$$\nabla_l K^{h}_{ijk} + \nabla_j K^{\hat{h}}_{ikl} + \nabla_k K^{\hat{h}}_{ilj} = 0, \tag{47}$$

$$\nabla_l K^{\hat{h}}_{ijk} + \nabla_j K^{\hat{h}}_{ikl} + \nabla_k K^{\hat{h}}_{\,il} = 0, \tag{48}$$

$$\nabla_l K^{h}_{ijk} + \nabla_j K^{h}_{ikl} + \nabla_k K^{h}_{ilj} = 0. \tag{49}$$

The Ricci tensor field is obtained easily according to its definition

$$K_{i\hat{j}} = K^{\hat{h}}_{\hat{i}h\hat{j}} - K^{h}_{\hat{i}\hat{j}h} = 0, \tag{50}$$

$$K_{\hat{i}j} = K^{\hat{h}}_{\hat{i}h\hat{j}} + K^{h} \quad = -K^{h}_{ijh} = -\frac{\partial \Phi^h_{hi}}{\partial y^j}, \tag{51}$$

$$K_{ij} = K^{\hat{h}}_{i\hat{h}j} + K^{h}_{ihj} = -\frac{\partial g^{hl}}{\partial y^i}\left(\frac{\delta g_{lh}}{\delta x^j} - g_{hm}\Gamma^m_{lj}\right) - \tag{52}$$

$$-g^{hl}\frac{\partial}{\partial y^h}\left(\frac{\delta g_{il}}{\delta x^j} - g_{lm}\Gamma^m_{ij}\right) = -\frac{\partial \Phi^h_{hj}}{\partial y^i},$$

$$K_{ij} = K^{\hat{h}}_{i\hat{h}j} + K^{h}_{ihr} = g^{hl}\frac{\partial}{\partial y^h}(R_{ijl} + R_{jil}). \tag{53}$$

Finally the scalar curvature is

$$K = 2K_{ij}g^{ij} = -2g^{ij}\frac{\partial \Phi^h_{hj}}{\partial y^i}. \tag{54}$$

REFERENCES

1. Bejan, C. L. and Oproiu, V.— *Embeddings of the tangent bundle in Riemannian manifolds*, An. şt. Univ. Iaşi, 31 (1985), 247–254.
2. Ichijyo, Y. — *On some G-structures defined on the tangent bundles*, Tensor, 42 (1985), 179 – 190.
3. Ikeda, S. — *Some physical aspects induced by the internal variable in the theory of fields in Finsler spaces*, Il Nuovo Cimento 61 B (2) (1981), 220 – 228.
4. Kern, J.— *Lagrange geometry*, Archiv Math. 25 (1974), 438 – 443.
5. Matsumoto, M.— *The induced and intrinsic Finsler connections of a hypersurface and Finslerian projective geometry*, J. Math. Kyoto Univ. 25 (1985), 107 – 144.
6. Miron, R.— *Metrical Finsler structures and metrical Finsler connections*, J. Math. Kyoto Univ. 23 (1983), 219 – 224.
7. Miron, R.— *A Langrangian theory of relativity*, Seminarul de Geometrie Topologie, Preprint Series of the Univ. Timişoara 84 (1985), 53 pp.
8. Oproiu, V.— *Some remarkable structures and connections defined on the tangent bundle*, Rend. di Mat. vol. 6 serie VI (1973), 503 – 540.
9. Oproiu, V.— *Some properties of the tangent bundle related to the Finsler geometry*, Proceed. Nat. Sem. Finsler Spaces, vol 1, Braşov 1980, 195 – 207.
10. Oproiu, V.— *A Riemannian structure in Lagrange geometry*, Rend. Sem. Fac. Sci. Univ. Cagliari 55 (2) (1985), 1–20.
11. Rund, H.— *The differential geometry of Finsler spaces*, Springer-Verlag, Berlin 1959.
12. Sanini, A.—*Su un tippo di struttura quasi hermitiana del fibrato tangente ad uno spazio di Finsler*, Rend. Semin. Mat. Torino 32 (1973/1974), 303 – 316.
13. Takano, Y.— *Theory of fields in Finsler spaces*, Progress in Theor. Physics, 40 (1968), 1159 – 1180.
14. Tamassy, L., Kiss, B.—*Relation between Finsler and affine connections*, Rend. Circ. Mat. Palermo 2 (1984) suppl. 3, 329 – 337.
15. Watanabe, S., Ikeda, F.—*On metrical Finsler connection of a metrical Finsler structure*, Tensor, 39 (1982), 37 – 41.

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Faculty of Mathematics
University of Iaşi
6 600 — Iaşi, România

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A GEOMETRIC PROOF OF HUREWICZ'S THEOREM

BY

R. AYALA, E. DOMINGUEZ, A. QUINTERO

1. **Introduction.** If we consider the homology groups as the bordism groups constructed with pseudomanifolds we obtain an easy geometric interpretation of the singular cycles. This fact permits of more intuitive proofs, from the geometric point of view, of many results of Algebraic Topology.

The aim of this note is, following the algebraic ideas developed in Spanier's book [7], to give a geometric proof of the well-known Hurewicz's Theorem that relates the basic algebraic invariants of a topological space: its homotopy and homology groups.

The references for the results of Polyhedral Topology and Homotopy Theory will be [6] and [7] respectively. A polyhedron is an Euclidean subspace covered by finite many simplices. Given the interval $I=[0, 1]$, we denote by I^n the ordinary n-cube with its canonical PL-structure and orientation. S^n is the quotient $I^n/\partial I^n$ with its canonical PL-structure and orientation. As usually, $\pi_n(X, x_0)$ is the n-th homotopy group of the pointed topological space (X, x_0), defined by the

$$f:(I^n, \partial I^n)\to(X, x_0),$$

and $\pi'_n(X, x_0)$ is the quotient of $\pi_n(X, x_0)$ by its commutator subgroup. It is known that

$$\pi_n(X, x_0)=\pi'_n(X, x_0) \quad \text{if } n\geqslant 2.$$

In general, we regard $\pi'_n(X, x_0)$ as $\pi_n(X, x_0)$ with its group operation made commutative.

A closed n-pseudomanifold is an n-dimensional polyhedron P which admits a subpolyhedron SP, called singular set of P, such that:

a) $\dim SP \leqslant n-2$,

b) $P-SP$ is an oriented PL-manifold without boundary.

An n-pseudomanifold P with boundary ∂P is an n-dimensional polyhedron verifying a) and the following conditions:

a') $P-SP$ is an oriented PL-manifold with boundary $\partial P-SP$.

b') P is a closed $(n-1)$-pseudomanifold such that $S(\partial P)=\partial P\cap SP$.

A singular n-pseudomanifold on a topological pair (X, A) is a pair (P, f) where P is an n-pseudomanifold with boundary ∂P, possibly empty, and

$$f:(P, \partial P)\to(X, A)$$

is a continuos mapping. If (P_1, f_1) and (P_2, f_2) are two singular n-pseudomanifolds on (X, A) we say that they are pseudobordant if there exists a pair (Q, F) such that:

1) Q is an $(n+1)$ - pseudomanifold with boundary

$$\partial Q=(P_1 \cup -P_2) \cup Q_0,$$

where the equality means an orientation-preserving PL-isomorphism, „$\cup$" is the disjoin union symbol, $-P_2$ is the pseudomanifold P_2 with the reversed orientation and Q_0 is an n-pseudomanifold such that

$$\partial Q_0=\partial P_1 \cup -\partial P_2=\partial Q_0 \cap (P_1 \cup -P_2).$$

2) $F:(Q, Q_0)\to(X, A)$ is a mapping which extends f_i $(i=1, 2)$.

Pseudobordism relation is an equivalence relation and the set of classes $[P, f]$ admits a structure of abelian group defining the sum of two classes as the class of the disjoin union of their representatives. This group is called the n-th pseudobordism group of (X, A) and it is denoted by $S\Omega_n(X, A)$.

A mapping $\alpha:(X, A)\to(Y, B)$ defines a morphism $\alpha_*: S\Omega_n(X, A)\to \to S\Omega_n(Y, B)$ given by $\alpha_*[P, f]=[P, f\alpha]$. In this way $S\Omega_*(-)$ is a covariant functor between the category of topological pairs and the category of abelian groups. This functor is an ordinary homology theory naturally equivalent to Singular Homology with coefficients in the group of integers. For details see [1], [2] and [3].

A ball complex K is a finite set of PL-balls such that (we denote K^q the set of q-balls):

1) If $\sigma, \beta \in K$, $\sigma \cap \beta=\partial\sigma \cap \partial\beta$.

2) If $\sigma \in K^q$, $\partial\sigma$ is an union of elements of K^{q-1}.

Given a ball complex K, the faces of the balls of K^q is a ball complex which we will denote K^q, if there is not doubt. The ball complex $K\times I$ is the complex $\{\sigma\times I, \sigma\times 0, \sigma\times 1\ ;\ \sigma\in K\}$. By $|K|$ we mean the underlying space of K.

We denote by I^n the canonical ball complex structure on the n-cube I^n.

If K is a ball complex it is easy to prove that $|K|$ is an n-pseudomanifold if and only if:

a) Every ball is face of an n-ball.

b) Every $(n-1)$-ball is face of just two n-balls.

c) There exists a compatible orientation on the n-balls of K.

Similarly, $|K|$ is an n-pseudomanifold with boundary if and only if the above conditions a) and c) are fulfilled and moreover:

b') Every $(n-1)$ - ball is face of one or two n-balls.

c') The faces of the $(n-1)$ -balls which lie in just an n-ball define a closed $(n-1)$-pseudomanifold.

2. Proof of Hurewicz's theorem. In this proof we will use

Lemma ([8; 6.40]). *Let K be a CW-complex, (X, x_0) a q-connected space and $f:|K|\to X$ a mapping. Let $L\subset K$ a subcomplex such that $f(|L^q|)=x_0$. Then there exists a homotopy $H:|K|\times I\to X$, relative to $|L|$, such that $H_0=f$ and $H_1(|K^q|)=x_0$.*

Hurewicz's Theorem. *There exists a canonical morphism*

$$\theta:\pi_n'(X, x_0)\to S\Omega_n(X, x_0)$$

defined by $\theta[f]=[I^n, f]$. Moreover, if (X, x_0) is $(n-1)$-connected θ is bijective.

Proof. It is immediate the good definition of θ. In order to prove that θ is morphism, we consider the subcubes

$$I_1^n=\{(x_1, x_2, \ldots, x_n)\in I^n\ ;\ x_1\leq 1/2\}, \qquad I_2^n=\{(x_1, x_2, \ldots, x_n)\in I^n\ ;\ x_1\geq 1/2\}.$$

Let Q be the $(n+1)$-pseudomanifold obtained when we glue $I^n\times I$ to $I_1^n\times I \cup I_2^n\times I$ identifying in the natural way $I^n\times 1$ to $I_1^n\times 0 \cup I_2^n\times 0$. Then, given $[f_1], [f_2]\in\pi_n'(X, x_0)$ is obvious the existence of a mapping $F:Q\to X$ showing that

$$[I^n, f_1*f_2]=[I^n, f_1]+[I^n, f_2]$$

in $S\Omega_n(X, x_0)$, where f_1*f_2 is the product of f_1 and f_2.

Now we suppose that (X, x_0) is $(n-1)$-connected.

a) We are going to see that θ is onto. Let $[P', f']\in S\Omega_n(X, x_0)$. We take $P=P'\cup c*(\partial P')$ and $f:P\to X$ the extension of f' such that $f(c*\partial P')=x_0$. It is easy to show that $[P', f']=[P, f]$. Then we can assume that every element of $S\Omega_n(X, x_0)$ admits a representative (P, f) with $\partial P=\emptyset$. Let K be an oriented triangulation of P. By the above lemma we can assume that $f(|K^{n-1}|)=x_0$ so we get a quotient mapping $\bar{f}:|K^n|/|K^{n-1}|\to X$. The space $|K^n|/|K^{n-1}|$ is a wedge, $S_1\vee S_2\vee\ldots\vee S_r$, of as many spheres as n-balls of K. Let g_i be the restriction of $\bar{f}$ to S_i $(1\leq i\leq r)$. We will prove that

$$[P, f]=[S_1, g_1]+\ldots+[S_r, g_r].$$

For this we consider the pseudomanifold obtained gluing the quotient $P\times I/|K^{n-1}|\times 1$ to each $S_i\times I$ by $S_i\times 0$.

On the other hand, because the spheres S_i are oriented, it is possible to choose based orientation-preserving PL-isomorphisms $h_i: S^n\to S_i$. Then

$$[P, f]=[S^n, g_1h_1]+\ldots+[S^n, g_rh_r]$$

Now, if $\pi: I^n\to S^n$ is the canonical projection it is obvious that $[I^n, g_ih_i\pi]=[S^n, g_ih_i]$ $(1\leq i\leq r)$. So

$$\theta([g_1h_1\pi]\ldots[g_rh_r\pi])=[P, f];$$

b) θ is injective. If $[f]\in\pi_n'(X, x_0)$ with $\theta[f]=0$ there exists a pair (Q, F') where Q is an $(n+1)$-pseudomanifold with boundary $\partial Q=I^n\cup Q_0$, Q_0 an n-pseudomanifold such that $\partial Q_0=\partial I^n=I^n\cap Q_0$ and F' is a mapping from (Q, Q_0) to (X, x_0) whose restriction to I^n is f.

Let $K'\subset L'\supset L_0'$ be triangulations of $I^n\subset Q\supset Q_0$ and K the ball complex constructed substituting $K'\times 1$ by I^n in $K'\times I$. Identifying K' to $K'\times 0$ we obtain a new ball complex L. It is easy to prove that $|L|$ is an $(n+1)$-pseudomanifold whose boundary is $\partial|L|=I^n\cup|L_0|$, where $L_0\subset L$ is subcomplex of K. Moreover we can extend F' to a mapping $F:(|L|, |L_0|)\to \to(X, x_0)$ and by the above lemma we can assume that $F(|L^{n-1}|)=x_0$. So we get a quotient mapping $\bar{F}: M=|L|/|L^{n-1}|\to X$.

Let $T=\{B_1, B_2, \ldots, B_r\}$ the projections of the $(n+1)$ - balls of L. Note that each B_i is an $(n+1)$-cell whose boundary, ∂B_i, is a wedge of n-spheres. Let $\Sigma=\{S_1, S_2, \ldots, S_K\}$ the set of n-spheres which are in the boundaries of the elements of T. The set Σ is in $1-1$ correspondence with the set of n-balls of L. Therefore, because $|L|$ is a pseudomanifold, each S_i belongs to the boundary of two elements of T at most, in such way that those S_j which belong to only one B_i are in $1-1$ correspondence with the n-balls of L. Then, because $\pi_n'(X, x_0)$ is commutative, it is easy to show that the restriction, f, of F to S^n is null-homotopic. So, $[f]=1\in\pi_n'(X, x_0)$.

Remarks. 1) We have just proved Hurewicz's Theorem for all $n\geq 1$. In the 1-dimensional case there is another way to show that θ is $1-1$. The proof is based on the classical surfaces classification and the fact that a 2-pseudomanifold is a surface with finite many point identifications ([4]). For more details see [5].

2) If we define pseudobordism without orientations we get a geometric interpretation of Singular Homology with coefficients in $Z/2Z$ (see [2]). Let $SN_*(-)$ be this bordism theory. The B o c k s t e i n sequence associated to the the sequence

$$0\to Z\to Z\to Z/2Z\to 0$$

leads us to prove that the forgetful morphism

$$\Phi : S\Omega_n(X)\to SN_n(X)$$

is onto and its kernel is $2S\Omega_n(X)$. This fact shows that the composition

$$\Phi\theta : \pi_1(X, x_0)\to SN_1(X, x_0)$$

is onto and its kernel is the subgroup generated by the elements a^2 and $aba^{-1}b^{-1}$; a, $b\in\pi_1(X, x_0)$.

As in 1) we can prove the above fact in a geometric way (see [5] for details).

REFERENCES

1. D o m i n g u e z, E. — *Grupos de seudobordismo*, Rev. Acad. Ciencias Zaragoza 30 (1975), 5—16.
2. D o m i n g u e z, E. — *Interpretación geométrica de la Homologia Singular*, Rev. Real Acad. Ciencias Madrid 69 (1975), 121—147.
3. D o m i n g u e z, E. — *Introducción geométrica a la Teoria de Bordismo*, Publ. Sem. Mat. „Garcia Galdeano" ser. II, sec. 3, 4 1985.
4. L u, C. K., — *Classification of 2-manifolds with singular points*, Bull. A. M. S. 55 (1949), 1093—1098.
5. Q u i n t e r o, A. — *Interpretación geométrica de algunos resultados de la Topologia Algebráica*, Univ. Sevilla, 1978.
6. R o u r k e, C. P., and S a n d e r s o n, B. J. — *Introduction to Piecewise Linear Topology*, Springer-Verlag, 1972.
7. S p a n i e r, E. H. — *Algebraic Topology*, McGraw Hill, 1966.
8. S w i t z e r, R. M. — *Algebraic Topology: Homotopy and Homology*, Springer-Verlag, 1975.

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R. Ayala, A. Quintero
Dpto. Geometria y Topologia
Facultad de Matematicas
c/Tarfia s/n
41012 — Sevilla, Spain

E. Dominguez
Dpto. Geometria y Topologia
Facultad de Cencias
Ciudad Universitaria
50009 — Zaragoza, Spain

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ON A HORIZONTAL LIFT OF A LINEAR CONNECTION INTO LINEARIZED TANGENT BUNDLE OF SECOND ORDER

BY

JAN KUREK

The purpose of this paper is to investigate a horizontal lift $\overset{H}{\nabla}$ of a linear connection Γ on a manifold M into the total space of linearized tangent bundle of second order ${}^2_0\pi : {}^2M\to M$. A connection Γ in the tangent bundle ${}^1_0\pi : TM\to M$ induces a pure-linear connection $\hat{\Gamma}=(\Gamma, \Gamma)$ in the linearized tangent bundle of second order ${}^2_0\pi : {}^2M\to M$. The connection $\hat{\Gamma}$ in the bundle ${}^2_0\pi : {}^2M\to M$ allows to introduce a horizontal lift X^H and vertical lifts X^{V_1} and X^{V_2} of type V_1 and V_2 of a vector field X on M, into the total space ${}^2M$.

We define a horizontal lift $\overset{H}{\nabla}$ on ${}^2M$ of a linear connection Γ on M similarly as in [2]. We find a torsion tensor and curvature tensor of the connection $\overset{H}{\nabla}$. We describe a geodesic curve on ${}^2M$ and a parallel displacement of: X^H, X^{V_1}, X^{V_2} on ${}^2M$ with respect to $\overset{H}{\nabla}$.

We state that if Γ is a Riemannian connection with respect to a metric tensor g on M, then a horizontal lift $\overset{H}{\nabla}$ is a metric connection for the diagonal lift G of g.

1. Let M be an n-dimensional Hausdorff manifold of class C^∞. Let Γ be a linear connection on M and ∇ its covariant derivative. We consider a linear connection Γ on M as a connection in the tangent bundle ${}^1_0\pi : TM\to M$.

Then Γ is a left splitting of the exact sequence of vector bundles over TM:

$$(1,1)\qquad 0\to V(TM)\underset{\iota}{\overset{\Gamma}{\rightleftarrows}} T(TM)\underset{\pi'}{\to} TM\underset{M}{\times}TM\to 0,\quad \Gamma\circ\iota=\mathrm{id}_{V(TM)}.$$

A connection map for Γ is of the form:

$$D : T(TM)\to TM,\quad D=p_2\circ\iota_{V(TM)}\circ\Gamma,$$

where $\iota_{V(TM)} : V(TM)\to TM\underset{M}{\times}TM$ is a canonical isomorphism of a vertical subbundle into the Whitney sum and $p_2 : TM\underset{M}{\times}TM\to TM$ is a projection on the second component.

We consider a tangent bundle of second order ${}^2_0\pi : {}^2M\to M$, where:

$$(1,2)\qquad {}^2M=\{A\in T(TM) : {}^1_0\pi_* A=T_\pi A\},\ {}^2_0\pi={}^1_0\pi\circ{}^1_0\pi_*|{}^2M.$$

The map ${}^1_0\pi_* : T(TM)\to TM$ is a differential of the natural projection