4. Let g be a metric tensor on M.

Definition 4. *The tensor G on the total space ${}^2M$ induced by the metric tensor g on M and defined in the following way:*

$$G(\tilde{X}, \tilde{Y}) = g({}^2_0\pi_* \tilde{X}, {}^2_0\pi_* \tilde{Y}) + g({}^1_0\pi_* \hat{D}\tilde{X}, {}^1_0\pi_* \hat{D}\tilde{Y}) + g(D\hat{D}\tilde{X}, D\hat{D}\tilde{Y}) \tag{4,1}$$

is called a diagonal lift of g.

In the adapted coframe (1,13) on ${}^2M$ the diagonal lift G of g is of the form:

$$G = g_{ij}\omega^{0i} \otimes \omega^{0j} + g_{ij}\omega^{1i} \otimes \omega^{1j} + g_{ij}\omega^{2i} \otimes \omega^{2j}. \tag{4,2}$$

The covariant derivative of the tensor G with respect to the connection $\overset{H}{\nabla}$ on ${}^2M$ is defined by the formula:

$$(\overset{H}{\nabla}_{\tilde{X}} G)(\tilde{Y}, \tilde{Z}) = \tilde{X}(G(\tilde{Y}, \tilde{Z})) - G(\overset{H}{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{Z}) - G(\tilde{Y}, \overset{H}{\nabla}_{\tilde{X}}\tilde{Z}). \tag{4,3}$$

Thus, we have the formulas for the covariant derivative of the diagonal lift G of g with respect to the horizontal lift $\overset{H}{\nabla}$ of ∇:

$$\begin{aligned} &(\overset{H}{\nabla}_{X^H} G)(Y^H, Z^H) = (\nabla_X g)(Y, Z), && (\overset{H}{\nabla}_{X^H} G)(Y^{V_a}, Z^H) = 0, \\ &(\overset{H}{\nabla}_{X^H} G)(Y^{V_a}, Z^{V_a}) = (\nabla_X g)(Y, Z), && (\overset{H}{\nabla}_{X^H} G)(Y^{V_a}, Z^{V_b}) = 0, \\ &(\overset{H}{\nabla}_{X^{V_a}} G)(Y^H, Z^H) = 0, && (\overset{H}{\nabla}_{X^{V_a}} G)(Y^{V_b}, Z^H) = 0, \\ &(\overset{H}{\nabla}_{X^{V_a}} G)(Y^{V_b}, Z^{V_c}) = 0, && a, b, c = 1, 2. \end{aligned} \tag{4,4}$$

Thus, we have:

Proposition 8. *If ∇ is the Riemannian connection with respect to a metric tensor g on M, then the horizontal lift $\overset{H}{\nabla}$ of a connection ∇ into ${}^2M$ is a metric connection for the diagonal lift G of the metric tensor g.*

REFERENCES

1. Bowman, R. H. — *Second Order Connections*, J. Differ. Geom. 7(1972), 549–561.
2. Dragomir, S. and Radivoiovici, M. — *Some algebraic properties concerning the tangent bundle of second order*, Demonstratio Math., (15) (1982), 993–1006.
3. Kurek, J. — *On some Riemannian structure on linearized tangent bundle of second order*, Annales U M C S, 35 (9) (1981), Sectio A, 81–86.
4. Yano, K. and Ishihara, S. — *Tangent and Cotangent Bundles*, New York, 1973.

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CONNECTIONS OF ORDER k ON M AND ASSOCIATED POLYNOMIAL STRUCTURES ON T^kM

BY

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0. Introduction. The differential geometry of the tangent bundles of higher order has been extensively studied by many authors, for example Morimoto [MO], Yano and Ishihara [YI2], Catz [C1] [C2], Tulczyjew [TU1] [TU2], de Leon and Villaverde [DLV1] [DLV2], de Leon and Rodrigues [DLR2], Crampin, Sarlet and Cantrijn [CSC]. Recently, the study of higher order mechanics in the framework of the theory of higher order tangent bundles has been undertaken by a number of authors (see [DL2], [DL3], [DLR1], [DLR2], [CSC]). On the other hand, it is well known the construction by Dombrowski [DO] of an almost complex structure on the tangent bundle TM of a Riemannian manifold M. Dombrowski construction has been generalized by Grifone [GR] for manifolds with a non homogeneous connection. Grifone shows that the corresponding almost complex structure on TM plays an important role in the developping of Classical Mechanics. The construction of Grifone has been extended in [DL1] to the tangent bundle of order 2, T^2M, by showing that there exists a f-structure on T^2M associated to each connection of order 2 on M.

Our aim in this paper is to generalize the constructions of Grifone and de León for the general case. In fact, we show that there exists an appropiate polynomial structure (said a $\Phi(k+1, k-1)$-structure) on the tangent bundle T^kM of order k of M associated to each connection of order k on M. This sort of polynomial structure has been studied in [DL4]. Here, we apply the results of that paper to obtain a sufficient condition for the partial integrability of the $\Phi(k+1, k-1)$-structure on T^kM (Theorem 4.2) in such a way that for $k=1$, 2 the results of [DO], [GR] and [DL1] are reobtained.

1. The tangent bundle of order k. Let M be an n-dimensional manifold. The tangent bundle of order k of M is the $(k+1)$-dimensional manifold T^kM of k-jets at $0 \in R$ of differentiable mappings $\sigma: R \to M$. We denote by $\rho^k: T^kM \to M$ the canonical projection defined by $\rho^k(j^k\sigma) = \sigma(0)$. Then T^kM has a bundle 0 structure over M. If $k=1$, $T^1M = TM$ is the tangent bundle of M. However, if $k>1$, $\rho^k: T^kM \to M$ is not a vector bundle. As well as being fibred over M, T^kM is fibred over T^rM, $0<r<k$. A projection map $\rho^k_r: T^kM \to T^rM$ is defined by $j^k_0\sigma \to j^r_0\sigma$.

These projection maps satisfy $\rho^k_s = \rho^r_s \rho^k_r$, for each r, s with $0 \leqslant s < r < k$; here ρ^k_0 is to be interpreted as ρ^k.

Notice that T^kM is associated with the principal bundle F^kM of the frames of order k on M (see [GA]). In fact, T^kM is the tangent bundle of the p^k-velocities of M, $p=1$, introduced by Ehresman (see [EH], [MO], [TU2]).

Remark 1. The fibration $T^{2k-1}M \to T^{k-1}M$, $k \geq 1$, plays an important role in the Generalized Classical Mechanics. We remit to the reader to [DLR1] and [DLR2].

Let (U, x^i) be a coordinate neighborhood of M and denote by $(z^i_0, z^i_1, \ldots, z^i_k)$ the induced system of coordinates in $(\rho^k)^{-1}(U)=T^kU$. We can define a canonical map $T: T^{k+1}M \to T(T^kM)$ by $j_0^{k+1}\sigma \to j_0^1\tau$, where $\tau: R \to T^kM$, $t \to \tau(t)=j_0^k\sigma_t$, and $\sigma_t(s)=\sigma(s+t)$. It is convenient to describe T in local coordinates. A simple computation (see [DLR2]) shows that T is locally given by

$$(z^i_0, z^i_1, \ldots, z^i_{k+1}) \to (z^i_0, \ldots, z^i_k ; z^i_1, 2z^i_2, \ldots, kz^i_k).$$

We use the map T to construct a differential operator d_T which maps each function on T^kM to a function on $T^{k+1}M$ (see [TU1], [CSC]). If f is a function on T^kM then d_Tf is the function an $T^{k+1}M$ defined by

$$(d_Tf)(j_0^{k+1}\sigma)=df(j_0^k\sigma)(T(j_0^{k+1}\sigma)).$$

Then d_Tf is locally expressed by

$$(d_Tf)(z^i_0, \ldots, z^i_{k+1})=\sum^k_{r=0}(r+1)z^i_{r+1}\,\partial f/\partial z^i_r.$$

So, we have

(1.1) $\qquad d_T(z^i_r)=(r+1)^i_{r+1}$, and (1.2) $\qquad d^r_T(z^i_0)=r!\, z^i_r$,

with $0 \leq r < k$.

From (1.1) and (1.2), we deduce

Proposition 1.1. *Let $\tilde X$ and $\tilde Y$ be vector fields on T^kM. Then $\tilde X=\tilde Y$ if and only if $\tilde X((\rho^k_r)^* d^r_T f)=\tilde Y((\rho^k_r)^* d^r_T f)$, for every function f on M, $0 \leq r \leq$ ≤ 0. (Here ρ^k_k is to be interpreted as the identity map.)*

Now, we shall describe a lifting operation which generalises the vertical lift in tangent bundle geometry, as described in [YI2] for example.

Definition 1.1. *Let X be a vector field on T^rM, $0 \leq r < k$. The vertical lift of X to T^kM (with respect to $\rho^k_{k-r-1}: T^kM \to T^{k-r-1}M$) is the unique vector field $\tilde X^k$ on T^kM defined*

$$\tilde X^k((\rho^k_s)^* d^s_T f)=\begin{cases} 0, & \text{if } 0 \leq s \leq k-r-1 \\ \dfrac{s!}{(s-(k-r))!} X((\rho^k_{s-(k-r)})^* d_T^{s-(k-r)} f), & \text{if } k-r \leq s \leq k \end{cases}$$

for every function f on M.

(We remark that the case $r=0$ was considered by Crampin, Sarlet and Cantrijn [CSC].)

If $X=\sum^r_{s=0} X^i_s\,\partial/\partial z^i_s$, then we deduce from (1.1) and (1.2) that $\tilde X^k$ is locally expressed by

(1.3) $$\tilde X^k=\sum^r_{s=0} X^i_s\,\partial/\partial z^i_{k-r+s}.$$

Using the vertical lift and the map $T: T^kM \to T(T^{k-1}M)$ we construct a canonical vector field C_1 on T^kM as follows:

$$C_1(j_0^k\sigma)=(\widetilde{T(j_0^k\sigma)})^k.$$

So, C_1 is locally expresed by

(1.4) $$C_1=\sum^k_{s=1} sz^i_s\,\partial/\partial z^i_s.$$

The vector field C_1 is a generali ion of the Liouville vector field (or dilation field) on TM (see [CSC], [DLR2]).

We may also use the vertical lift construction to define k tensor fields of type (1.1) (or vector 1-forms) on T^kM (see [CSC]). In fact, for every r, $1 \leq r \leq k$, we can define a linear endomorphism J_r of the tangent space $T_z(T^kM)$ of T^kM at z as follows:

$$(J_r)_z(X)=((\rho^k_{k-r})_* X)^k.$$

From (1.3), one easily deduces that J_r is locally expressed by

(1.5) $$((X^i_0, X^i_1, \ldots, X^i_k) \to (\overset{r}{0, \ldots, 0}, X^i_0, \ldots, X^i_{k-r}).$$

So, we have

Proposition 1.2. *J_r has constant rank $(k-r+1)n$ and satisfies*

$$(J_r)^s=0, \text{ if } rs \geq k+1 \text{ and } J_{rs}, \text{ if } rs < k+1.$$

Moreover, J_1 determines on T^kM an almost tangent structure of order k.

Remark 2. An alternative definition of J_r, $1 \leq r \leq k$, has been given by de León and Villaverde ([DLV1]) and de León and Rodrigues ([DLR2]) following work by Catz ([C]) on the case $k=2$.

If we denote by $C_r=J_{r-1}C_1$, $2 \leq r \leq k$, we have the following identities (see [DLV1], [DLR2]):

(1.6) $$J_rC_s=0, \text{ if } r+s \geq k+1 \text{ and } C_{r+s}, \text{ if } r+s < k+1,$$

(1.7) $$[C_r, J_s]=0, \text{ if } r+s > k+1 \text{ and } -sJ_{r+s-1}, \text{ if } r+s \leq k+1,$$

(1.8) $$[J_r, J_s]=0.$$

A direct consequence of (1.8) is the vanishing of the Nijenhuis tensor $N_{J_1}=$ $=1/2[J_1, J_1]$ of J_1. Therefore, the tangent complex structure of order k on T^kM determined by J_1 is always integrable (see [DLV1], [DLR2]).

2. **Homogeneous and semibasic forms.** Let us introduce the following definitions (see [DLV2], [DLR2]).

(a) Homogeneous forms

Definition 2.1. *A function f on T^kM is said to be homogeneous of degree α if and only if $\mathcal{L}_{C_1}f=\alpha f$. Always, $\mathcal{L}$ denotes the Lie derivative.*

Let $h_t: R \to R$ be the homothetia of ratio e^t and let $H_t: T^kM \to T^kM$ denote the fibre-preserving transformation deduced from h_t. Since C_1 generates the 1-parameter group of transformations H_t (see [DLV2], [DLR2]), then the condition in Definition 2.1 is equivalent to $f \circ H_t=e^{\alpha t}f$.

Definition 2.2. *A differentiable scalar p-form* ω *on* T^kM *is said to be homogeneous of degree* α *if* $L_{C_1}\omega=\alpha\omega$.

Definition 2.3. *A differentiable vector l-form* L *on* T^kM *is said to be homogeneous of degree* α *if* $[C_1, L]=(\alpha-1)L$.

(b) Semibasic forms

Definition 2.4. *A vector l-form* L *on* T^kM, *with* $l\geqslant 1$, *is said to be semibasic if* $J_kL=0$ *and* $i_{J_rX}L=0$, *for every vector field* X *on* T^kM.

3. **Connections of order k.** Our aim in this section is to get a characterization of the connections on tangent bundles of higher order by means of the canonical almost tangent structures of higher order and develop a similar study to the one elaborated by Grifone [GR] in the case $k=1$.

Let $V^{\rho^k}(T^kM)$ denote the vector bundle over T^kM of those vectors of $T(T^kM)$ which are projected to 0 by $(\rho^k)_*$.

Definition 3.1. *By a connection of order* k *on* M *we mean a vector bundle* H *over* T^kM *such that*

$$T(T^kM)=V^{\rho^k}(T^kM)\oplus H.$$

H is usually called the horizontal bundle. The tangent vectors to T^kM *which belong to* H *will be called the horizontal tangent vectors.*

Then a connection of order k on M defines two projection operators $h: T(T^kM)\to H$ and $v: T(T^kM)\to V^{\rho^k}(T^kM)$ which will be called the *horizontal* and *vertical projectors*, respectively.

Since $\operatorname{Im} J_1=V^{\rho^k}(T^kM)$, we have

Proposition 3.1. *The correspondence* $h=1/2(I+\Gamma)$ *determines a bijection between the connections of order* k *on* M *and the vector 1-forms* Γ *on* T^kM *such that*

$$J_k\Gamma=J_k,\quad \Gamma J_1=-J_1.$$

In the sequel, a connection of order k on M will be denoted by the corresponding vector 1-form Γ.

Remark 1. We may also consider connections on the fibration $\rho^k_{k-r}: T^kM\to T^{k-r}M$, $1\leqslant r\leqslant k$. Since $\operatorname{Im} J_{k-r+1}=V^{\rho^k_{k-r}}(T^kM)$, a connection on $\rho^k_{k-r}: T^kM\to T^{k-r}M$ is given by a vector 1-form Γ on T^kM such that $J_r\Gamma=J_r$, $\Gamma J_{k-r+1}=-J_{k-r+1}$. A connection on $\rho^k_{k-r}: T^kM\to T^{k-r}M$ is called a connection on M of order k and type r (see [DLR2]).

Let Γ be a connection of order k on M. Then a simple computation shows that Γ can be locally represented by the matrix

$$(3.1)\qquad \begin{pmatrix} I_n & 0 & 0\cdots\cdots\cdots 0\\ -2\Gamma^{(1)} & -I_n & 0\cdots\cdots\cdots 0\\ -2\Gamma^{(2)} & 0 & -I_n\cdots\cdots\cdots 0\\ \vdots & \vdots & \\ -2\Gamma^{(k)} & 0 & 0\cdots\cdots\cdots -I_n \end{pmatrix},$$

with respect to the induced coordinates.

Now, we shall express the non-homogeneity of a connection of order k on M.

Definition 3.2. *Let* Γ *be a connection of order* k *on* M. *We shall call tension of* Γ *the vector* 1-*form on* T^kM *given by* $H=1/2[C_1, \Gamma]$.

A simple computation from (3.1) shows that H can be locally represented by the matrix

$$(3.2)\qquad \begin{pmatrix} 0 & 0\cdots\cdots\cdots 0\\ -\sum_{s=1}^k sz_s\,\partial\Gamma^{(1)}/\partial z_s+\Gamma^{(1)} & 0\cdots\cdots\cdots 0\\ \cdots\cdots\cdots & \vdots\\ -\sum_{s=1}^k sz_s\,\partial\Gamma^{(k)}/\partial z_s+k\Gamma^{(k)} & 0\cdots\cdots\cdots 0 \end{pmatrix}.$$

From (3.2), we easily deduce

Proposition 3.2. *H is a semibasic vector 1-form.*

Definition 3.3. *A connection* Γ *of order* k *on* M *is said to be homogeneous if its tension vanishes, that is,* Γ *is an homogeneous vector 1-form of degree* 1.

From (3.2), we deduce that a connection Γ is homogeneous if and only if the functions $(\Gamma^{(r)})^i_j$ are also homogeneous of degree r, $1\leqslant r\leqslant k$.

Remark 2. An homogeneous connection of order 1 on M is a linear connection on M (see [GR]).

Remark 3. Since T^kM is an associate fiber bundle to F^kM, we easily deduce that a connection of order k on M is associated to a connection on the principal bundle F^kM (see [NO], [LI]).

To end this section, we introduce the curvature of a connection of order k on M.

Definition 3.4. *Let* Γ *be a connection of order* k *on* M. *The curvature of* Γ *is the vector 2-form* R *on* T^kM *defined by* $R=-\dfrac{1}{2}[h, h]$, h *being the horizontal projection of* Γ.

From a direct computation, we easily deduce.

Proposition 3.3. *The curvature* R *of a connection* Γ *of order* k *on* M *is a semibasic form.*

Proposition 3.4. (Bianchi's identities). *Let* Γ *be a connection of order* k *on* M. *Then we have*

1) $[J_r, R]=[h, [J_r, h]]$, $1\leqslant r\leqslant k$, 2) $[h, R]=0$.

Proof. Let us recall Jacobi's identity for vector 1-forms L_1, L_2, L_3 (see [FN]):

$$[L_1, [L_2, L_3]]+[L_2, [L_3, L_1]]+[L_3, [L_1, L_2]]=0.$$

If we put $L_1=J_r$, $L_2=L_3=h$, we obtain

$$[J_r,[h, h]]+[h, [h, J_r]]+[h,[J_r, h]]=0,$$

that is, $[J_r, R]=[h,[J_r, h]]$. Analogously, if we put $L_1=L_2=L_3=h$ then we have $[h,[h, h]]=0$, and thus $[h, R]=0$.

Proposition 3.5. *Let Γ be a connection of order k on M. Then we have* $[C_1, R] = -[h, H]$.

Proof. By using the general Jacobi's identity (see [FN]), we obtain $[C_1, [h, h]] = 2[h, [C_1, h]]$. But, since $[C_1, h] = H$, we deduce $[C_1, R] = -[h, H]$.

Corollary 3.1. *If Γ is homogeneous then its curvature R is also an homogeneous vector form.*

Remark. If Γ is an homogeneous connection of order 1, then Γ defines a linear connection ∇ on M and its curvature form R determines the curvature tensor of ∇ (see [GR]).

4. $\Phi(k+1, k-1)$-structures on T^kM associated to connections of order k on M. Firstly, let us recall the notion of $\Phi(k+1, k-1)$-structures introduced in [DL4].

Suppose that there is given in a $(k+1)n$-dimensional manifold V a tensor field Φ of type (1, 1) satisfying

$$\Phi^{k+1} + \Phi^{k-1} = 0 \ ; \ \operatorname{rank} \Phi = \begin{cases} 2n, & \text{if } k=1 \\ kn, & \text{if } k \geq 2 \end{cases} ; \ \operatorname{rank} \Phi^r = 2n, \ 2 \leq r \leq k.$$

We call such a structure Φ a $\Phi(k+1, k-1)$-structure on V. If we put

$$l = (-1)^{k/2}\Phi^k, \quad m = I - (-1)^{k/2}\Phi^k, \text{ if } k \text{ even},$$
$$l = (-1)^{(k-1)/2}\Phi^{k-1}, \quad m = I - (-1)^{(k-1)/2}\Phi^{k-2}, \text{ if } k \text{ odd},$$

then l and m are projection operators which determine on V complementary distributions D_l and D_m, respectively; D_l is $2n$-dimensional and D_m is $(k-1)n$-dimensional. Moreover, Φ acts on D_l as an almost complex structure operator and on D_m as an operator of almost tangent structure of order $k-2$.

Accordingly with [YI1] and [CG], the following terminology has been introduced in [DL4]:

(i) *The $\Phi(k+1, k-1)$-structure Φ is said to be c-partially integrable if the distribution D_l is integrable and the almost complex structure $\Phi' = \Phi|_{D_l}$ is integrable.*

(ii) *Φ is said to be t-partially integrable if the distribution D_m is integrable on the almost tangent structure of order $k-2$ defined by $\Phi'' = \Phi|_{D_m}$ is integrable.*

(iii) *Φ is said to be partially integrable if Φ is c-partially integrable and t-partially integrable simultaneously.*

(iv) *Φ is said to be integrable if there exists a coordinate system in wich Φ has the constant components*

$$\left(\begin{array}{cc|ccccc}
0 & -I_n & & & 0 & & \\
I_n & 0 & & & & & \\
\hline
 & & 0 & 0 & \cdots & 0 & 0 \\
 & & I_n & 0 & \cdots & 0 & 0 \\
 & 0 & 0 & I_n & \cdots & 0 & 0 \\
 & & \vdots & \vdots & & \vdots & \\
 & & 0 & 0 & \cdots & I_n & 0
\end{array}\right).$$

Let us remark that the structure is trivially t-partially integrable for $k=1, 2$, and has a non-trivial geometrical significance for $k \geq 3$, actually, a $\Phi(2, 0)$-structure is an almost complex structure and a $\Phi(3, 1)$-structure is a Φ-structure of rank $2n$ in the sense of Yano and Ishihara ([YI1] [YI2])

The following theorem ([DL4]) summarizes the integrability conditions for a $\Phi(k+1, k-1)$-structure.

Theorem 4.1. *Let Φ a $\Phi(k+1, k-1)$-structure on V and let N_Φ denote the Nijenhuis tensor of Φ. Then:*

(i) *Φ is c-partially integrable (resp., t-partially integrable) if and only if $N_\Phi(lX, lY) = 0$ (resp., $N_\Phi(mX, mY) = 0$) for any two vector fields X and A on V.*

(ii) *Φ is partially integrable if and only if*
$N_\Phi(X, Y) = N_\Phi(lX, mY) + N_\Phi(mX, lY)$ *for any two vector fields X and A on V.*

(iii) *Φ is integrable if and only if N_Φ vanishes identically.*

After this brief summary of definitions and results about $\Phi(k+1, k-1)$ structures, we shall pass again to the consideration of connections of order k on a manifold M.

Proposition 4.1. *Let Γ be a connection of order k on M, with horizontal projector h. Then there exists an unique vector 1-form Φ on T^kM such that*

$$\Phi J' = h, \quad \Phi h = -J', \quad \Phi J_r = J_{r+1}, \quad 2 \leq r \leq k-1, \quad \Phi J_k = 0,$$

where $J' = J_1 h$.

Obviously, Φ is well-defined from these identities and it is uniquely determined by its action on vertical and horizontal vector fields. A direct computation from (2.1) and Proposition 4.1 shows that Φ can be represented by the matrix

$$\begin{pmatrix}
\Gamma^{(1)} & I_n & 0 & \cdot & 0 & 0 \\
-I - \Gamma^{(1)}\Gamma^{(1)} & -\Gamma^{(1)} & 0 & \cdot & 0 & 0 \\
\Gamma^{(1)} - \Gamma^{(1)}\Gamma^{(2)} & -\Gamma^{(2)} & 0 & \cdot & 0 & 0 \\
2\Gamma^{(2)} - \Gamma^{(1)}\Gamma^{(2)}\Gamma^{(3)} + \Gamma^{(1)}\Gamma^{(1)} & -\Gamma^{(3)} + \Gamma^{(1)} & I_n & \cdot & 0 & 0 \\
\cdots\cdots\cdots\cdots & \cdots\cdots & & \vdots & \vdots & \vdots \\
2\Gamma^{(k-1)} - \Gamma^{(1)}\Gamma^{(k)} + \Gamma^{(1)}\Gamma^{(k-2)} & -\Gamma^{(k)} + \Gamma^{(k-2)} & 0 & \cdot\, 0 & I_n & 0
\end{pmatrix}.$$

Then we have

Proposition 4.2. *The vector 1-form Φ defines on T^kM a $\Phi(k+1, k-1)$-structure, which is said to be associated to the connection Γ.*

In order to study the integrability of Φ, let us remark, first of all, that the distribution D_m defined from Φ is always completely integrable because $D_m = V^{\rho^k_1}(T^kM)$.

Furthermore, we can state the following

Proposition 4.3. *The $\Phi(k+1, k-1)$-structure Φ associated to Γ is always t-partially integrable.*

Proof. Once more, remark that the result is trivially true for $k=1, 2$. If $k \geq 3$, we have

$$N_\Phi(J_2X, J_2Y)=[J_3X, J_3Y]-\Phi[J_3X, J_2Y]-\Phi[J_2X, J_3Y]+\Phi^2[J_2X, J_2Y].$$

Now, if X and Y are coordinates frames, we obtain $N_\Phi(J_2X, J_2Y)=0$, and, because the tensorial character of N_Φ and J_2, we finally get $N_\Phi(J_2X, J_2Y)=0$ for every two vector fields X and Y on T^kM. The results follows from Theorem 4.1 (i) and the identity $\mathrm{Im}\, J_2=D_m$.

Corollary 4.1. *Φ is partially integrable if and only if it is c-partially integrable.*

Before proceeding further, we shall prove two auxiliary lemmas.

Lemma 4.1. *The vector 2-forms $[J', h]$ and $N_{J'}$, both are semibasic.*

Proof. Let X, Y arbitrary vector fields on T^kM. Then, we have

$$[J', h](J_1X, Y)=J'[J_1X, Y]-J'[J_1X, hY]=J'[J_1X, Y-hY]=$$
$$=J'[J_1X, vY]=0.$$

$$J_k[J', h](X, Y)=J_k[J'X, hY]+J_k[hX, J'Y]-J_k[J'X, Y]-J_k[X, J'Y]=J_k([J'X, hY]-[J'X, Y])+J_k([hX, J'Y]-[X, J'Y])=$$
$$=J_k[J'X, vY]-J_k[vX, J'Y]=0.$$

This implies that $[J', h]$ is semibasic.

Now, since $[J_1X, J'Y]$ is a vertical vector field, we have

$$N_{J'}(J_1X, Y)=-J'[J_1X, J'Y]=0.$$

Moreover,

$$J_kN_{J'}(X, Y)=J_k[J'X, J'Y]=0,$$

and then $N_{J'}$ is also proved to be semibasic.

Lemma 4.2. *We have*

$$(J_1[J', h])(X, Y)=N_{J'}(X, Y)+J_2v[hX, hY].$$

for every two vector fields X and A on T^kM.

Proof. Since $[J', h]$ is semibasic, we have

$$[J', h](X, Y)=[J', h](hX, hY)=[J'X, hY]+[hX, J'Y]-J'[hX, hY]-$$
$$-h[J'X, hY]-h[hX, J'Y].$$

Then

$$(J_1[J', h])(X, Y)=J_1[J'X, hY]+J_1[hX, J'Y]-J_2h[hX, hY]-$$
$$-J'[J'X, hY]-J'[hX, J'Y].$$

Since J_1 is integrable, we have

$$0=N_{J_1}(hX, hY)=[J'X, J'Y]-J_1[J'X, hY]-J_1[hX, J'Y]+J_2[hX, hY].$$

So,

$$J_1[J'X, hY]+J_1[hX, J'Y]=[J'X, J'Y]+J_2[hX, hY].$$

On the other hand,

$$N_{J'}(X, Y)=N_{J'}(hX, hY)=[J'X, J'Y]-J'[J'X, hY]-J'[hX, J'Y],$$

and, now, the result follows immediately.

Theorem 4.2. *Let Γ be a connection of order k on M, with curvature form R. If the distribution D_1 is completely integrable, $R=0$ and $[J', h]=0$,*

then the $\Phi(k+1, k-1)$-structure Φ associated to Γ is c-partially integrable, or, equivalently, partially integrable.

Proof. Since $[J', h]$ is semibasic, we have

$$[J', h](X, Y)=[J', h](hX, hY)=[J'X, hY]+[hX, J'Y]-J'[hX, hY]$$
$$-h[J'X, hY]-h[hX, J'Y].$$

Therefore,

$$(\Phi[J', h])(X, Y)=\Phi[J'X, hY]+\Phi[hX, J'Y]-h[hX, hY]+$$
$$+J'[J'X, hY]+J'[hX, J'Y].$$

On the other hand,

$$(h^*N_\Phi)(X, Y)=[J'X, J'Y]+\Phi[J'X, hY]+\Phi[hX, J'Y]+\Phi^2[hX, hY],$$

$$N_{J'}(X, Y)=N_{J'}(hX, hY)=[J'X, J'Y]-J'[J'X, hY]-J'[hX, J'Y].$$

Now, $R=0$ implies $h[hX, hY]=[hX, hY]$, and then $\Phi^2[hX, hY]=-h[hX, hY]$.

Thus, we can write

$$(h^*N_\Phi)(X, Y)=(\Phi[J', h])(X, Y)+N_{J'}(X, Y),$$

that is $h^*N_\Phi=\Phi[J', h]+N_{J'}$, and, then, by using Lemma 4.2, we deduce $h^*N_\Phi=0$. We also have

$$(J')^*N_\Phi(X, Y)=[hX, hY]-\Phi[hX, J'Y]-\Phi[J'X, hY]+\Phi^2[J'X, J'Y],$$

and, on the other hand, $N_{J'}=0$ implies $[J'X, J'Y]\in \mathrm{Im}\, J'$. Therefore,

$$(J')^*N_\Phi(X, Y)=[hX, hY]-\Phi[hX, J'Y]-\Phi[J'X, hY]-[J'X, J'Y_n]=$$
$$=-(h^*N_\Phi)(X, Y),$$

that is, $(J')^*N_\Phi=-h^*N_\Phi=0$. Finally,

$$N_\Phi(J'X, hY)=-[hX, J'Y]-\Phi[hX, hY]+\Phi[J'X, J'Y]+\Phi^2[J'X, hY]$$

and, on the other hand, since the distribution D_1 is integrable, we have

$$(\Phi h^*N_\Phi)(X, Y)=\Phi[J'X, J'Y]+\Phi^2[J'X, hY]+\Phi^2[hX, J'Y]-\Phi[hX, hY]=$$
$$=\Phi[J'X, J'Y]+\Phi^2[J'X, hY]-[hX, J'Y]-\Phi[hX, hY].$$

Then, we deduce

$$N_\Phi(J'X, hY)=(\Phi h^*N_\Phi)(X, Y)=0,$$

which ends the proof.

Remark. For $k=1$, Φ is an almost complex structure on TM (see Grifone [GR]). In particular, if Γ is the homogenous connection defined by a Riemannian metric on M, we reobtain the well-known result of Dombrowski [DO]. The case $k=2$ has been considered by de Leon [DL1].

REFERENCES

[C1] Catz G. — *Sur le fibré tangent d'ordre 2*, C.R. Acad. Sci. Paris, 278 (1974), 277–280.

[C2] Catz G. — *Gerbes et connexions sur le fibré tangent d'ordre 2*, C.R. Acad. Sci. Paris 278 (1974), 347–349.

[CSC] Crampin M., Sarlet W., Cantrijn F. — *Higher order differential equations and higher order Lagrangian mechanics*, (1985) Preprint.

[DL1] De Leon M. — *Connections and f-structures on T^2M*, Kodai Math. J. 4 (1981), 189—216.

[DL2] De Leon M. — *Systèmes Lagrangiens règuliers d'ordre supérieur*, C.R. Acad. Sci. Paris 294 (1982), 451—453.

[DL3] De Leon M. — *Transformation de Legendre pour les systémes lagrangiens d'ordre supérieur*. C.R. Acad. Sci. Paris 295 (1982), 123—125.

[DL4] De Leon M. — *Existence and integrability conditions of $\Phi(k+1, k-1)$-structures on $(k+1)n$-dimensional manifolds*, Rev. Roumaine Math. Pures Appl. 29 (1984), 479—488.

[DLR1] De Leon M., Rodrigues P.R. — *Formalisme hamiltonien symplectique sur les fibrés tangents d'ordre supérieur*. C.R. Acad. Sci. Paris 301 (1985), 455—458.

[DLR2] De Leon M., Rodrigues P.R. — *Generalized Classical Mechanics and Field Theory*, North-Holland, Amsterdam, (1985).

[DLV1] De Leon M., Villaverde C. — *Calcul différentiel sur les fibrés tangents d'ordre supérieur*, C.R. Acad. Sci. Paris 292 (1981), 881—884.

[DLV2] De Leon M., Villaverde C. — *Sprays et connexions sur les fibrés tangents d'ordre supérieur*, C.R. Acad. Sci. Paris 293 (1981), 51—54.

[DO] Dombrowski P. — *On the geometry of the tangent bundle*, J. Reine Angew. Math. 210 (1962), 73—88.

[EH] Ehresmann Ch. — *Les prolongaments d'une space fibré différentielle*, C.R. Acad. Sci. Paris 240 (1955), 1 755—1 757.

[FN] Frolicher A., Nijenhuis A. — *Theory of vector-valued differential forms*, Proc. Kon. Akad. A 59 (1956), 338—359.

[CG] Gadea P.M., Cordero L.A. — *On integrability conditions of a structure Φ satisfying $\Phi^4+\Phi^2=0$*, Tensor N.S. 28 (1974), 78—82.

[GA] Gancarzewicz J. — *Liftings of functions and vector fields to natural bundles*, Dissertationes Mathematicae CCXII (1983).

[GO] Godbillon C. — *Géométrie différentielle et mecanique analytique*, Hermann, Paris (1969).

[GR] Grifone J. — *Structure presque-tangente et connexions I*, Ann. Inst. Fourier, Grenoble, 22, (1) (1972), 287—334.

[KA] Kandatu A. — *Tangent bundles of a manifold with non linear connection*, Kodai Math. Sem. Rep. 18 (1966), 250—270.

[LI] Libermann P. — *Calcul tensoriel et connexions d'ordre supérieur*, Annals Acad. Brasileira de Ciencias 37 (1965), 17—29.

[MO] Morimoto A. — *Prolongations of geometric structures*, Lect. Notes, Math. Inst. Nagoya Univ. (1969).

[NO] Nomizu K. — *Lie groups and differential geometry*, Publ. of Math. Soc. Japan, Nagoya (1956).

[R] Roux A. — *Jets et connexions*, Pub. Math. Univ. Lyon 7 (7) (1970), 1—42.

[TO] Tachibana S., Okumura M. — *On the almost complex structure of tangent bundles of Riemannian spaces*, Tohoku Math. J. 14 (1962), 156—161.

[TU1] Tulczyjew W. — *Sur la différentielle de Lagrange*, C.R. Acad. Sci. Paris 280 (1975), 1 295—1 298.

[TU2] Tulczyjew W. — *Les jets généralisés*, C.R. Acad. Sci. Paris 281 (1975), 349—352.

[V1] Vilms J. — *Connections on tangent bundles*, J. Diff. Geometry 1 (1967), 235—243.

[V2] Vilms J. — *Curvature of non linear connections*, Proc. Amer. Math. Soc. 19 (1968), 1125—1129.

[YI1] Yano K., Ishihara S. — *On integrability conditions of a structure satisfying $f^3+f=0$*, Quart. Math. Oxford, 15 (1964), 217—222.

[YI2] Yano K., Ishihara S. — *Tangent and cotagent bundles*, Marcel Dekker, New York (1973).

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ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
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RECENZII

F. CAJORI: *History of Mathematics*, 4th Ed., Chelsea Publishing Company, New York, 1985, VIII + 524 p., $ 22.50

A patra ediţie a istoriei matematicii scrisă în 1893 de F. Cajori, apărută în condiţii grafice excelente, frumos scrisă şi plăcută la lectură, acoperă o perioadă imensă, din antichitate pînă înainte de primul război mondial. În afara părţii referitoare la matematica babiloniană, rescrisă de editor după recentele descoperiri ale lui O. Neugebauer, A. Sachs şi alţii, cartea a rămas aşa cum era în 1980. Ea conţine capitole speciale, consacrate matematicii din Babilon, Egipt, Grecia, Roma, China, Japonia, India, matematicii din Europa Evului Mediu, din Europa secolelor X—XX.

Mirela Ştefănescu

W. PURKERT, H. J. ILGAUDS: *Georg Cantor 1845—1918*, Colecţia „Vita Mathematica", Band 1, Birkhäuser Verlag, Basel, Boston, Stuttgart, 1987, 262 p. sFr 48.00

Una din cele mai impresionante vieţi şi activităţi din istoria matematicii — a lui Georg Cantor — este descrisă cu talent şi pasiune de doi tineri cercetători în istorie, care în acelaşi timp inaugurează o serie nouă — Vita Mathematica — a editurii Birkhäuser. Cartea cuprinde capitole privitoare la personalitatea lui Cantor, dar, cel mai cuprinzător capitol se referă, cum era de aşteptat, la geneza teoriei mulţimilor. Cititorul va afla, de asemenea, multe alte date interesante privitoare la societăţile şi la periodicele matematice pentru crearea cărora au militat mulţi din clasicii ştiinţei.

Adolf Haimovici

Die Werke von DANIEL BERNOULLI, Bd. 3, Mechanik, Birkhäuser Verlag, Basel, Boston, Stuttgart, 1987, XXVII + 457 p., 188 sFr.

În colecţia operelor familiei Bernoulli, editată la Birkhauser de Naturforschende Gesellschaft din Basel, volumul de faţă este consacrat lucrărilor de Mecanică ale lui Daniel Bernoulli. El este prelucrat şi comentat de D. Speiser, A. de Baenst-Vandenbroucke, J. L. Pietenpol, P. Radelet — de Grave şi are contribuţia lui H. Straub. O introducere generală prezintă lucrările inserate în volum, locul lui D. B. în tradiţia ştiinţifică, unele date personale. Lucrările sînt precedate de comentarii substanţiale care lămuresc textele originale, acestea, de multe ori prezentînd dificultăţi pentru un cititor neavizat.

Adolf Haimovici

H. KURKE, M. ROCZEN (Editors): *Proceedings of the Conference on Algebraic Geometry*, Berlin, November 13—19, 1985, on the occasion of the 175th anniversary of the Humboldt University Berlin, Teubner-Texte zur Mathematik, Band 92, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1986, 308 p., 32,00 M

Lucrările publicate în prezenta colecţie arată dezvoltările şi progresele recente în geometria algebrică şi domenii conexe. Cîteva exemple alese la întîmplare dintre cele 26 de lucrări: Hauser, H., Müller, G. — Singularităţi armo-