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EXISTENCE THEOREMS FOR HAMMERSTEIN TYPE INTEGRAL EQUATIONS

BY

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1. **Introduction.** Let $(B, \|\cdot\|)$ be a Banach space and $x=(x_1, \ldots, x_r)$ denote a point of the Euclidean space E^r, $r \geqslant 1$. Let Ω be an open and bounded subset of E^r. For $x, y \in E^r$, we define $|x-y|=[\sum_{i=1}^{r}(x_i-y_i)^2]^{1/2}$ and $|x-x_S|=\inf_{y \in S}|x-y|$, where S is the boundary of Ω. Consider the following general Hammerstein type integral equation

$$u(x)+\int_\Omega f(x, y, u(y), \int_\Omega g(y, z, u(z))dz)dy=h(x), \tag{1}$$

where u is an unknown function, $h \in C[\Omega, B]$, $g \in C[\Omega^2 \times B, B]$ and $f \in C[\Omega^2 \times B^2, B]$.

The questions of existence and uniqueness of the solutions of various special forms of equation (1) have been studied by many authors since the appearance of the celebrated paper of A. Hammerstein [6] (see, [1—3, 5, 7—10, 12—16]). During the past few years several authors have successfully applied the theory of monotone operators to study the Hammerstein type integral equations (see, [1—3, 7—10, 16]). Here we are interested in the study of equation (1) without using monotone conditions. Our interest in equation (1) is motivated by the Hammerstein type integral equation studied by Nesterenko [12].

In this paper we present a set of conditions on functions f, g, h which are sufficient to guarantee the existence and uniqueness of unbounded and continuous solutions of equation (1) and a slightly different form of equation (1). The well known Banach fixed point theorem is used to establish our results. An application to elliptic boundary value problems is also given.

2. **Main results.** In this section we establish our main results on the existence and uniqueness of unbounded and continuous solutions of equation (1). For convenience we first list the following hypotheses.

(H_1) For $(x, y, u_1), (x, y, u_2) \in \Omega^2 \times B$, $(x, y, u_1, u_2), (y, x, \bar{u}_1, \bar{u}_2) \in \Omega^2 \times B^2$,

$$\|g(x, y, u_1)-g(x, y, u_2)\| \leqslant q(x, y)\|u_1-u_2\|,$$

$$\|f(x, y, u_1, u_2)-f(x, y, \bar{u}_1, \bar{u}_2)\| \leqslant p(x, y)[\|u_1-\bar{u}_1\|+\|u_2-\bar{u}_2\|],$$

where $p, q \in C[\Omega^2, B]$ such that

$$\int_\Omega p(x, y)[|y-y_S|^{-\alpha}+\int_\Omega q(y, z)|z-z_S|^{-\alpha}dz]dy \leq \mu|x-x_S|^{-\alpha}$$

in which α, μ are nonnegative constants such that $\mu<1$.

(H_2) There exists a constant $P\geq 0$ such that

$$\|h(x)\|+\int_\Omega \|f(x, y, 0, \int_\Omega g(y, z, 0)dz)\|dy \leq P|x-x_S|^{-\alpha},$$

for $x\in\Omega$, where 0 is the zero element in B and α is the constant defined in (H_1).

Let G be the space of those functions $\varphi:\Omega\to B$ which are continuous in Ω and fulfil the condition

$$\|\varphi(x)\|\leq M|x-x_S|^{-\alpha},\quad x\in\Omega, \tag{2}$$

where $M>0$ is a constant. In the space G we define the norm (see, [4])

$$\|\varphi\|_G=\sup_{x\in\Omega}[|x-x_S|^{\alpha}\|\varphi(x)\|], \tag{3}$$

and metric

$$\|\varphi-\psi\|_G=\sup_{x\in\Omega}[|x-x_S|^{\alpha}\|\varphi(x)-\psi(x)\|]. \tag{4}$$

It is easy to verify that G with the metric defined in (4) is complete. From (2) and (3) we observe that

$$\|\varphi\|_G\leq M. \tag{5}$$

Now we shall prove

Theorem 1. *Assume that the hypotheses (H_1) and (H_2) hold, and $M\mu+ +P\leq M$. Then there exists a unique solution $u(x)\in G$ continuous in Ω and unbounded for $|x-x_S|\to 0$ of equation* (1).

Proof. For $u\in G$, we define the operator Tu such that

$$Tu(x)=h(x)-\int_\Omega f(x, y, u(y), \int_\Omega g(y, z, u(z))dz)dy. \tag{6}$$

Now we shall prove that T maps G into itself. Evidently T is continuous in Ω. We verify that (2) is fulfilled. From (6) and hypotheses (H_1) and (H_2) we obtain

$$\|Tu(x)\|\leq\|h(x)\|+\int_\Omega(\|f(x, y, u(y), \int_\Omega g(y, z, u(z))dz)-$$
$$-f(x, y, 0, \int_\Omega g(y, z, 0)dz)\|)dy+\int_\Omega\|f(x, y, 0, \int_\Omega g(y, z, 0)dz)\|dy\leq$$
$$\leq\|u\|_G\int_\Omega p(x, y)[|y-y_s|^{-\alpha}+\int_\Omega q(y, z)|z-z_s|^{-\alpha}dz]dy+P(x-x_s|^{-\alpha}\leq$$
$$\leq[M\mu+P]|x-x_s|^{-\alpha}.$$

This proves that T maps G into itself.

Next we verify that the operator T is a contraction. Let u, $v\in G$, then from (6) and hypothesis (H_1) we obtain

$$\|Tu(x)-Tv(x)\|\leq\int_\Omega(\|f(x, y, u(y), \int_\Omega g(y, z, u(z))dz)-f(x, y, v(y),$$
$$\int_\Omega g(y, z, v(z))dz)\|dy\leq\|u-v\|_G\int_\Omega p(x, y)[|y-y_S|^{-\alpha}+\int_\Omega q(y, z)|z-z_S|^{-\alpha}dz]dy\leq$$
$$\leq\mu\|u-v\|_G|x-x_S|^{-\alpha}.$$

Consequently, we have

$$\|Tu-Tv\|_G\leq\mu\|u-v\|_G.$$

Since $\mu<1$, it follows from the Banach fixed point theorem that T has a unique fixed point in G. The fixed point of T is a solution of equation (1). Clearly, this solution is unbounded as $|x-x_S|\to 0$. This completes the proof of the theorem.

Remark 1. The existence and uniqueness of solutions of equation (1) when $h(x)=0$, $g(y, z, u)=k_1(y, z)u$ and $f(x, y, u, v)=k(x, y)f_1(y, u, v)$, have been studied by N e s t e r e n k o [12] by using the method of degenerate kernels. The integral equation studied in [12] is a further generalization of Hammerstein type equations studied by many authors in the literature.

The technique used in proving Theorem 1 can be applied equally well to study the existence and uniqueness of unbounded and continuous solutions of the following Hammerstein type integral equation

$$u(x)+\int_\Omega f(x, y, u(y))dy=h(x, u(x), \int_\Omega g(x, y, u(y))dy), \tag{7}$$

where u is an unknown function, f, $g\in C[\Omega^2\times B, B]$ and $h\in C[\Omega\times B^2, B]$.

Concerning the nonlinear functions f, g and h in (7) we assume the following conditions.

(H_3) For $(x, y, u_i)\in\Omega^2\times B$, $(i=1, 2)$, (x, u_1, u_2), $(x, \bar{u}_1, \bar{u}_2)\in\Omega\times B^2$,

$$\|f(x, y, u_1)-f(x, y, u_2)\|\leq a(x, y)\|u_1-u_2\|,$$
$$\|g(x, y, u_1)-g(x, y, u_2)\|\leq b(x, y)\|u_1-u_2\|,$$
$$\|h(x, u_1, u_2)-h(x, \bar{u}_1, \bar{u}_2)\|\leq L_1\|u_1-\bar{u}_1\|+L_2\|u_2-\bar{u}_2\|,$$

where $a, b\in C[\Omega^2, B]$ such that

$$\int_\Omega a(x, y)|y-y_s|^{-\alpha}dy\leq M_1|x-x_S|^{-\alpha},\quad x\in\Omega,$$
$$\int_\Omega b(x, y)|y-y_s)^{-\alpha}dy\leq M_2|x-x_S|^{\alpha},\quad x\in\Omega,$$

in which L_1, L_2, M_1, M_2 are nonnegative constants such that $L_1+L_2M_2+ +M_1<1$.

(H_4) There exists a constant $P\geq 0$ such that

$$\|h(x, 0, \int_\Omega g(x, y, 0)dy)\|+\int_\Omega\|f(x, y, 0)\|dy\leq P|x-x_s|^{-\alpha},\ \text{for}\ x\in\Omega.$$

The following theorem gives sufficient conditions for the existence and uniqueness of unbounded and continuous solution of equation (7).

Theorem 2. *Assume that the hypotheses (H_3) and (H_4) hold, and $M(L_1+ +L_2M_2+M_1)+P\leq M$. Then there exists a unique solution $u(x)\in G$ continuous in Ω and unbounded for $|x-x_S|\to 0$ of equation* (7).

The details of the proof of this theorem follow by the same argument as in the proof of Theorem 1 with suitable modifications. We omit the details.

Remark 2. The equation (7) in the special case when $f(x, y, u) = k(x, y) f_1(y, u)$ and $h(x, u, \int_\Omega g(x, y, u)dy) = h_1(x)$ is studied by many authors in the literature by using different techniques, see [1—3, 5—10, 12, 15, 16]. However, the sufficient conditions obtained here are different from those obtained by various authors in [1—3, 5—10, 12, 15, 16].

3. **Application to elliptic equations.** As an application of the results in Theorems 1 and 2, we consider the existence and uniqueness of unbounded and continuous solutions of nonlinear elliptic boundary value problem of the form

$$-\Delta z(x) = f(x, z(x)),\ \ x \in Q, \tag{8}$$

$$z(x) = b(x),\ \ x \in \partial Q, \tag{9}$$

where Δ denotes the Laplace operator. Here we shall assume that $Q \subset E^r$ is a bounded domain whose boundary ∂Q is of class $C^{1+\lambda}$ for some $\lambda > 0$ (see, [11]) and let $\bar{Q}$ denote the closure of Q, $b \in C(\partial Q)$ and $f \in C[Q \times E^r, E^r]$ and f is Hölder continuous at every point in $Q \times E^r$. It is well known that the boundary value problem (8)—(9) is equivalent to the integral equation (see [15])

$$z(x) = g(x) + \int_Q G(x, y) f(y, z(y))dy,\ \ x \in Q, \tag{10}$$

where $g(x)$ is defined by $\Delta g(x) = 0$ on Q and $g(x) = b(x)$ on ∂Q and G denotes the Green function for the problem. The equation (10) can be considered as a special case of equation (1) or (7). Hence Theorem 1 or Theorem 2 can be applied to assert the existence and uniqueness of unbounded and continuous solutions of equation (10) under suitable modifications in the assumptions (H_1)—(H_2) or (H_3)—(H_4). The precise formulation of such a result for equation (10) is similar to that of our results in section 2 and hence we omit the details.

Remark 3. We note that the existence and uniqueness of solutions of boundary value problem (8)—(9) is recently studied by the present author [13] by using different techniques. Here we can assert the existence and uniqueness of unbounded and continuous solution of (8)—(9) under different conditions than those obtained in [13].

In concluding this paper, we note that an interesting special form of equation (7) when $f = 0$ and the function h in (7) depends on the real parameter is recently studied by the present author in [14]. The difference between the main result established in [14] and our Theorem 2 is that here we insure in addition to the existence and uniqueness the unboundedness of the solution of equation (7).

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