int_{R^p}\rho_p(\theta)d\theta \subseteq \overline{\text{co}}\{\nabla F(u) ; \|u - z\|_p \leqslant \varepsilon^2\}.$$

Since $y^\varepsilon \to y$ strongly in $L^1(0, T; R^p)$ there exists a subsequence, renoted $\{y^\varepsilon\}$, such that $y^\varepsilon(t) \to y(t)$ a.e. $t \in]0, T[$.

Since $\nabla F^\varepsilon(y^\varepsilon(.)) \to \xi$ weakly in $L^1(0, T; R^{p\times q})$ there exists a sequence $\omega^\varepsilon \to \xi$ strongly in $L^1(0, T; R^{p\times q})$ such that $\omega^\varepsilon(.) \in \text{co}\{\nabla F^\eta(y^\eta(.)).; \eta \in (0, \varepsilon]\}$.

Analogously, there exists a subsequence, denoted $\omega^{\bar\varepsilon}$, with the property $\omega^{\bar\varepsilon}(t) \to \xi(t)$ a.e. $t \in]0, T[$.

Let be $t \in [0, T]$, so that $y^\varepsilon(t) \to y(t)$ and $\omega^{\bar\varepsilon}(t) \to \xi(t)$.

Let be $\delta > 0$ arbitrary. There exists $\varepsilon_0 > 0$ so that $\varepsilon^2 + \|y^\varepsilon(t) - y(t)\|_p \leqslant \delta$ for $\varepsilon \in (0, \varepsilon_0]$.

If $\bar\varepsilon \in (0, \varepsilon_0]$, then

$$\omega^{\bar\varepsilon}(t) \in \text{co}\{\nabla F^\eta(y^\eta(t)) ; \eta \in (0, \bar\varepsilon]\} \subseteq \text{co}\left(\bigcup_{\eta\in(0,\bar\varepsilon]} \overline{\text{co}}\{\nabla F(u) ;\right.$$

$$\left.\|u - y^\eta(t)\| \leqslant \eta^2\}\right) \subseteq \text{co}\left(\bigcup_{\eta\in(0,\bar\varepsilon]} \overline{\text{co}}\{\nabla F(u) ; \|u - y(t)\| \leqslant \eta^2 + \|y(t) - y^\eta(t)\|\}\right) \subseteq$$

$$\subseteq \overline{\text{co}}\{\nabla F(u) ; \|u - y(t) \leqslant \delta\}.$$

From the convergence of the subsequence $\{\omega^{\bar\varepsilon}\}$ and from the fact that the right-side set is closed there results that

$$\xi(t) \subseteq \overline{\text{co}}\{\nabla F(u), \|u - y(t)\| \leqslant \delta\} = \partial F(y(t)) \text{ a.e. } t \in]0, T[$$

(see [4]).

Taking the scalar product of equation (1.15) with $p^\varepsilon(t)$, integrating on the interval $[t, T]$ and using Gronwall's lemma, we get $\|p^\varepsilon\|_{C([0,T]; R^n)} \leqslant C$ for every $\varepsilon > 0$.

Taking the scalar product of equation (1.15) with $q = (\operatorname{sgn} p_1^\varepsilon, \ldots, \operatorname{sgn} p_n^\varepsilon)$ and integrating on the interval $[0, T]$ we obtain the inequality:

(1.23) $$\sum_{i=1}^n \int_0^T |\dot\beta_i^\varepsilon(y_i^\varepsilon(t))p_i^\varepsilon(t)|dt \leqslant C, \text{ for every } \varepsilon > 0,$$

that is the sequence $\{\nabla\gamma^\varepsilon(y^\varepsilon)p^\varepsilon\}_{\varepsilon>0}$ is bounded in $L^1(0, T; R^n)$.

Since the sequence $\{dp^\varepsilon/dt\}_{\varepsilon>0}$ is bounded in $L^1(0, T; R^n)$ and the sequence $\{p^\varepsilon\}$ is bounded in $L^\infty(0, T; R^n)$ we infer, by Helly's theorem, that there exists $p \in BV([0, T]; R^n)$ and a subsequence of $\{\varepsilon\}$, renoted $\{\varepsilon\}$, so that $p^\varepsilon(t) \to p(t)$ for every $t \in [0, T]$.

We may infer that

$$(1.24) \qquad p^\varepsilon \to p \text{ strongly in } L^r(0, T; R^n), \ 1 \leq r < \infty.$$

Thus,

$$(1.22') \qquad \left(\frac{\partial F}{\partial y}(y^*(t))\right)^* p(t) \to (\partial F(y^*(t)))^* p(t) \text{ a.e. } t \in [0, T]$$

from relations (1.22) and (1.24).

By (1.23) it follows that there exists a measure $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in ((L^\infty(0, T))^*)^n$ so that on a generalized subsequence of $\{\varepsilon\}$, denoted again $\{\varepsilon\}$ we have:

$$(1.25) \qquad \dot\beta_i^\varepsilon(y_i^\varepsilon) p_i^\varepsilon \to \mu_i \text{ weakly}^* \text{ in } (L^\infty(0, T))^* \text{ for } i = 1, 2, \ldots, n.$$

Then, tending ε to 0 in equations (1.15), (1.16), (1.17), it follows that p satisfies equations (1.13), (1.14) and

$$p_i(t) - [(\partial F(y^*(t)))^* p(t)]_i - \mu_i \in L^\infty(0, T) \text{ a.e. } t \in]0, T[, \ i = 1, 2, \ldots, n$$

$$p'(t) - (\partial F(y^*(t)))^* p(t) - \mu \in g(t, y^*(t)) \text{ a.e. } t \in]0, T[.$$

In this case we can specify the measure μ.

If we take $t \in A = \{t \in [0, T]; 0 < y_i^*(t) < a_i\}$ then there exists $\delta_i > 0$ such that $2\delta_i < y_i^* < a_i - 2\delta_i$. Since $y^\varepsilon \to y^*$ uniformly for every $t \in [0, T]$ for $\delta_i > 0$ there exists $\varepsilon_0(\delta_i) > 0$ with $|y_i^\varepsilon - y_i^*| < \delta_i$ for all $\varepsilon < \varepsilon_0(\delta_i)$.

Then we have the inequalities:

$$\delta_i < y_i^* - \delta_i < y_i^\varepsilon < y_i^* + \delta_i < a_i - 2\delta_i + \delta_i = a_i - \delta_i.$$

Thus, $\delta_i < y_i^\varepsilon < a_i - \delta_i$ for all $\varepsilon < \varepsilon_0(\delta_i)$. It results that $\varepsilon^{-2}(y_i^\varepsilon - a_i) < -\varepsilon^{-2}\delta_i < -1$ for all $\varepsilon^2 < \delta_i$ and $\varepsilon^{-2} y_i^\varepsilon > \varepsilon^{-2}\delta_i > 1$.

We note that $A = \bigcup_i A_{\delta_i} = \bigcup_i \{t \mid \delta_i < y_i^*(t) < a_i - \delta_i\}$.

Thus, on A we have, by the definition of $\dot\gamma^\varepsilon$ $\dot\beta_i^\varepsilon(y_i^\varepsilon) p_i^\varepsilon \to 0$ strongly in $L^1(0, T)$ for $i = 1, 2, \ldots, n$.

Using the convergences (1.25) and (1.18) we have $\mu_i y_i^* = 0$ that is:

$$(1.26) \qquad (p_i'(t) - [(\partial F(y^*(t)))^* p(t)]_i - \xi_i(t)) y_i^*(t) = 0, \text{ a.e. } t \in]0, T[, \ i = 1, 2, \ldots, n,$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in L^\infty(0, T; R^n)$ satisfies the equation $\xi(t) \in \partial g(t, y^*(t))$ a.e. $t \in]0, T[$. From (1.26) it results (1.9) on the set A.

Using the expressions of γ^ε and $\nabla\gamma^\varepsilon$ we obtain:

$$(1.27) \qquad \begin{aligned} E_i^\varepsilon &= |y_i^\varepsilon \dot\beta_i^\varepsilon(y_i^\varepsilon) p_i^\varepsilon - p_i^\varepsilon \beta_i^\varepsilon(y_i^\varepsilon)| = |p_i^\varepsilon| \, |\varepsilon^{-1} a_i \int_{-\infty}^{\varepsilon^{-2}(y_i^\varepsilon - a_i)} \rho(\theta) d\theta - \varepsilon\Big(\int_0^1 \theta\rho(\theta) d\theta - \\ &- \int_{-\infty}^{\varepsilon^{-2}(y_i^\varepsilon - a_i)} \theta\rho(\theta) d\theta - \int_{\varepsilon^{-2} y_i^\varepsilon}^{\infty} \theta\rho(\theta) d\theta\Big)|, \text{ a.e. } t \in]0, T[; \ (i = 1, 2, \ldots, n). \end{aligned}$$

If $t \in B = \{t \in [0, T]; y_i^*(t) = 0\}$, by the convergence (1.18) we can write (1.27) as:

$$(1.27') \qquad E_i^\varepsilon = \varepsilon |p_i^\varepsilon| \, |\int_0^1 \theta\rho(\theta) d\theta - \int_{\varepsilon^{-2} y_i^\varepsilon}^{1} \theta\rho(\theta) d\theta| = \varepsilon |p_i^\varepsilon| \int_0^{\varepsilon^{-2} y_i^\varepsilon} \theta\rho(\theta) d\theta \leq \varepsilon |p_i^\varepsilon|.$$

Since the sequence $\{p_i^\varepsilon\}$ is bounded and $\varepsilon \to 0$ it follows that

$$(1.28) \qquad E_i^\varepsilon \to 0, \text{ strongly for } t \in B \text{ and } i = 1, 2, \ldots, n.$$

From (1.18) and from the bounding of the sequence $\{\nabla\gamma^\varepsilon(y^\varepsilon) p^\varepsilon\}_{\varepsilon>0}$ in $L^1(0, T; R^n)$ we get:

$$(1.29) \qquad y_i^\varepsilon \dot\beta_i^\varepsilon(y_i) p_i^\varepsilon \to 0 \text{ strongly in } L^1(B), \ i = 1, 2, \ldots, n.$$

By (1.27), (1.28), (1.29) it results:

$$(1.30) \qquad p_i^\varepsilon \beta_i^\varepsilon(y_i^\varepsilon) \to 0 \text{ strongly in } L^1(B), \ i = 1, 2, \ldots, n.$$

Relations (1.19), (1.24) and (1.30) lead to:

$$p_i(t)[(Bu^*(t))_i + f_i(t) - (F(0))_i] = 0, \ i = 1, 2, \ldots, n.$$

The last relation, for $(Bu^*(t))_i + f_i(t) \neq (F(0))_i$ implies (1.14).

Next, let be $t \in C = \{t \in [0, T]; y_i^*(t) = a_i\}$.

We have:

$$(1.31) \qquad \begin{aligned} F_i^\varepsilon &= |(y_i^\varepsilon - a_i)\dot\beta_i^\varepsilon(y_i) p_i^\varepsilon - p_i^\varepsilon \beta_i^\varepsilon(y_i^\varepsilon)| = |p_i^\varepsilon| \, |-\varepsilon^{-1} a_i \int_{\varepsilon^{-2} y_i^\varepsilon}^{\infty} \rho(\theta) d\theta - \\ &- \varepsilon\Big(\int_0^1 \theta\rho(\theta) d\theta - \int_{-\infty}^{\varepsilon^{-2}(y_i^\varepsilon - a_i)} \theta\rho(\theta) d\theta - \int_{\varepsilon^{-2} y_i^\varepsilon}^{\infty} \theta\rho(\theta) d\theta\Big)| = \varepsilon |p_i^\varepsilon| \, |\int_0^1 \theta\rho(\theta) d\theta - \\ &- \int_{-\infty}^{\varepsilon^{-2}(y_i^\varepsilon - a_i)} \theta\rho(\theta) d\theta| \leq 2\varepsilon |p_i^\varepsilon|. \end{aligned}$$

Similarly,

$$(1.32) \qquad F_i^\varepsilon \to 0 \text{ strongly for } t \in C, \ i = 1, 2, \ldots, n.$$

By an analogous argument we obtain (1.12).

This completes the proof.

Remark 1.1. We can study a more general case, such as:

Problem (P): Minimize:

$$\int_0^T [g(t, y(t)) + h(v(t))] dt + \varphi_0(y(T)) \qquad 0 < T < \infty$$

over the set of all functions $y \in W^{1,2}([0, T]; R^n)$ solutions of the variational inequality:

$$\begin{cases} y'(t) + Fy(t) + \beta(y(t)) \ni Bu(t) + f(t) \text{ a.e. } t \in]0, T[, \\ y(0) = y_0 \end{cases}$$

written for all $u \in W^{1,2}([0, T]; R^m)$ solutions of the problem:

$$\begin{cases} \alpha u'(t) + Gu(t) = v(t) \quad \text{a.e. } t \in]0, T[, \\ u(0) = u_0 \end{cases}$$

Here $G: R^m \to R^m$ is a locally Lipschitzian function and $(Gu, u)_m \geq -F_1\|u\|_m^2 - F_2$ for all $u \in R^m$, where $F_1, F_2 \in R_+^*$, $\alpha \in R_+$ and $\beta: R^n \to R^n$ ia a maximal monotone operator.

If $v \in L^2(0, T; R^m)$ there exists a unique solution $u \in W^{1,2}([0, T]; R^m)$ for which we can prove in the hypotheses (a)—(c) the uniqueness of a solution $y \in W^{1,2}([0, T]; R^n)$.

Moreover, the control problem (P) has, at least, one optimal pair.

Remark 1.2. The particular case, where $K = \{z \in R^n; z_i \geq 0, i = 1, 2, \ldots, n\}$, A, B matrices of dimensions $n \times n$ and $n \times m$ respectively and $f = (f_1, f_2, \ldots, f_n) \in L^2(0, T; R^n)$ is solved one the whole (see [1], p. 198).

2. **Examples.** *Example* 2.1. Consider the following problem:

$$(P_1) \text{ Maximize } \int_0^T y(t)dt$$

over the set of all functions $y \in W^{1,2}([0, T]; R)$ and $u \in U$ subject to:

$$(2.1) \qquad \begin{cases} y'(t) + y(t) + \partial I_K(y(t)) \ni u(t), \text{ a.e. } t \in]0, T[, \\ y(0) = y_0 \geq 0, \end{cases}$$

where $U = \{u \in L^2(0, T; R);\ 0 \leq u(t) \leq \gamma;\ \forall t \in [0, T];\ \int_0^T u(t)dt = \delta\}$, with $\delta \leq \gamma T$ and $K = R_+$.

This problem represents a particular case of the problem (P_K), for $n = m = 1$, $F \equiv B \equiv \mathrm{Id}_R$ which is the identity function, $f \equiv \varphi_0 \equiv 0$, $g(t, y(t)) = -y(t)$ on $[0, T]$ and $h \equiv I_U$ — the indicator function — that is $h(u) = 0$ if $u \in U$ and $h(u) = +\infty$ if $u \notin U$.

If $(y, u) \in W^{1,2}([0, T]; R) \times L^2(0, T; R)$ is an optimal pair for the problem (P_1), then

$$(2.2) \qquad y'(t) + y(t) = u(t) \qquad \text{a.e. } \{t \in [0, T];\ y(t) > 0\}$$

$$(2.3) \qquad u(t) = 0 \qquad \text{a.e. } \{t \in [0, T];\ y(t) = 0\}$$

$$(2.4) \qquad y(t) \geq 0 \text{ on } [0, T]$$

$$(2.5) \qquad y(0) = y_0 \geq 0.$$

From the maximum principle (see theorem (1.1)) we have assured the existence of an element $p \in BV([0, T]; R)$ which satisfies the adjoint state system:

$$(2.6) \qquad p'(t) - p(t) + 1 = 0 \qquad \text{a.e. } \{t \in [0, T];\ y(t) > 0\},$$

$$(2.7) \qquad p(t) = 0 \qquad \text{a.e. } \{t \in [0, T];\ y(t) = 0, \quad u(t) \neq 0\}$$

$$(2.8) \qquad p(T) = 0$$

$$(2.9) \qquad p(t) \in \partial I_U(u(t)) \qquad \text{a.e. }]0, T[.$$

From (2.9), using the definition of the subdifferential of the indicator function, there results the existence of an element $\rho \in R$ so that:

$$(2.10) \qquad \begin{aligned} u(t) &= 0, && \text{for } t \in \{t \in [0, T];\ p(t) < \rho\}, \\ 0 < u(t) &< \gamma, && \text{for } t \in \{t \in [0, T];\ p(t) = \rho\}, \\ u(t) &= \gamma, && \text{for } t \in \{t \in [0, T];\ p(t) > \rho\}. \end{aligned}$$

Case 1. If $y(0) = y_0 > 0$, then $0 \in \{t \in [0, T];\ y(t) > 0\}$ and from the continuity of y there exists a maximal interval $[0, t_1]$, so that $y(t) > 0$ on it.

In this case, the linear differential equation

$$\begin{cases} y'(t) + y(t) = u(t), \text{ on } [0, t_1], \\ y(0) = y_0 \end{cases}$$

has the general solution:

$$(2.11) \qquad y(t) = y_0 e^{-t} + \int_0^t e^{-(t-s)} u(s)ds \text{ for } t \in [0, t_1].$$

On the other hand, from (2.6) we get $p(t) = Ce^t + 1$ for $t \in [0, t_1]$.

Case 1.1. If $C < 0$, then p have the following graph:

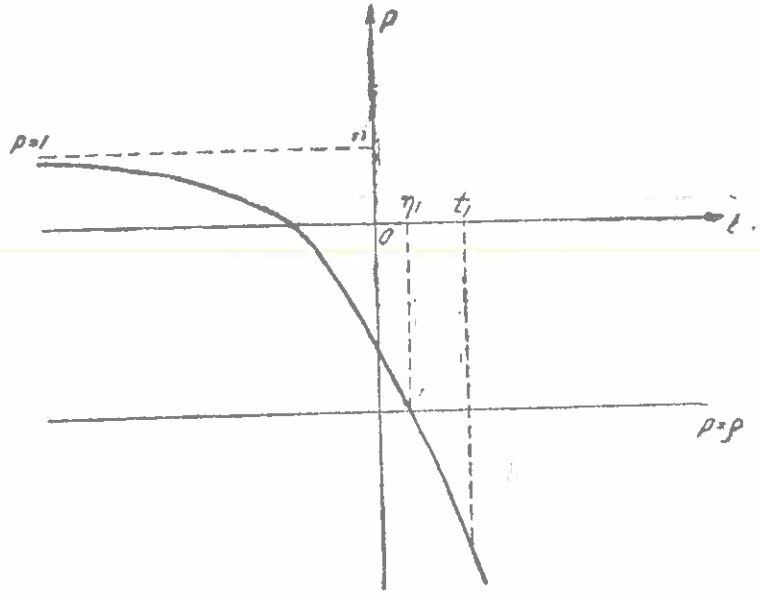

Fig. 1

From (2.10) there results the existence of $\eta_1 \in [0, t_1]$ so that: $u(t) = \gamma$ for $t \in [0, \eta_1[$, $0 < u(t) < \gamma$ for $t = \eta_1$ $u(t) = 0$ for $t \in]\eta_1, t_1]$.

Then (2.11) implies:

$$y(t) = \begin{cases} C_1 e^{-t} + \gamma \text{ on } [0, \eta_1] \\ C_2 e^{-t} \text{ on }]\eta_1, t_1]. \end{cases}$$

From the initial condition $y(0) = y_0$ and from the continuity of the function y we get:

$$y(t) = \begin{cases} (y_0 - \gamma)e^{-t} + \gamma \text{ on } [0, \eta_1] \\ (y_0 - \gamma + \gamma e^{\eta_1})e^{-t} \text{ on } [\eta_1, t_1]. \end{cases}$$

We note that $y_0 + \gamma(e^{\eta_1} - 1) > 0$ for every $\eta_1 \in [0, t_1]$ which implies $t_1 = T$.

In conclusion, the optimal pair is:

$$u(t)=\begin{cases}\gamma & \text{for } t\in[0,\delta/\gamma],\\ 0 & \text{for } t\in]\delta/\gamma, T],\end{cases}$$

and

$$y(t)=\begin{cases}(y_0-\gamma)e^{-t}+\gamma & \text{for } t\in[0,\delta/\gamma]\\ (y_0-\gamma+\gamma e^{\delta/\gamma})e^{-t} & \text{for } t\in[\delta/\gamma, T].\end{cases}$$

where γ_1 was determined from the condition $\int_0^T u(t)dt=\delta$.

From (2.6) and (2.8) relation we have $C=-e^{-T}$, thus $p(t)=1-e^{t-T}$ for $t\in[0,T]$.

Case 1.2. If $C>0$, then the function p have the graph:

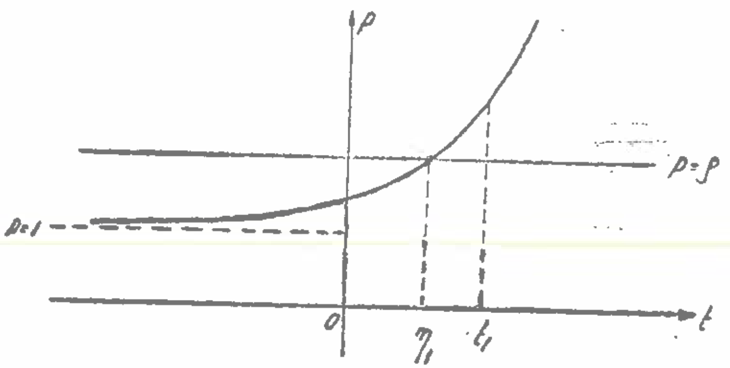

Fig. 2

There exists, $\gamma_1\in[0,t_1]$ so that:

$u(t)=0$ for $t\in[0,\gamma_1[$, $0<u(t)<\gamma$ for $t=\gamma_1$, $u(t)=\gamma$ for $t\in]\gamma_1,t_1[$.

Analogously, we obtain $t_1=T$, which is absurd since it contradicts the (2.8) relation.

Case 2. Let $y(0)=y_0=0$ and $y(t)=0$ on the closed interval $[0,\bar{t}]$, then $y(t)=0$ and $y(t)>0$ in a neighborhood of the point t.

In this case, from the relation (2.3) it follows that:

$$u(t)=\begin{cases}0 & \text{on } [0,\bar{t}]\\ \gamma & \text{on }]\bar{t},\bar{\bar{t}}].\end{cases}$$

Along with (2.11) we obtain:

$$y(t)=\begin{cases}0 & \text{on } [0,\bar{t}]\\ C_3e^{-t}+\gamma & \text{on }]\bar{t},\bar{\bar{t}}].\end{cases}$$

From the continuity of y we have:

$$y(t)=\begin{cases}0 & \text{on } [0,\bar{t}]\\ -e^{t-\bar{t}}+ & \text{on } [\bar{t},\bar{\bar{t}}].\end{cases}$$

We note that the function y is increasing and concave on the closed interval $[\bar{t},\bar{\bar{t}}]$ which implies $\bar{\bar{t}}=T$.

In conclusion the optimal pair is:

$$u(t)=\begin{cases}0 & \text{on } [0,T-\delta/\gamma]\\ \gamma & \text{on } [T-\delta/\gamma, T]\end{cases}$$

$$y(t)=\begin{cases}0 & \text{on } [0,T-\delta/\gamma]\\ -\gamma e^{T-\frac{\delta}{\gamma}-t}+\gamma & \text{on } [T-\delta/\gamma, T].\end{cases}$$

where $\bar{t}$ was determined from the condition $\int_0^T u(t)dt=\delta$.

Example 2.2. We consider the following optimal control problem: Problem (P_2): Minimize

$$\int_0^L dx\int_0^T [g(t,y(t,x))+h(u(t,x))]dt+\int_0^L \varphi_0(y(T,x))dx,$$

over the set of all functions $y\in W^{1,2}(D)$ and $u\in L^2(D)$ subject to:

$$\frac{\partial y(t,x)}{\partial t}-b^2\frac{\partial^2 y(t,x)}{\partial x^2}+\partial I_K(y(t,x))\ni u(t,x),\ \text{a.e. } (t,x)\in D, \tag{2.12}$$

where $D=[0,T]\times[0,L]$ and $K=[0,a]$, with the initial condition:

$$y(0,x)=y^0(x) \text{ for } x\in[0,L], \tag{2.13}$$

and the boundary conditions:

$$y(t,0)=y(t,L)=0 \text{ for } t\in[0,T]. \tag{2.14}$$

Using the finite differences method this problem can be expressed as a particular case of the (P_K) problem.

Divide the space interval $[0,T]$ by the equidistance points $0=x_1<x_2<\ldots<x_n=L$ in $(n-1)$ equal parts $\Delta x=L/(n-1)$ in length; i.e. $x_i=[(i-1)L]/(n-1)$ for $i=1,2,\ldots,n$.

In this way, the $y: D\to R$ function is approximated by $y(t)=(y_i(t))_{i=\overline{1,n}}$, where $y_i(t)=y(t,x_i)$ for $i=1,2,\ldots,n$.

Analogously, $u(t)=(u_i(t))_{i=\overline{1,n}}$.

With the finite differences of the second order, the equation (2.12) becomes:

$$\frac{dy_i(t)}{dt}-\frac{b^2}{(\Delta x)^2}[y_{i+1}(t)-2y_i(t)+y_{i-1}(t)]+\partial I_K(y_i(t))\ni u_i(t) \tag{2.12'}$$

for $t\in[0,T]$ and $i=2,3,\ldots,n-1$.

The initial condition and the boundary condition become

$$y_i(0)=y^0(x_i) \text{ for } i=1,2,\ldots,n.$$

respectively

$$y_1(t)=y_n(t)=0 \text{ for } t\in[0,T].$$

We obtain a first order linear differential system of n equations and n unknown factors such as:

$$
(2.13) \qquad \begin{cases} y'(t)+Ay(t)+\partial I_K(y(t)) \ni u(t) & \text{for } t\in[0,T] \\ y(0)=y_0 \end{cases}
$$

where

$$
A=\frac{b^2}{(\Delta x)^2}\begin{vmatrix} 2 & -1 & 0 & 0 \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 \dots & 0 & -1 & 2 \end{vmatrix}
$$

The relation (2.13) can be written as:

$$
(2.14) \qquad \begin{cases} y_i'(t)+(Ay(t))_i=u_i(t) & \text{on } \{t\in[0,T]\,;\ 0<y_i(t)<a\} \\ u_i(t)\leq 0 & \text{on } \{t\in[0,T]\,;\ y_i(t)=0\} \\ u_i(t)\geq 0 & \text{on } \ t\in[0,T]\,;\ y_i(t)=a\} \\ 0\leq y_i(t)\leq a & \text{on } [0,T] \\ y_i(0)=y^0(x_i)\in[0,a] \end{cases}
$$

for $i=2, 3, \dots, n-1$.

We approximate the cost functional by:

$$
\Delta x\left[\int_0^T\sum_{i=1}^n (g(t, y_i(t))+h(u_i(t)))dt+\sum_{i=1}^n \varphi_0(y_i(T))\right].
$$

If $(y, u)\in W^{1,2}([0,T]\,;R^n)$ is an optimal pair for the (P_2) problem then, from the maximum principle (see theorem 1.1), we have assured the existence of some elements $p\in BV([0,T]\,;R^n)$ and $\xi\in L^\infty(0,T\,;R^n)$ satisfying:

$$
(2.15) \qquad \begin{cases} p_i'(t)-(Ap(t))_i=\xi_i & \text{a.e. } \{t\in[0,T]\,;\ 0<y_i(t)<a\} \\ \xi(t)\in\partial g(t, y(t)) \\ p_i(t)=0 & \text{a.e. } \{t\in[0,T]\,;\ y_i(t)=0,\ u_i(t)\neq 0\} \\ p_i(t)=0 & \text{a.e. } \{t\in[0,T]\,;\ y_i(t)=a,\ u_i(t)\neq 0\} \\ p(T)+\partial\varphi_0(y(T))\ni 0 \\ p(t)\in\partial h(u(t)) & \text{a.e. } t\in[0,T] \end{cases}
$$

where $i=\overline{2, n-1}$, $A^*=A$ (A is a symmetric matrix).

REFERENCES

1. B a r b u, V. — *Optimal Control of Variational Inequalities*, Pitman Advanced Publishing Program Boston, London, Melbourne, 1984.
2. B a r b u, V., P r e c u p a n u, Th. — *Convexity and Optimization in Banach Spaces*, Ed. Acad. Bucureşti, 1978.
3. H i r i a r t-U r r u t y, J., B T h i b a u l t, L. — *Existence et charactérisation de différentielles généralisées d'applications localement lipschitziennes d'un espace de Banach separable dans un espace de Banach réflexif separable*, C.R. Acad. Sc. Paris, t. 290, série A (19) 1091—1094.
4. P o u r c i a u, B H. — *Global properties of proper Lipschitzian maps*, SIAM J. Math. Anal., Vol. 14, (1983).
5. R o c k a f e l l a r, R.T. — *Convex Analysis*, Princeton University Press, N.J., 1970.

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Institutul Politehnic
6600—Iaşi, România

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ON THE EXISTENCE AND UNIQUENESS OF INSTATIONARY CONVECTION IN POROUS MEDIA

BY

DAN POLIŠEVSKI

1. Preliminaries. Let Ω be an open connected bounded set in $\mathbf{R}^n$ ($n=2$ or 3) locally located on one side of the boundary $\partial\Omega$, which is a Lipschitz manifold composed of a finite number of connected components; let real $\theta>0$ be a fixed and let us denote with Q the cylinder $\Omega\times(0, \theta)$.

With the assumptions and approximations which are frequently used for the thermal convection in a homogeneous porous medium saturated with an incompressible fluid, the governing evolution system of the Darcy-Boussinesq equations for the filtration velocity u, the pressure p and the temperature T may be written in a non-dimensional form as:

$$
(1.1) \qquad \operatorname{div} u=0 \text{ in } Q
$$

$$
(1.2) \qquad \gamma\partial_t u+u=-\nabla p+aTe \text{ in } Q\ \left(\partial_t=\frac{\partial}{\partial t}\right),
$$

$$
(1.3) \qquad \partial_t T+u\nabla T=\Delta T \text{ in } Q,
$$

where $\gamma>0$ and $a>0$ (the Rayleigh number) are two fixed real numbers, while e is the fixed versor of the gravitational acceleration. As we are in the non-dimensional case, we can assume that Ω can be included in a n-cube of edge-length 1.

Let Γ_0 be a subset of $\partial\Omega$ with positive surface measure and let us denote with $\Gamma_1=\partial\Omega\setminus\Gamma_0$; the boundary conditions are:

$$
(1.4) \qquad u\cdot\nu=0 \quad \text{on } \partial\Omega\times(0, \theta),
$$

$$
(1.5) \qquad \frac{\partial T}{\partial\nu}=0 \quad \text{on } \Gamma_1\times(0, \theta),
$$

$$
(1.6) \qquad T=\tau \quad \text{on } \Gamma_0\times(0, \theta),
$$

where ν denotes the unit outward normal on $\partial\Omega$ and $\tau=\tau(x, t)$ is the most important datum of our problem.

Also, u and T have to obey in some sense to the initial conditions

$$
(1.7) \qquad u(0)=u^\circ \text{ in } \Omega, \qquad\qquad (1.8) \qquad T(0)=T^\circ \text{ in } \Omega.
$$