Iași was invited to deliver lectures at different imposing academic institutions in France, Great Britain, the Soviet Union, Hungary, Italy, Germany, Japan etc. Beginning with 1972 he was charged with Ph. D. programmes. Over six candidates passed under his guidance, having the thesis inspired by their master's researches.

Peerless teacher, having the gift of speech, his classes and his lectures are full of point and charm. His speeches are impressive by the exquisite exposing and rich information. Loved and esteemed by his students and by all his colleagues, professor Radu Miron gains the admiration for his enthusiasm for science, for the quality of his researches and for his major contribution to the progress of the old School of mathematics in Iași as a teacher and as a scientist.

He contributed to the ascent of the secondary and higher education by many conferences, lectures, lessons, textbooks and monographs. Each of them reflects his genuine spirit and his great care for the novelty and accessibility of the matters.

Together with some other distinguished professors of the department, in 1968–1969 he drafted the textbook comprising the courses of analytical and differential geometry in two volumes, destined to the first two-years students of the Mathematics Department.

Together with I. Pop he published, in 1974 at Editura Academici R.S.R, the monograph entitled Algebraic Topology, *entirely written in the language of categories and in which the authors brought several original contributions out of which we mention the new concept of* Abstract Cech Cohomology associated to a projective system *which had the merit of considerably simplifying the study of Cech cohomology with coefficients in a presheaf. This paper destined to Ph D programmes doctor's degree and to the last-year students of the Mathematics Department was also a starting point for several candidates' theses.*

The celebration of Professor Radu Miron when having reached six decades of life and over 35 years of scientific activity is indeed a festive moment for the Seminary of Mathematics of Iași that had the major part in his training up and in his consecration in the field of Science and Teaching.

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NATURAL CONTINUITY OF A H-MAP

BY

EUGEN POPA

§0. **Introduction.** In a fundamental paper [4], H. Cartan discovered the so-called quasi-continuity property of a superharmonic function, namely that for each positive superharmonic function s on X, and $\varepsilon>0$, there exists an open set $\omega\subseteq X$, of capacity $<\varepsilon$, such that $s|_{X\setminus\omega}$ is continuous.

Fuglede [7] (see also Brelot [3]) proposes axiomatisations of this property. In [6] it is shown that, on harmonic spaces, the fact that every positive, superharmonic function is quasi-continuous, is equivalent with the axiom of domination.

In the setting of a H-cone, the notion of quasi-continuous element was introduced recently [1], and only as a technical tool.

One defines [5] the harmonic morphisms between harmonic spaces, as a *continuous* map, which preserves (super-)harmonicity. However, between the fine structures considered by Fuglede [5] or between H-cones, one is naturally led to abandone the condition of continuity (see [9], [2], [13]).

In this paper, we analyse the continuity property of the morphisms between H-cones of functions. The main result is contained in th. 1, which is an adaptation, to the frame of H-cones, of the results of Fuglede [9].

All the notions and notations about H-cones are as in [2]. If S is an H-cone of functions on the set X and $A\subseteq X$, then by: cl A, int A, ∂A we shall denote respectively: the *fine closure*, the *fine interior*, the *fine boundary* of the set A. If S and T are H-cones of functions on the sets X and Y, then the function $\varphi: Y\to X$ is called H-map if, for any bounded $s\in S$, $s\circ\varphi\in T$ (see [13]).

§1. **Quasi-continuous elements in a H-cone.** Let S be a standard H-cone of functions on the set X. For each $s\in S$ we denote:

$\mathcal{R}(s)=\{K\subseteq X \mid K$ *naturally compact*, $s|_K$ *naturally continuous*

$s\in S$ is called *quasi-continuous* if, for each $p\in S_0$ we have:

$$\bigwedge_{K\in\mathcal{R}(s)} B^{X\setminus K}p=0.$$

We denote by Q the set of all quasi-continuous elements of S. Then, the same proof as in [6] shows:

Proposition 1. a) *Q is a min-stable, convex subcone of S.*

b) *Q is a specific band,*

c) *Let $F\subseteq S$ be an increasing family, such that:*

$$\sup\{t(x) \mid t\in F\}=+\infty\ \forall,\ x\in X.$$

If $s\in S$ is such that $s\wedge t\in Q$, $\forall t\in F$, then $s\in Q$.

We need the following:

Lemma ([1, prop. 1.2]). *Let $(A_i)_{i\in I}$ be a decreasing family of finely closed sets from X. For any finite $s\in S$:*

$$\bigwedge_{i\in I} B^{A_i} s(x) = \inf_{i\in I} B^{A_i}s(x), \qquad \forall x\in X\setminus \bigcap_{i\in I} A_i.$$

Proposition 2. *For each $s\in Q$ there exists a polar, finely closed set $A\subseteq X$, such that each $x\in X\setminus A$ has a naturally compact, fine neighbourhood $K\in\mathcal{R}(s)$.*

Proof. Let us denote:

$$A = X\setminus \bigcap_{K\in\mathcal{R}(s)} \operatorname{int} K.$$

Since: $B^{X\setminus K}=B^{\mathrm{cl}(X\setminus K)}$, the lemma shows that $B^A p=0$, $\forall p\in S_0$.

Let us suppose that $Q\neq\varnothing$. This fact is equivalent with : $0\in Q$ or with : $S_0\subseteq Q$. Under this assumption, prop. 1 shows:

Corollary 3. *Any nearly continuous element from S is quasi-continuous.*

Corollary 4. *If S satisfies the axiom of nearly continuity, then $Q=S$.*

§ 2. **The quasi-continuity of a H-map.** Let S and T be standard H-cones of functions on the sets X and Y. Let $\varphi: Y\to X$ be a H-map. We denote:

$R(\varphi)=\{K\subseteq Y \mid K$ *naturally compact,* $\varphi|_K$ *naturally continuous*$\}$.

Theorem 1 [9]. a) *The following properties are equivalent:*

(i) $s\in S_0 \Rightarrow s\circ\varphi$ *is quasi-continuous;*

(ii) $\forall t\in T_0$:

$$\bigwedge_{K\in\mathcal{R}(\varphi)} B^{X\setminus K}t=0;$$

b) *Each of the properties (i) or (ii) above implies:*

$\exists A\subseteq Y$ polar, such that each $y\in Y\setminus A$ has a naturally compact, fine neighbourhood $K\in\mathcal{R}(\varphi)$.

Proof. a) i$\Rightarrow$ii. We shall prove that : $\forall\mu\in T_0^*$: $\inf\limits_{K\in\mathcal{R}(\varphi)} \mu(B^{X\setminus K}t)=0$.

Let $\varepsilon>0$ and $\{s_n \mid n\in\mathbf{N}\}$ be a countable, dense part of S_0. For each $n\in\mathbf{N}$, we can find $K_n\in R(s_n\circ\varphi)$ and $\mu(B^{X\setminus K_n}t)<\varepsilon\cdot 2^{-n-1}$.

If we denote: $K=\bigcap\limits_{n\in N} K_n$, then K is naturally compact, and $K\in\mathcal{R}(\varphi)$.

Moreover:

$$\mu(B^{X\setminus K}t)\leqslant \sum_{n\in N}(B^{X\setminus K_n}t)<\varepsilon.$$

ii$\Rightarrow$i is immediate.

b) Let $\{s_n \mid n\in \mathbf{N}\}$ be a countable, dense part of S_0. There exists $\alpha_n>0$, such that:

$$q=\sum_{n\in N}\alpha_n\cdot s_n\in S^{**}.$$

Applying prop. 1.2 to the function $q\circ\varphi$, we find $A\subseteq Y$ polar, such that each $y\in Y\setminus A$ has a naturally compact, fine neighbourhood $K\in\mathcal{R}(q\circ\varphi)$. Since $s_n\leqslant\alpha_n^{-1}\cdot q$, $\forall n\in\mathbf{N}$ it follows that $K\in\mathcal{R}(s_n\circ\varphi)$, hence $K\in\mathcal{R}(\varphi)$.

Remarks. a) The property (i) holds, for exemple, if T satisfies the axiom of nearly continuity.

b) If S satisfies the axiom of nearly continuity and $0\in S$ is quasi-continuous, then (i) is equivalent to:

$s\in S$ quasi-continuous$\Rightarrow s\circ\varphi$ quasi-continuous.

This result allows a reformulation of the theorem on *fine openess* of a H-map [14], by replacing the hypotheses: „$\varphi(Y)$ Borel" or „φ naturally continuous". More precisely:

Corollary. *If the following hold:*

(i) *X and Y are nearly saturated and Suslin;*

(ii) *the points of X are totally thin;*

(iii) *$\forall s\in S_0$, $s\circ\varphi$ is quasi-continuous;*

(iv) *for any naturally open set $U\subseteq X$ and any bounded $f\in S'(U)$ we have: $f\circ\varphi\in\mathcal{T}'(\varphi^{-1}(U))$.*

Then, the following are equivalent:

1) *$\exists A\subseteq Y$ semi-polar, such that $\varphi|_{Y\setminus A}$ is a finely open map;*

2) *$\forall A\subseteq X$ totally thin$\Rightarrow\varphi^{-1}(A)$ is totally thin;*

3) *$\forall x\in X$, $\varphi^{-1}(\{x\})$ is nowhere finely dense in Y.*

Proof. There exists a polar set $A'\subseteq Y$ such that each $y\in Y\setminus A'$ has a naturally compact, fine neighbourhood $K_y\in\mathcal{R}(\varphi)$. Hence $\varphi(K_y)$ is naturally compact. Since:

$$Y\setminus A'=\bigcup_{y\in Y}\operatorname{int} K_y,$$

the Doob property shows that there exists a semi-polar set $A\subseteq Y$ and a countable part $\{y_n\in Y \mid n\in\mathbf{N}\}$ such that:

$$Y\setminus A'=\bigcup_{n\in N}\operatorname{int} K_{y_n}.$$

Hence $\varphi(Y\setminus A')=\bigcup\limits_{n\in N}\varphi(K_{y_n})$ is Borel. From now on, the proof follows [14].

Remark. If moreover, T satisfies the axiom of nearly continuity, then we can abandon the condition (iii), and the set A is polar.

§ 3. **Remarks on the continuity of a H-map.** Let S and T be standard H-cones of functions on the sets X and Y. We recall two known results.

Proposition 1. [12] *If Y is completely metrisable in the natural topology, then the set of points of discontinuity of any H-map $\varphi: Y\to X$ is of the first category.*

Proposition 2 [15]. *Suppose that:*

$\forall x\in X$ $\forall V$ natural neighbourhood of x, $\exists s\in S$ such that:

$$\sup_{y\in X\setminus V} s(y)<s(x).$$

Then, any H-map $\varphi: Y\to X$ is naturally continuous.

Remark. This condition is fulfilled, for instance, when X is a harmonic space, for which there exists $g: X\times X\to\mathbf{R}_+$, with the properties:

(i) $\forall x\in X$, $x'\to g(x,x')$ is in S and is continuous on $X/\{x\}$;

(ii) $g(x,x)=+\infty$, $\forall x\in X$;

(iii) $\forall x\in X$, $\lim\limits_{x'\to\infty} g(x,x')=0$.

Strengthening the conditions on T, one gets:

Corollary 3. *Let us suppose that T satisfies axiom D, and Y is saturated. If $\forall x\in X$ $\forall V$ naturally neighbourhood of x, $\exists s\in S$ such that:*

$$\sup_{y\in\partial V} s(y)<s(x),$$

then any H-map $Y \to X$ is naturally continuous.

Proof. Let us fix $y_0 \in Y$ and let V be a naturally open neighbourhood for $\varphi(y_0)$. Let $s \in S$ be the function such that:

$$\sup_{u \in \partial V} s(u) < s(\varphi(y_0)).$$

If $\alpha \in (\sup_{u \in \partial V} s(u),\ s(\varphi(y_0)))$, we denote: $W = \{y \in Y \mid (s \circ \varphi)(y) > \alpha\}$.
W is a naturally open neighbourhood of y_0. Let W' be a naturally connected component of W. $\varphi(W')$ is finely connected, contained in $V \cup (X \setminus \mathrm{cl}\, V)$. Hence: or $\varphi(W') \subseteq V$, or $\varphi(W') \subseteq X \setminus \mathrm{cl}\, V$. It follows that $\varphi^{-1}(V)$ is naturally open, as a union of naturally open connected components of W (cf. [2, cor. 5.6.16]).

Remark. The above result covers [11, th. 2.1.5], [16, lemme 2] as well as the fact that the *H*-maps between Riemannian varieties are naturally continuous [10].

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A CHARACTERIZATION OF THE MEAN VALUE EQUALITY IN CONVEX ANALYSIS

BY

SEBASTIAN ANIŢA

§0. It is well known that for vector valued functions the mean value theorem does not hold generally equality, the theorem being usually expressed with inequalities. From mean value inequalities we especially remark the form given by Dieudonné [3] and its extension by Flett [5].

For the vectorial case, as we can see in [8], the perturbation of the equality arises because generally the differential of a vector valued function does not conserve the convexity. Thus, McLeod [8] had obtained various extensious of the mean value theorem in the equality form replacing the value of the differential at a point by an element of the closed convex hull of the values of the differential on the considered segment.

On the other hand, in the case of real-valued convex functions one may obtain natural extensions replacing the differential by the subdifferential. The first result obtained in this way was established, for the finite dimensional case, by Wegge [12]; later on this result was extended by Gwinner [6] for the topological linear spaces.

A detailed discussion of the mean value theorems in non-smooth convex analysis was presented by Hiriart-Urruty [7], especially for Lipschitzian functions.

In the present paper we consider convex functions with values in a separated topological vector space which at the same time is completely ordered [9] by a closed convex cone and we establish a simple necessary and sufficient condition such that the mean value theorem can be expressed in equality form with the aid of a single element of the subdifferential.

The proof is based on the theorem of Elster-Nehse [4] concerning the separation of sets in a product space.

§1. Let X be a real linear space and let Y be a topological linear space. We suppose that Y is an ordered complete [10] topological linear space (i.e. every subset which is majorized in Y has a supremum belonging to Y) such that its positive cone is convex and closed. Let E be a convex set in X and let $f : E \to Y$, be a convex function. A linear operator $T : X \to Y$ is called an *algebraic subgradient* of f at the point $x_0 \in E$ if

$$T(x - x_0) \leqslant f(x) - f(x_0), \quad \forall x \in E.$$

The set of all such T is called the *algebraic subdifferential* [10], [11] of f at x_0, written $\partial_a f(x_0)$. If X is a linear topological space, the set