the set of n-balls of L. Therefore, because $|L|$ is a pseudomanifold, each S_i belongs to the boundary of two elements of T at most, in such way that those S_j which belong to only one B_i are in $1-1$ correspondence with the n-balls of L. Then, because $\pi_n(X, x_0)$ is commutative, it is easy to show that the restriction, f, of F to S^n is null-homotopic. So, $[f]=1\in\pi_n'(X, x_0)$.

Remarks. 1) We have just proved Hurewicz's Theorem for all $n\geq 1$. In the 1-dimensional case there is another way to show that θ is $1-1$. The proof is based on the classical surfaces classification and the fact that a 2-pseudomanifold is a surface with finite many point identifications ([4]). For more details see [5].

2) If we define pseudobordism without orientations we get a geometric interpretation of Singular Homology with coefficients in $Z/2Z$ (see [2]). Let $SN_*(-)$ be this bordism theory. The Bockstein sequence associated to the the sequence

$$0\to Z\to Z\to Z/2Z\to 0$$

leads us to prove that the forgetful morphism

$$\Phi : S\Omega_n(X)\to SN_n(X)$$

is onto and its kernel is $2S\Omega_n(X)$. This fact shows that the composition

$$\Phi\theta : \pi_1(X, x_0)\to SN_1(X, x_0)$$

is onto and its kernel is the subgroup generated by the elements a^2 and $aba^{-1}b^{-1}$; $a, b\in\pi_1(X, x_0)$.

As in 1) we can prove the above fact in a geometric way (see [5] for details).

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ON A HORIZONTAL LIFT OF A LINEAR CONNECTION INTO LINEARIZED TANGENT BUNDLE OF SECOND ORDER

BY

JAN KUREK

The purpose of this paper is to investigate a horizontal lift $\overset{H}{\nabla}$ of a linear connection Γ on a manifold M into the total space of linearized tangent bundle of second order ${}^2_0\pi : {}^2M\to M$. A connection Γ in the tangent bundle ${}^1_0\pi : TM\to M$ induces a pure-linear connection $\hat{\Gamma}=(\Gamma, \Gamma)$ in the linearized tangent bundle of second order ${}^2_0\pi : {}^2M\to M$. The connection $\hat{\Gamma}$ in the bundle ${}^2_0\pi : {}^2M\to M$ allows to introduce a horizontal lift X^H and vertical lifts X^{V_1} and X^{V_2} of type V_1 and V_2 of a vector field X on M, into the total space ${}^2M$.

We define a horizontal lift $\overset{H}{\nabla}$ on ${}^2M$ of a linear connection Γ on M similarly as in [2]. We find a torsion tensor and curvature tensor of the connection $\overset{H}{\nabla}$. We describe a geodesic curve on ${}^2M$ and a parallel displacement of: X^H, X^{V_1}, X^{V_2} on ${}^2M$ with respect to $\overset{H}{\nabla}$.

We state that if Γ is a Riemannian connection with respect to a metric tensor g on M, then a horizontal lift $\overset{H}{\nabla}$ is a metric connection for the diagonal lift G of g.

1. Let M be an n-dimensional Hausdorff manifold of class C^∞. Let Γ be a linear connection on M and ∇ its covariant derivative. We consider a linear connection Γ on M as a connection in the tangent bundle ${}^1_0\pi : TM\to M$.

Then Γ is a left splitting of the exact sequence of vector bundles over TM:

$$(1,1)\qquad \begin{tikzcd} 0 \arrow[r] & V(TM) \arrow[r, "\iota"'] & T(TM) \arrow[l, bend right, "\Gamma"'] \arrow[r, "\pi'"'] & TM\times_M TM \arrow[r] & 0, \end{tikzcd}\quad \Gamma\circ\iota=\mathrm{id}_{V(TM)}.$$

A connection map for Γ is of the form:

$$D : T(TM)\to TM,\quad D=p_2\circ\iota_{V(TM)}\circ\Gamma,$$

where $\iota_{V(TM)} : V(TM)\to TM\times_M TM$ is a canonical isomorphism of a vertical subbundle into the Whitney sum and $p_2 : TM\times_M TM\to TM$ is a projection on the second component.

We consider a tangent bundle of second order ${}^2_0\pi : {}^2M\to M$, where:

$$(1,2)\qquad {}^2M=\{A\in T(TM) : {}^1_0\pi_* A=T_\pi A\},\ {}^2_0\pi={}^1_0\pi\circ{}^1_0\pi_{*|{}^2M}.$$

The map ${}^1_0\pi_* : T(TM)\to TM$ is a differential of the natural projection

${}^1_0\pi : TM \to M$ and $T_\pi : T(TM) \to TM$ is the natural projection for the tangent bundle over TM.

A vector bundle structure for the bundle $\tau : TTM \to M$, where $\tau = {}^1_0\pi \circ T_\pi$, is defined by a diffeomorphism :

$$(1,3) \qquad T_\pi \times {}^1_0\pi_* \times D : T(TM) \to TM \underset{M}{\times} TM \underset{M}{\times} TM$$

and it induces a vector bundle structure for the tangent bundle of second order by the diffeomorphism [1] :

$$(1,4) \qquad {}^1_0\pi_* \times D : {}^2M \to TM \underset{M}{\times} TM.$$

A local chart (U, x^{0i}) on M induces a local chart $({}^1_0\pi^{-1}(U), x^{0i}, x^{1i})$ on TM and a local chart $({}^2_0\pi^{-1}(U), x^{0i}, x^{1i}, x^{2i})$ on ${}^2M$. A local chart for the vector tangent bundle of second order ${}^2M$ is of the form $({}^2_0\pi^{-1}(U), z^{0i}, z^{1i}, z^{2i})$, where :

$$(1,5) \qquad z^{0i} = x^{0i}, \quad z^{1i} = x^{1i}, \quad z^{2i} = x^{2i} + \Gamma^i_{jk} x^{1j} x^{1k}.$$

The vector tangent bundle of second order ${}^2_0\pi : {}^2M \to M$ is the Whitney sum ${}^2M = {}^2M^H \underset{M}{\times} {}^2M^V$ of a horizontal subbundle ${}^2M^H$ and a vertical subbundle ${}^2M^V$.

A basis of local sections of ${}^2M$ consist of :

$$(1,6) \qquad \overset{0}{E}_{1i|(x^0)} = \frac{\partial}{\partial x^{0i}} - \Gamma^k_{ij}\delta^j_l \frac{\partial}{\partial x^{1k}}\bigg|_{(x^{0j},\, \delta^j_l)}, \quad \overset{0}{E}_{2i|(x^0)} = \frac{\partial}{\partial x^{1i}}\bigg|_{(x^{0j},\, 0)}.$$

The sections $\overset{0}{E}_{1i}$ and $\overset{0}{E}_{2i}$ span the subbundles ${}^2M^H$ and ${}^2M^V$ respectively.

A linear connection Γ in the tangent bundle ${}^1_0\pi : TM \to M$ induces a pure-linear connection $\hat{\Gamma} = (\Gamma, \Gamma)$ in the linearized tangent bundle of second order ${}^2_0\pi : {}^2M \to M$. The pure-linear connection $\hat{\Gamma} = (\Gamma, \Gamma)$ induced by a linear connection Γ may be regarded as a left splitting of the exact sequence of vector bundles over ${}^2M$:

$$(1,7) \qquad \begin{tikzcd}
0 \arrow[r] & V({}^2M) \arrow[r, "\iota"] \arrow[d, "{}^2_1\pi_*"'] & T({}^2M) \arrow[r] \arrow[l, bend right, "\hat{\Gamma}"'] \arrow[d, "{}^2_1\pi_*"] & {}^2M \underset{M}{\times} TM \arrow[r] & 0 \\
0 \arrow[r] & V(TM) \arrow[r, "\iota"] & T(TM) \arrow[r] \arrow[l, bend left, "\Gamma"] & TM \underset{M}{\times} TM \arrow[r] & 0
\end{tikzcd}$$

$$\Gamma \circ \iota = \mathrm{id}_{V({}^2M)},$$

$$(1,8) \qquad \begin{gathered} \hat{\Gamma}(z^{0i}, z^{1i}, z^{2i}, y^{0i}, y^{1i}, y^{2i}) = (z^{0i}, z^{1i}, z^{2i}\,;\, 0, y^{1i} + \Gamma^i_{jk} z^{1k} y^{0j}, y^{2i} + \\ + \Gamma^i_{jk} z^{2k} y^{0j}) \\ \Gamma(x^{0i}, x^{1i}\,;\, y^{0i}, y^{1i}) = (x^{0i}, x^{1i}\,;\, 0, y^{1i} + \Gamma^i_{jk} x^{1k} y^{0j}). \end{gathered}$$

A connection map for the connection $\hat{\Gamma}$ is of the form :

$$(1,9) \qquad \hat{D} : T({}^2M) \to {}^2M, \qquad \hat{D} = p_2 \circ \iota_{V({}^2M)} \circ \hat{\Gamma},$$

where $\iota_{V({}^2M)} : V({}^2M) \to {}^2M \underset{M}{\times} {}^2M$ is an isomorphism of a vertical subbundle into the Whitney sum and $p_2 : {}^2M \underset{M}{\times} {}^2M \to M$ is a projection on the second component.

Then for any $A \in {}^2M$ we have a decomposition into the direct sum :

$$(1,10) \qquad T_A({}^2M) = H_A({}^2M) \oplus V_A({}^2M),$$

where : $H_A({}^2M) = \operatorname{Ker} \hat{D}$ and $V_A({}^2M) = \operatorname{Ker}({}^2_0\pi_*)$.

Moreover, for the cotangent bundle $T^*({}^2M)$ we define a decomposition into the direct sum corresponding to (1,10) :

$$(1,11) \qquad T^*({}^2M) = H({}^2M)^\perp \oplus V({}^2M)^\perp,$$

where : $\quad H({}^2M)^\perp = \{\omega \in T^*({}^2M) : \omega(X) = 0, \ X \in H({}^2M)\}$,

$\qquad\qquad V({}^2M)^\perp = \{\omega \in T^*({}^2M) : \omega(X) = 0, \ X \in V({}^2M)\}$.

A system of $3n$-vectors (D_{0i}) and (D_{1i}, D_{2i}) that spans $H({}^2M)$ and $V({}^2M)$ respectively and locally is defined by :

$$(1,12) \qquad D_{0i} = \frac{\partial}{\partial z^{0i}} - \Gamma^k_{ij} z^{1j} \frac{\partial}{\partial z^{1k}} - \Gamma^k_{ij} z^{2j} \frac{\partial}{\partial z^{2k}}, \quad D_{1i} = \frac{\partial}{\partial z^{1i}}, \quad D_{2i} = \frac{\partial}{\partial z^{2i}}$$

is called an *adapted frame* on ${}^2M$ with respect to the induced connection $\hat{\Gamma} = (\Gamma, \Gamma)$ in the bundle ${}^2_0\pi : {}^2M \to M$.

A system of $3n$ 1-forms (ω^{0i}) and $(\omega^{1i}, \omega^{2i})$ that spans $V({}^2M)^\perp$ and $H({}^2M)^\perp$ respectively and locally is defined by :

$$(1,13) \qquad \omega^{0i} = dz^{0i}, \quad \omega^{1i} = dz^{1i} + \Gamma^i_{jk} z^{1k} dz^{0j}, \quad \omega^{2i} = dz^{2i} + \Gamma^i_{jk} z^{2k} dz^{0j}$$

is called *an adapted coframe* on ${}^2M$.

For adapted frames, adapted coframes and the natural frames on ${}^2M$ with respect to the coordinates (z^{0i}, z^{1i}, z^{2i}) and $(z^{0i'}, z^{1i'}, z^{2i'})$, where :

$$(1,14) \qquad z^{0i'} = z^{0i'}(z^{0i}), \quad z^{1i'} = A^{i'}_i z^{1i}, \quad z^{2i'} = A^{i'}_i z^{2i}, \quad A^{i'}_i = \frac{\partial z^{0i'}}{\partial z^{0i}},$$

we have :

$$(D_{0i}, D_{1i}, D_{2i}) = (D_{0i'}, D_{1i'}, D_{2i'}) \begin{bmatrix} A^{i'}_i & 0 & 0 \\ 0 & A^{i'}_i & 0 \\ 0 & 0 & A^{i'}_i \end{bmatrix},$$

$$(1,15) \qquad \begin{bmatrix} \omega^{0i'} \\ \omega^{1i'} \\ \omega^{2i'} \end{bmatrix} = \begin{bmatrix} A^{i'}_i & 0 & 0 \\ 0 & A^{i'}_i & 0 \\ 0 & 0 & A^{i'}_i \end{bmatrix} \begin{bmatrix} \omega^{0i} \\ \omega^{1i} \\ \omega^{2i} \end{bmatrix},$$

$$(D_{0i}, D_{1i}, D_{2i}) = \left(\frac{\partial}{\partial z^{0j}}, \frac{\partial}{\partial z^{1j}}, \frac{\partial}{\partial z^{2j}}\right) \begin{bmatrix} \delta^j_i & 0 & 0 \\ -\Gamma^j_{ik} z^{1k} & \delta^j_i & 0 \\ -\Gamma^j_{ik} z^{2k} & 0 & \delta^j_i \end{bmatrix}.$$

Definition 1. *A horizontal lift of a vector field X on M into the total space ${}^2M$ is a section $X^H \in H({}^2M)$ such that :*

$$(1,16) \qquad {}^2_0\pi_*(X^H) = X, \quad \hat{D}(X^H) = 0.$$

Definition 2. *A vertical lift of type V_1 or V_2 of a vector field X on M into the total space ${}^2M$ is a section $X^{V_1} \in V({}^2M)$ or $X^{V_2} \in V({}^2M)$ respectively such that:*

$$(1,17) \qquad \begin{aligned} &{}^{2}_{0}\pi_*(X^{V_1})=0, \ {}^{1}_{0}\pi_*(\hat{D}(X^{V_1}))=X, \ D(\hat{D}(X^{V_1}))=0, \\ &{}^{2}_{0}\pi_*(X^{V_2})=0, \ {}^{1}_{0}\pi_*(\hat{D}(X^{V_2}))=0, \ D(\hat{D}(X^{V_2}))=X. \end{aligned}$$

For a vector field $X=X^i \dfrac{\partial}{\partial x^i}$ the horizontal lift X^H and the vertical lifts X^{V_1}, X^{V_2} in the adapted frame are of the form:

$$(1,18) \qquad X^H=X^iD_{0i}, \ X^{V_1}=X^iD_{1i}, \ X^{V_2}=X^iD_{2i},$$

and in the natural frame have the form:

$$(1,19) \qquad X^H=\begin{bmatrix} X^i \\ -\Gamma^i_{jk}X^jz^{1k} \\ -\Gamma^i_{jk}X^jz^{2k} \end{bmatrix}, \quad X^{V_1}=\begin{bmatrix} 0 \\ X^i \\ 0 \end{bmatrix}, \quad X^{V_2}=\begin{bmatrix} 0 \\ 0 \\ X^i \end{bmatrix}.$$

Proposition 1. *For the horizontal lift X^H and the vertical lifts X^{V_1}, X^{V_2} into ${}^2M$ of vector fields X, Y on M we have the relations:*

$$(1,20) \qquad \begin{aligned} &[X^H, Y^H]=[X, Y]^H-\gamma_1R(X, Y)-\gamma_2R(X, Y), \\ &[X^{V_1}, Y^H]=[X, Y]^{V_1}-(\nabla_XY)^{V_1}, \ [X^{V_2}, Y^H]=[X, Y]^{V_2}-(\nabla_XY)^{V_2}, \\ &[X^{V_1}, Y^{V_1}]=0, \ [X^{V_1}, Y^{V_2}]=0, \ [X^{V_2}, Y^{V_2}]=0, \end{aligned}$$

where $\gamma_1R(X, Y)$ and $\gamma_2R(X, Y)$ are vertical fields of type V_1 and V_2 respectively of the form:

$$\gamma_1R(X, Y)=R^i_{jkt}X^jY^kz^{1t}D_{1i}, \ \gamma_2R(X, Y)=R^i_{jkt}X^jY^kz^{2t}D_{2i}.$$

2. Let Γ be a linear connection on M and ∇ its covariant derivative.

Definition 3 [2]. *A horizontal lift of a linear connection ∇ on M into the total space ${}^2M$ is a linear connection $\overset{H}{\nabla}$ defined by the formulas:*

$$(2,1) \qquad \begin{aligned} &\overset{H}{\nabla}_{X^H}Y^H=(\nabla_XY)^H, \ \overset{H}{\nabla}_{X^H}Y^{V_a}=(\nabla_XY)^{V_a}, \\ &\overset{H}{\nabla}_{X^{V_a}}Y^H=0, \ \overset{H}{\nabla}_{X^{V_a}}Y^{V_b}=0 \text{ for } a, b=1, 2. \end{aligned}$$

Thus the components of the horizontal lift $\overset{H}{\nabla}$ of a linear connection ∇ with components Γ^k_{ij} in the adapted frame $(D_\alpha)=(D_{0i}, D_{1i}, D_{2i})$ are of the form

$$\overset{H}{\nabla}_{D_\alpha}D_\beta=\overset{H}{\Gamma}{}^\gamma_{\alpha\beta}D_\gamma,$$

$$(2,2) \qquad \begin{aligned} &\overset{H}{\Gamma}{}^{0k}_{0i\,0j}=\Gamma^k_{ij}, \ \overset{H}{\Gamma}{}^{1k}_{0i\,1j}=\Gamma^k_{ij}, \ \overset{H}{\Gamma}{}^{2k}_{0i\,2j}=\Gamma^k_{ij}, \\ &\overset{H}{\Gamma}{}^\gamma_{\alpha\beta}=0, \text{ for other } \alpha, \beta, \gamma=0i, 1i, 2i. \end{aligned}$$

Moreover we have the relations (1, 15): $(D_\alpha)=(\partial_J)[B^J_\alpha]$ between the adapted frame $(D_\alpha)=(D_{0i}, D_{1i}, D_{2i})$ and the natural frame $(\partial_J)=\left(\dfrac{\partial}{\partial z^{0j}}, \dfrac{\partial}{\partial z^{1j}}, \dfrac{\partial}{\partial z^{2j}}\right)$. Thus the components of the linear connection $\overset{H}{\nabla}$ in the natural frame (∂_J) are of the form: $\nabla_{\partial_J}\partial_K=F^L_{JK}\partial_L$

$$(2,3) \qquad \begin{aligned} &\overset{H}{F}{}^L_{JK}=B^L_\alpha\overset{H}{\Gamma}{}^\alpha_{\beta\gamma}\tilde{B}^\beta_J\tilde{B}^\gamma_K-D_\beta B^L_\alpha\tilde{B}^\alpha_J\tilde{B}^\beta_K, \ B^L_\alpha\tilde{B}^\alpha_J=\delta^L_J, \\ &\overset{H}{F}{}^{0l}_{0j\,0k}=\Gamma^l_{jk}, \ \overset{H}{F}{}^{1l}_{0j\,1k}=\overset{H}{F}{}^{1l}_{1j\,0k}=\Gamma^l_{jk}, \ \overset{H}{F}{}^{2l}_{0j\,2k}=\overset{H}{\Gamma}{}^{2l}_{2j\,0k}=\Gamma_{jk}, \\ &\overset{H}{F}{}^{1l}_{0j\,0k}=\partial_j\Gamma^l_{ki}z^{1i}-\Gamma^l_{si}\Gamma^s_{jk}z^{1i}+\Gamma^l_{js}\Gamma^s_{ki}z^{1i}, \\ &\overset{H}{F}{}^{2l}_{0j\,0k}=\partial_j\Gamma^l_{ki}z^{2i}-\Gamma^l_{si}\Gamma^s_{jk}z^{2i}+\Gamma^l_{js}\Gamma^s_{ki}z^{2i}, \\ &\overset{H}{F}{}^L_{JK}=0 \text{ for other } J, K, L=0j, 1j, 2j. \end{aligned}$$

A torsion tensor $\overset{H}{T}$ of the connection $\overset{H}{\nabla}$ is of the form:

$$(2,4) \qquad \begin{aligned} &\overset{H}{T}(X^H, Y^H)=(T(X, Y))^H+\gamma_1R(X, Y)+\gamma_2R(X, Y), \\ &\overset{H}{T}(X^{V_a}, Y^H)=(T(X, Y))^{V_a}, \ \overset{H}{T}(X^{V_a}, Y^{V_b})=0, \ a, b=1, 2. \end{aligned}$$

A curvature tensor $\overset{H}{R}$ of the horizontal lift $\overset{H}{\nabla}$ is of the form:

$$(2,5) \qquad \begin{aligned} &\overset{H}{R}(X^H, Y^H)Z^H=(R(X, Y)Z)^H, \ \overset{H}{R}(X^H, Y^H)Z^{V_a}=(R(X, Y)Z)^{V_a}, \\ &\overset{H}{R}(X^{V_a}, Y^H)Z^H=0, \ R(X^{V_a}, Y^H)Z^{V_b}=0, \ \overset{H}{R}(X^{V_a}, Y^{V_b})Z^H=0. \end{aligned}$$

Thus, we have

Proposition 2. *If a linear connection ∇ on M is torsionless, i.e. $T=0$, then the horizontal lift $\overset{H}{\nabla}$ on ${}^2M$ has nonzero torsion tensor $\overset{H}{T}$ with $\overset{H}{T}(X^H, Y^H)=\gamma_1R(X, Y)+\gamma_2R(X, Y)$, $\overset{H}{T}(X^{V_a}, Y^H)=0$, $\overset{H}{T}(X^{V_a}, Y^{V_b})=0$. If a linear connection ∇ on M is locally flat (i.e. torsion tensor $T=0$ and curvature tensor $R=0$), then the horizontal lift $\overset{H}{\nabla}$ on ${}^2M$ is torsionless.*

Proposition 3. *If the curvature tensor R of a linear connection ∇ on M vanishes, then the curvature tensor $\overset{H}{R}$ of the horizontal lift $\overset{H}{\nabla}$ on ${}^2M$ vanishes too.*

3. Let $\tilde{C}$ be a geodesic curve on the total space ${}^2M$ with respect to the horizontal lift $\overset{H}{\nabla}$. In local coordinates (z^{0i}, z^{1i}, z^{2i}) the equations of a geodesic curve $\tilde{C}: R\supset I\to{}^2M$, $\tilde{C}: t\to\tilde{C}(t)=(z^{0i}(t), z^{1i}(t), z^{2i}(t))$ are of the form:

$$(3,1) \qquad \frac{d^2z^L}{dt^2}+\overset{H}{F}{}^L_{JK}\frac{dz^J}{dt}\frac{dz^K}{dt}=0, \ L=0i, 1i, 2i.$$

Thus, using the formulas (2, 3) for $\overset{H}{F}{}^L_{JK}$, we get:

$$(3,2)\qquad \begin{aligned}
&\frac{d^2z^{0i}}{dt^2}+\Gamma^i_{jk}\frac{dz^{0j}}{dt}\frac{dz^{0k}}{dt}=0,\\
&\frac{d^2z^{1i}}{dt^2}+(\partial_j\Gamma^i_{ks}+\Gamma^i_{jr}\Gamma^r_{ks}-\Gamma^i_{ks}\Gamma^r_{jk})z^{1s}\frac{dz^{0j}}{dt}\frac{dz^{0k}}{dt}+\\
&\qquad+\Gamma^i_{jk}\frac{dz^{0j}}{dt}\frac{dz^{1k}}{dt}+\Gamma^i_{jk}\frac{dz^{1j}}{dt}\frac{dz^{0k}}{dt}=0,\\
&\frac{d^2z^{2i}}{dt^2}+(\partial_j\Gamma^i_{ks}+\Gamma^i_{jr}\Gamma^r_{ks}-\Gamma^i_{rs}\Gamma^r_{jk})z^{2s}\frac{dz^{0j}}{dt}\frac{dz^{0k}}{dt}+\\
&\qquad+\Gamma^i_{jk}\frac{dz^{0j}}{dt}\frac{dz^{2k}}{dt}+\Gamma^i_{jk}\frac{dz^{2j}}{dt}\frac{dz^{0k}}{dt}=0.
\end{aligned}$$

We denote:

$$(3,3)\qquad \frac{Dz^{1i}}{dt}=\frac{dz^{1i}}{dt}+\Gamma^i_{jk}z^{1k}\frac{dz^{0j}}{dt},\quad \frac{Dz^{2i}}{dt}=\frac{dz^{2i}}{dt}+\Gamma^i_{jk}z^{2k}\frac{dz^{0j}}{dt}.$$

Then, if we assume that:

$$\Gamma^i_{jk}=\Gamma^i_{kj},\quad \frac{d^2z^{0i}}{dt^2}+\Gamma^i_{jk}\frac{dz^{0j}}{dt}\frac{dz^{0k}}{dt}=0,$$

we obtain:

$$\frac{D^2z^{1i}}{dt^2}=\frac{D}{dt}\left(\frac{Dz^{1i}}{dt}\right)=\frac{d^2z^{1i}}{dt^2}+(\partial_l\Gamma^i_{jk}+\Gamma^i_{lr}\Gamma^r_{jk}-\Gamma^i_{rk}\Gamma^r_{lj})z^{1k}\frac{dz^{0l}}{dt}\frac{dz^{0j}}{dt}+$$
$$+\Gamma^i_{jk}\frac{dz^{1k}}{dt}\frac{dz^{0j}}{dt}+\Gamma^i_{jk}\frac{dz^{0k}}{dt}\frac{dz^{1j}}{dt}.$$

Thus, we have

Proposition 4. *A geodesic curve $\tilde{C}$ on ${}^2M$ with respect to the horizontal lift $\overset{H}{\nabla}$ of a linear connection ∇ on M with torsion tensor $T=0$, has, in coordinates (z^{0i}, z^{1i}, z^{2i}), equations of the form:*

$$(3,4)\qquad \frac{d^2z^{0i}}{dt^2}+\Gamma^i_{jk}\frac{dz^{0j}}{dt}\frac{dz^{0k}}{dt}=0,\quad \frac{D}{dt}\left(\frac{Dz^{1i}}{dt}\right)=0,\quad \frac{D}{dt}\left(\frac{Dz^{2i}}{dt}\right)=0.$$

Proposition 5. *A curve $\tilde{C}$ on the total space ${}^2M$ is a geodesic curve with respect to the horizontal lift $\overset{H}{\nabla}$, if its projection $\bar{C}=\pi^2_0(\tilde{C})$ on M is a geodesic curve with respect to a connection Γ on M and if $\tilde{C}$ is horizontal curve with respect to the induced connection $\hat{\Gamma}=(\Gamma, \Gamma)$ in the tangent bundle of second order $\pi^2_0: {}^2M\to M$.*

Proposition 6. *If C is a geodesic curve on a manifold M with respect to a connection ∇, then its canonical lift $\tilde{C}$ into ${}^2M$ is a geodesic curve with respect to the horizontal lift $\overset{H}{\nabla}$.*

Let $\tilde{X}$ be a vector field defined along a curve $\tilde{C}$ on ${}^2M$. The vector field $\tilde{X}=\tilde{X}^J(\partial/\partial z^J)$ along the curve $\tilde{C}(t)=(Z^J(t))$ is parallel with respect to the horizontal lift $\overset{H}{\nabla}$ on ${}^2M$ if the equations are satisfied:

$$(3,5)\qquad \frac{d\tilde{X}^J}{dt}+F^J_{KL}\frac{dz^K}{dt}\tilde{X}^L=0.$$

Thus, using the formulas (2, 3) for the components $\overset{H}{F}{}^J_{KL}$ of $\overset{H}{\nabla}$ in local coordinates (z^{0i}, z^{1i}, z^{2i}), we get equations of the form:

$$(3,6)\qquad \begin{aligned}
&\frac{d\tilde{X}^{0i}}{dt}+\Gamma^i_{jk}\tilde{X}^{0k}\frac{dz^{0j}}{dt}=0,\\
&\frac{d\tilde{X}^{1i}}{dt}+(\partial_j\Gamma^i_{ks}-\Gamma^i_{rs}\Gamma^r_{jk}+\Gamma^i_{jr}\Gamma^r_{ks})z^{1s}\tilde{X}^{0k}\frac{dz^{0j}}{dt}+\\
&\qquad+\Gamma^i_{jk}\frac{dz^{0j}}{dt}\tilde{X}^{1k}+\Gamma^i_{jk}\tilde{X}^{0k}\frac{dz^{1j}}{dt}=0\\
&\frac{d\tilde{X}^{2i}}{dt}+(\partial_j\Gamma^i_{ks}-\Gamma^i_{rs}\Gamma^r_{jk}+\Gamma^i_{jr}\Gamma^r_{ks})z^{2s}\tilde{X}^{0k}\frac{dz^{0j}}{dt}+\\
&\qquad+\Gamma^i_{jk}\frac{dz^{0j}}{dt}\tilde{X}^{2k}+\Gamma^i_{jk}\tilde{X}^{0k}\frac{dz^{2j}}{dt}=0.
\end{aligned}$$

For the vertical lifts X^{V_a} of type V_a, $a=1, 2$ of a vector field $X=X^i\,\partial/\partial x^i$, the equations (3,6) reduce to:

$$(3,7)\qquad \frac{dX^i}{dt}+\Gamma^i_{jk}X^k\frac{dz^{0j}}{dt}=0.$$

For the horizontal lift X^H of a vector field $X=X^i\,\partial/\partial x^i$ the equations (3, 6) reduce to:

$$(3,8)\qquad \begin{aligned}
&\frac{dX^i}{dt}+\Gamma^i_{jk}X^k\frac{dz^{0j}}{dt}=0,\\
&-\Gamma^i_{rs}z^{1s}\left(\frac{dX^r}{dt}+\Gamma^r_{jk}X^k\frac{dz^{0j}}{dt}\right)=0,\\
&-\Gamma^i_{rs}z^{2s}\left(\frac{dX^r}{dt}+\Gamma^r_{jk}X^k\frac{dz^{0j}}{dt}\right)=0.
\end{aligned}$$

Thus, we have:

Proposition 7. *A parallel displacement of the horizontal lift A^H or the vertical lifts A^{V_a}, $a=1, 2$ of a vector $A\in T_pM$ along the curve $\tilde{C}$ on ${}^2M$ with respect to the horizontal lift $\overset{H}{\nabla}$ is equal to the horizontal lift or the vertical lifts of type V_a, $a=1, 2$ respectively of the parallel displacement of a vector A along the curve $C=\pi^2_0(\tilde{C})$ on M with respect to a linear connection ∇:*

$$(3,9)\qquad \tilde{C}(A^H)=(C(A))^H,\quad \tilde{C}(A^{V_a})=(C(A))^{V_a},\quad a=1, 2.$$

4. Let g be a metric tensor on M.

Definition 4. *The tensor G on the total space ${}^2M$ induced by the metric tensor g on M and defined in the following way:*

$$G(\tilde{X},\tilde{Y})=g({}^2_0\pi_*\tilde{X},{}^2_0\pi_*\tilde{Y})+g({}^1_0\pi_*\hat{D}\tilde{X},{}^1_0\pi_*\hat{D}\tilde{Y})+g(D\hat{D}\tilde{X},D\hat{D}\tilde{Y}) \tag{4,1}$$

is called a diagonal lift of g.

In the adapted coframe (1,13) on ${}^2M$ the diagonal lift G of g is of the form:

$$G=g_{ij}\omega^{0i}\otimes\omega^{0j}+g_{ij}\omega^{1i}\otimes\omega^{1j}+g_{ij}\omega^{2i}\otimes\omega^{2j}. \tag{4,2}$$

The covariant derivative of the tensor G with respect to the connection $\overset{H}{\nabla}$ on ${}^2M$ is defined by the formula:

$$(\overset{H}{\nabla}_{\tilde{X}}G)(\tilde{Y},\tilde{Z})=\tilde{X}(G(\tilde{Y},\tilde{Z}))-G(\overset{H}{\nabla}_{\tilde{X}}\tilde{Y},\tilde{Z})-G(\tilde{Y},\overset{H}{\nabla}_{\tilde{X}}\tilde{Z}). \tag{4,3}$$

Thus, we have the formulas for the covariant derivative of the diagonal lift G of g with respect to the horizontal lift $\overset{H}{\nabla}$ of ∇:

$$\begin{array}{ll}
(\overset{H}{\nabla}_{X^H}G)(Y^H,Z^H)=(\nabla_Xg)(Y,Z), & (\overset{H}{\nabla}_{X^H}G)(Y^{V_a},Z^H)=0,\\
(\overset{H}{\nabla}_{X^H}G)(Y^{V_a},Z^{V_a})=(\nabla_Xg)(Y,Z), & (\overset{H}{\nabla}_{X^H}G)(Y^{V_a},Z^{V_b})=0,\\
(\overset{H}{\nabla}_{X^{V_a}}G)(Y^H,Z^H)=0, & (\overset{H}{\nabla}_{X^{V_a}}G)(Y^{V_b},Z^H)=0,\\
(\overset{H}{\nabla}_{X^{V_a}}G)(Y^{V_b},Z^{V_c})=0, & a,b,c=1,2.
\end{array} \tag{4,4}$$

Thus, we have:

Proposition 8. *If ∇ is the Riemannian connection with respect to a metric tensor g on M, then the horizontal lift $\overset{H}{\nabla}$ of a connection ∇ into ${}^2M$ is a metric connection for the diagonal lift G of the metric tensor g.*

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CONNECTIONS OF ORDER k ON M AND ASSOCIATED POLYNOMIAL STRUCTURES ON T^kM

BY

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0. Introduction. The differential geometry of the tangent bundles of higher order has been extensively studied by many authors, for example Morimoto [MO], Yano and Ishihara [YI2], Catz [C1] [C2], Tulczyjew [TU1] [TU2], de Leon and Villaverde [DLV1] [DLV2], de Leon and Rodrigues [DLR2], Crampin, Sarlet and Cantrijn [CSC]. Recently, the study of higher order mechanics in the framework of the theory of higher order tangent bundles has been undertaken by a number of authors (see [DL2], [DL3], [DLR1], [DLR2], [CSC]). On the other hand, it is well known the construction by Dombrowski [DO] of an almost complex structure on the tangent bundle TM of a Riemannian manifold M. Dombrowski construction has been generalized by Grifone [GR] for manifolds with a non homogeneous connection. Grifone shows that the corresponding almost complex structure on TM plays an important role in the developping of Classical Mechanics. The construction of Grifone has been extended in [DL1] to the tangent bundle of order 2, T^2M, by showing that there exists a f-structure on T^2M associated to each connection of order 2 on M.

Our aim in this paper is to generalize the constructions of Grifone and de León for the general case. In fact, we show that there exists an appropiate polynomial structure (said a $\Phi(k+1, k-1)$-structure) on the tangent bundle T^kM of order k of M associated to each connection of order k on M. This sort of polynomial structure has been studied in [DL4]. Here, we apply the results of that paper to obtain a sufficient condition for the partial integrability of the $\Phi(k+1, k-1)$-structure on T^kM (Theorem 4.2) in such a way that for $k=1$, 2 the results of [DO], [GR] and [DL1] are reobtained.

1. The tangent bundle of order k. Let M be an n-dimensional manifold. The tangent bundle of order k of M is the $(k+1)$-dimensional manifold T^kM of k-jets at $0\in R$ of differentiable mappings $\sigma: R\to M$. We denote by $\rho^k: T^kM\to M$ the canonical projection defined by $\rho^k(j^k\sigma)=\sigma(0)$. Then T^kM has a bundle 0 structure over M. If $k=1$, $T^1M=TM$ is the tangent bundle of M. However, if $k>1$, $\rho^k: T^kM\to M$ is not a vector bundle. As well as being fibred over M, T^kM is fibred over T^rM, $0<r<k$. A projection map $\rho^k_r: T^kM\to T^rM$ is defined by $j^k_0\sigma\to j^r_0\sigma$.

These projection maps satisfy $\rho^k_s=\rho^r_s\rho^k_r$, for each r, s with $0\leqslant s<r<k$; here ρ^k_0 is to be interpreted as ρ^k.