then any H-map $Y \to X$ is naturally continuous.

Proof. Let us fix $y_0 \in Y$ and let V be a naturally open neighbourhood for $\varphi(y_0)$. Let $s \in S$ be the function such that:

$$\sup_{u \in \partial V} s(u) < s(\varphi(y_0)).$$

If $\alpha \in (\sup_{u \in \partial V} s(u),\ s(\varphi(y_0)))$, we denote: $W = \{y \in Y \mid (s \circ \varphi)(y) > \alpha\}$. W is a naturally open neighbourhood of y_0. Let W' be a naturally connected component of W. $\varphi(W')$ is finely connected, contained in $V \cup (X \setminus \mathrm{cl}\, V)$. Hence: or $\varphi(W') \subseteq V$, or $\varphi(W') \subseteq X \setminus \mathrm{cl}\, V$. It follows that $\varphi^{-1}(V)$ is naturally open, as a union of naturally open connected components of W (cf. [2, cor. 5.6.16]).

Remark. The above result covers [11, th. 2.1.5], [16, lemme 2] as well as the fact that the H-maps between Riemannian varieties are naturally continuous [10].

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Seminarul Matematic
Universitatea „Al. I. Cuza"
6600—Iași, România

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A CHARACTERIZATION OF THE MEAN VALUE EQUALITY IN CONVEX ANALYSIS

BY

SEBASTIAN ANIȚA

§0. It is well known that for vector valued functions the mean value theorem does not hold generally equality, the theorem being usually expressed with inequalities. From mean value inequalities we especially remark the form given by Dieudonné [3] and its extension by Flett [5].

For the vectorial case, as we can see in [8], the perturbation of the equality arises because generally the differential of a vector valued function does not conserve the convexity. Thus, McLeod [8] had obtained various extensions of the mean value theorem in the equality form replacing the value of the differential at a point by an element of the closed convex hull of the values of the differential on the considered segment.

On the other hand, in the case of real-valued convex functions one may obtain natural extensions replacing the differential by the subdifferential. The first result obtained in this way was established, for the finite dimensional case, by Wegge [12]; later on this result was extended by Gwinner [6] for the topological linear spaces.

A detailed discussion of the mean value theorems in non-smooth convex analysis was presented by Hiriart-Urruty [7], especially for Lipschitzian functions.

In the present paper we consider convex functions with values in a separated topological vector space which at the same time is completely ordered [9] by a closed convex cone and we establish a simple necessary and sufficient condition such that the mean value theorem can be expressed in equality form with the aid of a single element of the subdifferential.

The proof is based on the theorem of Elster-Nehse [4] concerning the separation of sets in a product space.

§1. Let X be a real linear space and let Y be a topological linear space. We suppose that Y is an ordered complete [10] topological linear space (i.e. every subset which is majorized in Y has a supremum belonging to Y) such that its positive cone is convex and closed. Let E be a convex set in X and let $f: E \to Y$, be a convex function. A linear operator $T: X \to Y$ is called an *algebraic subgradient* of f at the point $x_0 \in E$ if

$$T(x - x_0) \leq f(x) - f(x_0), \quad \forall x \in E.$$

The set of all such T is called the *algebraic subdifferential* [10], [11] of f at x_0, written $\partial_a f(x_0)$. If X is a linear topological space, the set

$$\partial f(x_0) = \partial_a f(x_0) \cap L(X, Y), \tag{1}$$

where $L(X, Y)$ is the space of all linear continuous operators from X into Y, is called the *(topological) subdifferential* of f at x_0. If $A \subset X$, ${}^iA$ denotes the relative algebraic interior of A ([4]).

Theorem 1.1. *Let $f : E \to Y$ be a convex function such that $f/[a, b]$ is continuous, where a, b are two elements of the convex set E with $]a, b[\subset {}^iE$. Then, there exist $c \in]a, b[$ and $T \in \partial_a f(c)$ such that*

$$f(b) - f(a) = T(b-a) \tag{2}$$

if and only if for any $t_1, t_2 \in [0, 1]$ there exists $t_3 \in [0, 1]$ such that

$$f(a+t_3(b-a)) - t_3[f(b)-f(a)] \leqslant f(a+t_i(b-a)) - t_i[f(b)-f(a)], \ i = 1, 2. \tag{3}$$

Proof. Consider the function $\varphi : [0, 1] \to Y$ defined by

$$\varphi(t) = f(a+t(b-a)) - t[f(b)-f(a)], \ t \in [0, 1].$$

Using the property (3) it follows that $\varphi([0, 1])$ is a (left) directed set in Y. Also, the family of level sets $A_t = \{u \in [0, 1] ; \varphi(u) \leqslant \varphi(t)\}$, $t \in [0, 1]$, is directed. Since the positive cone of Y is closed and the function φ is continuous, the sets A_t are compact. Therefore $\bigcap_{t \in [0,1]} A_t \neq \varnothing$, and so, there exists $t_0 \in [0, 1]$ such that

$$\varphi(t_0) \leqslant \varphi(t), \quad \forall t \in [0, 1]. \tag{4}$$

Moreover, t_0 can be taken in $]0, 1[$ since by convexity of φ we have

$$\varphi(u) = \varphi(u \cdot 1 + (1+u) \cdot 0) \leqslant u\varphi(1) + (1-u)\varphi(0) = \varphi(0) = \varphi(1) \quad \forall u \in [0, 1].$$

Now, let us consider the following two convex sets of $X \times Y$:

$$A = \{(x, y) ; x \in E, \ y \geqslant f(x) - f(x_0)\},$$

$$B = \{(a+t(b-a), \ (t-t_0)[f(b)-f(a)]) ; t \in [0, 1]\},$$

where $x_0 = a + t_0(b-a) \in E$. It is clear that

$${}^iP_X(A) \cap {}^iP_X(B) \neq \varnothing,$$

since $a+t(b-a) \in {}^iP_X(B)$, $\forall t \in]0, 1[$ and $P_X(A) = E$. Further, if $(x, y) \in A$ and $(x, z) \in B$ then there exists $t \in [0, 1]$ such that $x = a+t(b-a)$, $z = (t-t_0)(f(b)-f(a))$ and $y \geqslant f(a+t(b-a)) - f(x_0)$. By (4) we obtain $y \geqslant z$. Applying the separation theorem of Elster-Nehse [4] we find $T_1 \in L(X, Y)$ and $y_0 \in Y$ such that

$$T_1(x_1) - y_1 \leqslant y_0 \leqslant T_1(x_2) - y_2, \ \forall (x_1, y_1) \in A, \ (x_2, y_2) \in B.$$

Since $(x_0, 0) \in A \cap B$ one may take $y_0 = T_1(x_0)$. Also, $(x, f(x) - f(x_0)) \in A$ implies that

$$T_1(x) - [f(x) - f(x_0)] \leqslant y_0 = T_1(x_0)$$

and so

$$T_1(x - x_0) \leqslant f(x) - f(x_0), \ \forall x \in E.$$

From $(a+t(b-a), (t-t_0)[f(b)-f(a)]) \in B$ it follows that

$$T_1(x_0) \leqslant y_0 \leqslant T_1(a+t(b-a)) - (t-t_0)[f(b)-f(a)]$$

and so

$$(t-t_0)[f(b)-f(a)] \leqslant (t-t_0)T_1(b-a), \ \forall t \in [0, 1].$$

Consequently, we have $T_1(b-a) \leqslant f(b) - f(a)$.

Let $X = X_1 \oplus \mathbf{R}(b-a)$, where X_1 is an algebraic complement of $\mathbf{R}(b-a)$ in X. We define the linear operator $T : X \to Y$ by

$$T(x) = T_1(x_1) + t[f(b)-f(a)], \ x = x_1 + t(b-a), \ x_1 \in X_1, \quad t \in \mathbf{R}.$$

It is obvious that

$$T(b-a) = f(b) - f(a) \text{ and } T(x-x_0) \leqslant f(x) - f(x_0), \quad \forall x \in E,$$

i.e. $T \in \partial_a f(x_0)$. Hence the mean equality (2) holds.

Conversely, if there exist $x_0 \in]a, b[$ and $T \in \partial_a f(x_0)$ such that $f(b) - f(a) = T(b-a)$, it follows

$$T(x - x_0) \leqslant f(x) - f(x_0), \ \forall x \in X.$$

If $x = a+t(b-a)$, $x_0 = a+t_0(b-a)$, $t, t_0 \in]0, 1[$, we obtain

$$(t-t_0)T(b-a) \leqslant f(a+t(b-a)) - f(a+t_0(b-a)), \ \forall t \in]0, 1[.$$

According to mean equality we have

$$f(a+t_0(b-a)) - t_0[f(b)-f(a)] \leqslant f(a+t(b-a)) - t[f(b)-f(a)], \ \forall t \in [0, 1],$$

which shows that the property (3) is fulfilled.

This result of algebraic type can be easily completed by the following topological form.

Theorem 1.2. *Let X be a topological linear space and let Y be a completely ordered topological linear space, ordered by a closed normal cone. If $f : E \to Y$ is a convex function on a convex set $E \subset X$ which is continuous on a closed interval $[a, b] \subset \operatorname{int} E$, then there exist $x_0 \in]a, b[$ and $T \in \partial f(x_0)$ such that $f(b) - f(a) = T(b-a)$ if and only if the property (3) is fulfilled.*

Proof. By the above theorem it is sufficient to observe that the continuity of f in each point of $[a, b]$ provides the continuity of algebraic subgradients, i.e. $\partial f_a(x) = \partial f(x)$ for all $x \in [a, b]$.

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Faculty of Mathematics
University of Iaşi
6600—Iaşi, România

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EXISTENCE THEOREMS FOR HAMMERSTEIN TYPE INTEGRAL EQUATIONS

BY

B. G. PACHPATTE

1. **Introduction.** Let $(B, \|\cdot\|)$ be a Banach space and $x=(x_1, \ldots, x_r)$ denote a point of the Euclidean space E^r, $r\geq 1$. Let Ω be an open and bounded subset of E^r. For $x, y\in E^r$, we define $|x-y|=[\sum_{i=1}^{r}(x_i-y_i)^2]^{1/2}$ and $|x-x_S|=\inf_{y\in S}|x-y|$, where S is the boundary of Ω. Consider the following general Hammerstein type integral equation

$$u(x)+\int_\Omega f(x, y, u(y), \int_\Omega g(y, z, u(z))dz)dy=h(x), \tag{1}$$

where u is an unknown function, $h\in C[\Omega, B]$, $g\in C[\Omega^2\times B, B]$ and $f\in C[\Omega^2\times B^2, B]$.

The questions of existence and uniqueness of the solutions of various special forms of equation (1) have been studied by many authors since the appearance of the celebrated paper of A. Hammerstein [6] (see, [1—3, 5, 7—10, 12—16]). During the past few years several authors have successfully applied the theory of monotone operators to study the Hammerstein type integral equations (see, [1—3, 7—10, 16]). Here we are interested in the study of equation (1) without using monotone conditions. Our interest in equation (1) is motivated by the Hammerstein type integral equation studied by Nesterenko [12].

In this paper we present a set of conditions on functions f, g, h which are sufficient to guarantee the existence and uniqueness of unbounded and continuous solutions of equation (1) and a slightly different form of equation (1). The well known Banach fixed point theorem is used to establish our results. An application to elliptic boundary value problems is also given.

2. **Main results.** In this section we establish our main results on the existence and uniqueness of unbounded and continuous solutions of equation (1). For convenience we first list the following hypotheses.

(H_1) For $(x, y, u_1), (x, y, u_2)\in\Omega^2\times B$, (x, y, u_1, u_2), $(y, x, \overline{u}_1, \overline{u}_2)\in\Omega^2\times B^2$,

$$\|g(x, y, u_1)-g(x, y, u_2)\|\leq q(x, y)\|u_1-u_2\|,$$

$$\|f(x, y, u_1, u_2)-f(x, y, \overline{u}_1, \overline{u}_2)\|\leq p(x, y)[\|u_1-\overline{u}_1\|+\|u_2-\overline{u}_2\|],$$

where $p, q\in C[\Omega^2, B]$ such that