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JUSTIFICATION OF THE AVERAGING METHOD FOR A SYSTEM OF FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH VARIABLE STRUCTURE AND IMPULSES

BY

D. D. BAINOV and S. D. MILUSHEVA

1. **Introduction.** Recently, a number of papers have developed the fundamenthal theory of differential equations with impulses, see [1]–[3]. Besides the pure mathematical interest, a reason for such considerations seems to be the considerable importance of such equations in practice. For example, the concept of switching curve (or switching hypersurface) is a classical concept in the control theory. Introducing relaxed controls, the modern control theory generalizes the concept of motion provided that the solution of the system (the state trajectory) may be discontinuous, that is, it may have instantaneous jumps according to the control law. Such an approach essentially enriches the mathematical structure of the differential systems and has numerous applications.

In this paper we present a justification of the averaging method for a system of differential equations of neutral type with impulses where the perturbation is represented by a small parameter in the right hand side of the equation. The technique used is related to that in [4] where the averaging method for a system of differential equations with fast and slow variables and with impulses is investigated.

2. **Statement of the problem.** We consider the following problem:

a) Given a set of hypersurfaces $\sigma_i : t=t_i(x)$, $i=1, 2, \dots$, which for $x \in D \subset R^n$ lie in the half-space $t>0$ of the $(n+1)$-dimensional space (t, x) and satisfy the condition $t_i(x) < t_{i+1}(x)$, $i=1, 2, \dots$;

b) Given a set of functions $\Phi_i(t, x)$, $i=1, 2, \dots$ defined on the hypersurfaces σ_i respectively;

c) Given a set of ordered pairs of vector functions $X_i^{(1)}(t, x, y, z)$, $X_i^{(2)}(t, x, y, z)$, $i=0, 1, 2, \dots$ defined in the domain $\{t \geq 0, x, y \in D, z \in D_1 \subset R^n\}$;

d) Given a set of vector-functions $I_i(x)$, $i=1, 2, \dots$ defined in the domain D;

A mapping point P_t with current coordinates $(t, x(t))$ moves in the domain $\{t \geq 0, x \in D\}$. We assume that the motion law of the point P_t is characterized by

e) the system of differential equations of neutral type

$$(1)\qquad \begin{aligned} &\dot{x}(t)=\varepsilon X(t,\, x(t),\, x(\Delta(t,\, x(t))),\, \dot{x}(\Delta(t,\, x(t)))),\ t>0,\ t\neq t_i(x),\\ &x(t)=\varphi(t,\, \varepsilon),\quad t\in[-\delta,\, 0],\qquad \dot{x}(t)=\dot{\varphi}(t,\, \varepsilon),\qquad t\in[-\delta,\, 0]\,, \end{aligned}$$

where $\varepsilon>0$ is a small parameter, δ is a positive constant, $\Delta(t, x)$ is the transformed argument which satisfies the condition

$$(2)\qquad t-\delta\leqslant\Delta(t,\, x)\leqslant t,\ (t,\, x)\in\{t\geqslant-\delta,\ x\in D\}\,,$$

$\varphi(t,\, \varepsilon)\in D$ is the initial function defined and differentiable with respect to t in the domain $\{t\in[-\delta,\, 0],\ \varepsilon\in(0,\ \mathcal{E}],\ \mathcal{E}=\text{const.}>0\}$.

The motion itself can be described as follows. Departing from the point $(t=\tau_0=0,\ x_0=\varphi(0,\, \varepsilon))$ the point P_t moves along the trajectory $(t,\, x(t))$ governed by the solution $x(t)$ of the system (1), where $X(t,\, x,\, y,\, z)\equiv\equiv X_0^{(1)}(t,\, x,\, y,\, z)\equiv X_0^{(2)}(t,\, x,\, y,\, z)$ untill the moment $\tau_1>0$ at which the trajectory meets the hypersurface σ_1 at the point $(\tau_1,\ x_1^-=x(\tau_1))$. Then the point P_t instantly moves from the position $(\tau_1,\ x_1^-)$ to the position $(\tau_1,\ x_1^+=x_1^-+\varepsilon I_1(x_1^-))$ and further on it follows the trajectory $(t,\, x(t))$ described by the solution $x(t)$ of the system (1), where

$$X(t,\, x,\, y,\, z)=\begin{cases} X_1^{(1)}(t,\, x,\, y,\, z) & \text{if } \Phi_1(\tau_1,\ x_1^-)\geqslant0,\\ X_1^{(2)}(t,\, x,\, y,\, z) & \text{if } \Phi_1(\tau_1,\ x_1^-)<0,\end{cases}$$

untill it meets the hypersurface σ_2 etc.

Note that the point $(\tau_i,\ x_i^+)$, where

$$(3)\qquad x_i^+=x_i^-+\varepsilon I_i(x_i^-),\ i=1,\, 2,\, \dots$$

does not necessarily belong to the hypersurface σ_i, $i=1,\ 2,\, \dots$.

The relations a)—d) and e) in which the vector-function X of the equations (1) has the form

$$(4)\qquad X(t,\, x,\, y,\, z)=\begin{cases} X_0^{(1)}(t,\, x,\, y,\, z)\equiv X_0^{(2)}(t,\, x,\, y,\, z) & \text{for } \tau_0<t\leqslant\tau_1,\\ X_i^{(1)}(t,\, x,\, y,\, z) & \text{for } \tau_i<t\leqslant\tau_{i+1},\ \text{if } \Phi_i(\tau_i,\, x_i^-)\geqslant0,\\ X_i^{(2)}(t,\, x,\, y,\, z) & \text{for } \tau_i<t\leqslant\tau_{i+1},\ \text{if } \Phi_i(\tau_i,\, x_i^-)<0,\ i=1,\, 2,\, \dots\end{cases}$$

will be called a system of functional-differential equations with variable structure and with impulses and will be denoted shortly by (1). The curve described by the point P_t during its motion will be called an integral curve or a trajectory of this system in the space $(t,\ x)$.

Suppose that the following limits exist

$$(5)\qquad \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}X(\theta,\, x,\, x,\, 0)\, d\theta=X_0(x),\qquad \lim_{T\to\infty}\frac{1}{T}\sum_{t<t_i<t+T}I_i(x)=I_0(x).$$

Then one can introduce the following averaged system of ordinary differential equations associated with the system (1) considered

$$(6)\qquad \dot{\bar{x}}(t)=\varepsilon[X_0(\bar{x}(t))+I_0(\bar{x}(t))]\,.$$

The initial condition is

$$(7)\qquad \bar{x}(0)=x_0.$$

We note that if $x=(x_1,\, \dots,\, x_n)$ and $A=(a_{ij})_{nm}$ then, by definition

$$\|x\|=\left[\sum_{i=1}^{n}x_i^2\right]^{1/2},\qquad \|A\|=\left[\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}^2\right]^{1/2}.$$

We denote by $\overline{1,n}$ the set of natural numbers $\{1,\, 2,\, \dots,\, n\}$.

3. **Main result.** The following theorem concerned with the closeness between the solutions of the system (1) and the averaged system (6) with initial condition (7) holds true:

Theorem 1. *Let the following conditions be fulfilled:*

1. *The functions $t_i(x)$, $i=1,\, 2,\, \dots$ are defined, positive and satisfy the condition $t_i(x)<t_{i+1}(x)$, $i=1,\, 2,\, \dots$ for $x\in D\subset R^n$. The function $\Phi_i(t,\ x)$ is defined on the hypersurface $t=t_i(x)$, $x\in D$, $i=1,\, 2,\, \dots$. The function $X_i^{(k)}(t,\, x,\, y,\, z)$ is continuous in the domain $\{t_i(x)\leqslant t\leqslant t_{i+1}(x),\ x,\, y\in D,\ z\in D_1\subset R^n\}$, $t_0(x)\equiv0$, $k=1,\ 2$, $i=0,\ 1,\ 2,\, \dots$. The function $\Delta(t,\ x)$ is continuous and satisfies the condition $t-\delta\leqslant\Delta(t,\, x)\leqslant t$, $\delta=\text{const.}>0$ in the domain $\{t\geqslant0,\ x\in D\}$. The functions $\varphi(t,\, \varepsilon)$, $\dot{\varphi}(t,\, \varepsilon)$ are continuous in the domain $\{t\in[-\delta,\ 0],\ \varepsilon\in(0,\ \mathcal{E}],\ \mathcal{E}=\text{const.}>0\}$ and $\varphi(t,\, \varepsilon)\in D$, $\dot{\varphi}(t,\, \varepsilon)\in D_1$. The functions $I_i(x)$, $i=1,\, 2,\, \dots$ are continuous in D.*

2. *There exist positive constants M, N, K and a function $\gamma(\varepsilon)$ such that*

$$\|X_i^{(k)}(t,\, x,\, y,\, z)\|+\|I_i(x)\|\leqslant M,\ \ |t_i(x)-t_i(x')|\leqslant N\|x-x'\|,$$

$$\|X_i^{(k)}(t,\, x,\, y,\, z)-X_i^{(k)}(t,\, x',\, y',\, z')\|\leqslant K(\|x-x'\|+\|y-y'\|+\|z-z'\|),$$

$$\|I_i(x)-I_i(x')\|\leqslant K\|x-x'\|$$

for all $t\geqslant0$, $x,\, x',\, y,\, y'\in D$, $z,\, z'\in D_1$, $k=1,\, 2$, $i=0,\ 1,\ 2,\, \dots$ and $\|\dot{\varphi}(t,\, \varepsilon)\|\leqslant\leqslant\gamma(\varepsilon)$ for $\varepsilon\in(0,\ \mathcal{E}]$, where $\lim_{\varepsilon\to0}\gamma(\varepsilon)/\varepsilon>0$ and $\sup_{\varepsilon\in(0,\mathcal{E}]}\gamma(\varepsilon)/\varepsilon>0$.

3. *The following limit*

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t<t_i<t+T}1=d,\ d=\text{const.}>0$$

and the limit (5) are finite uniformly in $t\geqslant0$ and $x\in D$.

4. *For every $\varepsilon\in(0,\ \mathcal{E}]$ the system of functional-differential equations (1) with variable structure and with impulses has a continuous solution $x(t)$ for $t>0$, $t\neq\tau_i$, $i=1,\, 2,\, \dots$ which satisfies the matching condition $x(0+0)==\varphi(0,\ \varepsilon)=x_0$, $\dot{x}(0+0)=\dot{\varphi}(0,\ \varepsilon)$.*

5. *For every $\varepsilon\in(0,\ \mathcal{E}]$ the averaged system (6) with initial condition (7) has a solution $\bar{x}(t)$ which belongs to the domain D for $t\geqslant0$ together with its neighbourhood of radius $\rho=\text{const.}>0$.*

6. *For every $\varepsilon\in(0,\mathcal{E}]$ for $t>0$ the trajectory $(t,\, x(t))$ meets once the hypersurface $t=t_i(x)$, $i=1,\, 2,\, \dots$.*

Then for every $\eta>0$ and $L>0$ there exists $\varepsilon_0\in(0,\ \mathcal{E}]$ $(\varepsilon_0=\varepsilon_0(\eta,\ L))$ such that for $\varepsilon\leqslant\varepsilon_0$ the inequality

$$\| x(t)-\bar{x}(t) \| < \gamma_i \quad \textit{holds} \quad \textit{for } 0 \leqslant t \leqslant L\varepsilon^{-1}.$$

In the proof of Theorem 1 we shall use the following lemma:

Lemma 1. *Let the conditions of the Theorem 1 be fulfilled. Let* $T > \delta$ *be a sufficiently large and fixed number. Then for every natural number* $p \geqslant 1$ *the following inequality holds*

$$\|x(pT)-\bar{x}(pT)\| \leqslant \varepsilon \sum_{i=0}^{p-1} [1+\varepsilon(2+d)\,KT]^i\,[\alpha(T)\,T+\varepsilon\bar{M}], \tag{8}$$

where $\bar{M}=(1+d)\,(2+d)\,KMT^2+\max_{i=\overline{1,p}} M_i$, *and* $M_i = M_i(T, d_1, \ldots, d_i)$ *are constants depending on* T *and on the constants* $d_j > 0$, $j=\overline{1,\, i}$.

Proof of Lemma 1. The condition 3 of Theorem 1 implies existence of a function $\alpha(t)$ which is monotonically decreasing to zero for t tending to infinity, such that for all $t \geqslant 0$ and $x \in D$ the following inequalities hold

$$\begin{gathered} \left\| \int_t^{t+T} [X(0, x, x, 0)-X_0(x)]\, d\theta \right\| \leqslant \alpha(T)\; T/2, \\ \left\| \sum_{t<t_i<t+T} I_i(x)-I_0(x)\; T \right\| \leqslant \alpha(T)\; T/2. \end{gathered} \tag{9}$$

In the proof we use the method of the complete mathematical induction. We prove the inequality (8) for $p=1$. Consider the system with impulses for $t \in [0,\ T]$.

Let the points $t_1(x_0)=t_1^{(0)}, \ldots, t_{d_1}(x_0)=t_{d_1}^{(0)}$ lie in the interval $(0,\ T)$ so that $t_i^{(0)} < t_{i+1}^{(0)}$ for $i=\overline{1,\ (d_1-1)}$.

We denote as $x_0^{(0)}(t, 0, x_0)$ the solution of the system

$$x_0^{(0)}(t, 0, x_0) = \begin{cases} x_0+\varepsilon \displaystyle\int_0^t X_0^{(1)}(0, x_0^{(0)}(0, 0, x_0),\ x_0^{(0)}(\Delta_0^{(0)}(\theta), 0,\ x_0), \\ \quad \dot{x}_0^{(0)}(\Delta_0^{(0)}(0),\ 0,\ x_0))\, d\theta,\ t>0, \\ \varphi(t,\ \varepsilon),\ -\delta \leqslant t \leqslant 0, \end{cases} \tag{10}$$

$$\dot{x}_0^{(0)}(t, 0, x_0)=\dot{\varphi}(t,\ \varepsilon),\ -\delta \leqslant t \leqslant 0,$$

where $\Delta_0^{(0)}(t)=\Delta(t,\ x_0^{(0)}(t,\ 0,\ x_0))$.

The solution of (10) coincides with the solution $x(t)$ of the system (1) till to the moment τ_1 at which the trajectory of this system metts the hypersurface σ_1, i.e. $x(t)=x_0^{(0)}(t, 0, x_0)$ for $t \in [-\delta,\ \tau_1]$.

Consider the function

$$\bar{x}_0^{(0)}(t,\ 0,\ x_0)=x_0+\varepsilon \int_0^t X_0^{(1)}(\theta,\ x_0,\ x_0,\ 0)\, d\theta.$$

Let us estimate the norm of the difference

$$R_0^{(0)}(t,\ 0,\ x_0,\ \varepsilon)=x_0^{(0)}(t,\ 0,\ x_0)-\bar{x}_0^{(0)}(t,\ 0,\ x_0).$$

For $0<t \leqslant T$ we have

$$\begin{aligned} &\| R_0^{(0)}(t, 0, x_0, \varepsilon) \| \leqslant \varepsilon \int_0^t \| X_0^{(1)}(0,\ x_0^{(0)}(0,\ 0,\ x_0),\ x_0^{(0)}(\Delta_0^{(0)}(\theta),\ 0,\ x_0), \\ &\quad \dot{x}_0^{(0)}(\Delta_0^{(0)}(\theta),\ 0,\ x_0))-X_0^{(1)}(0,\ x_0,\ x_0,\ 0) \|\, d\theta \leqslant \\ &\leqslant \varepsilon K \int_0^t \{\| x_0^{(0)}(0,\ 0,\ x_0)-x_0 \| + \| x_0^{(0)}(\Delta_0^{(0)}(\theta),\ 0,\ x_0)-x_0 \| + \\ &+ \| \dot{x}_0^{(0)}(\Delta_0^{(0)}(\theta),\ 0,\ x_0) \|\}\, d\theta \leqslant \varepsilon^2 K \int_0^t d\theta \int_0^\theta \| X_0^{(1)}(l,\ x_0^{(0)}(l,\ 0,\ x_0), \\ &x_0^{(0)}(\Delta_0^{(0)}(l),\ 0,\ x_0),\ \dot{x}_0^{(0)}(\Delta_0^{(0)}(l),\ 0,\ x_0)) \|\, dl+\varepsilon K\{ \int_{J_{0,t}^-} \| \varphi(\Delta_0^{(0)}(\theta),\ \varepsilon)-\varphi(0,\ \varepsilon) \|\, d\theta+ \\ &+ \varepsilon \int_{J_{0,t}^+} d\theta \int_0^{\Delta_0^{(0)}(\theta)} \| X_0^{(1)}(l,\ x_0^{(0)}(l,\ 0,\ x_0),\ x_0^{(0)}(\Delta_0^{(0)}(l),\ 0,\ x_0),\ \dot{x}_0^{(0)}(\Delta_0^{(0)}(l),\ 0,\ x_0)) \|\, dl\} + \\ &+ \varepsilon K\{ \int_{J_{0,t}^-} \| \dot{\varphi}(\Delta_0^{(0)}(\theta),\ \varepsilon) \|\, d\theta+\varepsilon \int_{J_{0,t}^+} \| X_0^{(1)}(\Delta_0^{(0)}(\theta),\ x_0^{(0)}(\Delta_0^{(0)}(\theta),\ 0,\ x_0), \\ &\quad x_0^{(0)}(\Delta_0^{(0)}(\Delta_0^{(0)}(\theta)),\ 0,\ x_0),\ \dot{x}_0^{(0)}(\Delta_0^{(0)}(\Delta_0^{(0)}(\theta)),\ 0,\ x_0)) \|\, d\theta\} \leqslant \\ &\leqslant \varepsilon^2 KM \int_0^t \theta\, d\theta+\varepsilon\gamma(\varepsilon)\sqrt{n}K \int_{J_{0,t}^-} | \Delta_0^{(0)}(\theta) |\, d\theta+\varepsilon^2 KM \int_{J_{0,t}^+} \Delta_0^{(0)}(\theta)\, d\theta + \\ &+\varepsilon\gamma(\varepsilon)\, K \int_{J_{0,t}^-} d\theta+\varepsilon^2 KM \int_{J_{0,t}^+} d\theta \leqslant \varepsilon^2 KM[(T+0)^2+T]+\varepsilon\gamma(\varepsilon)\,(\delta\sqrt{n}+1) \\ &\qquad KT_0^{(0)} \equiv \omega_0^{(0)}(\varepsilon^2,\ T), \end{aligned}$$

where

$$J_{0,t}^- \cup J_{0,t}^+=(0,\ t],\ J_{0,t}^-=\{\theta : \theta \in (0,\ t] \wedge \Delta_0^{(0)}(\theta) \in [-\delta,\ 0]\},\ J_{0,t}^+=(0,\ t] \setminus J_{0,t}^-.$$

The obtained estimate shows that the function $\bar{x}_0^{(0)}(t,\ 0,\ x_0)$ approximates the solution $x_0^{(0)}(t,\ 0,\ x_0)$ of the system (10) in the interval $(0,\ T]$ with accuracy of order ε^2.

The instant of time τ_1 at which the trajectory $(t,\ x(t))$ meets the hypersurface σ_1 is the unique solution of the equation $t=t_1(x_0^{(0)}(t,\ 0,\ x_0))$.

Since for $0<t \leqslant T$ the following inequalities hold

$$| t_1(x_0^{(0)}(t,\ 0,\ x_0))-t_1(x_0) | \leqslant N \| x_0^{(0)}(t,\ 0,\ x_0)-x_0 \| \leqslant \varepsilon MNT,$$

then there exists a constant, $\lambda_1^{(0)} \in (-1,\ 1)$ such that

$$\tau_1=t_1^{(0)}+\varepsilon\lambda_1^{(0)}\Theta_1^{(0)},\quad \Theta_1^{(0)}=MNT.$$

From the inequality $t_1^{(0)}>0$, it follows that $\tau_1 > \tau_0$, for ε sufficiently small. Hence

$$x(t)=x_0^{(0)}(t,\ 0,\ x_0)=\bar{x}_0^{(0)}(t,\ 0,\ x_0)+R_0^{(0)}(t,\ 0,\ x_0,\ \varepsilon)$$

for $\tau_0 < t \leqslant \tau_1=t_1^{(0)}+\varepsilon\lambda_1^{(0)}\Theta_1^{(0)}$. Furthermore, we find that

$$\begin{aligned} x_1^+=x_0^{(0)}(\tau_1,\ 0,\ x_0)+\varepsilon I_1(x_0^{(0)}(\tau_1,\ 0,\ x_0))=x_0+\varepsilon \int_0^{\tau_1} X_0^{(1)}(\theta,\ x_0,\ x_0,\ 0)\, d\theta + \\ +\varepsilon I_1^{(0)}+R_0^{(0)}(\tau_1,\ 0,\ x_0,\ \varepsilon), \end{aligned}$$

where $I_1^{(0)} = I_1(x_0^{(0)}(\tau_1, 0, x_0))$.

In the general case $(S = \overline{1, d_1})$, we denote by $x_s^{(0)}(t, \tau_s, x_s^+)$ the solution of the system

$$(11)\quad x_s^{(0)}(t, \tau_s, x_s^+) = \begin{cases} x_s^+ + \varepsilon \int_{\tau_s}^{t} X_s^{(K_s)}(\theta, x_s^{0}(\theta, \tau_s, x_s^+), x_s^{(0)}(\Delta_s^{0}(\theta), \tau_s, x_s^+), \\ \dot{x}_s^{(0)}(\Delta_s^{(0)}(\theta), \tau_s, x_s^+))\, d\theta, \quad t > \tau_s, \\ x_{s-1}^{(0)}(t, \tau_{s-1}, x_{s-1}^+), \quad -\delta \leq t \leq \tau_s, \end{cases}$$

$$\dot{x}_s^{(0)}(t, \tau_s, x_s^+) = \dot{x}_{s-1}^{(0)}(t, \tau_{s-1}, x_{s-1}^+), \quad -\delta \leq t \leq \tau_s,$$

where K_s is equal to 1 or 2 depending on whether the number $\Phi_s(\tau_s, x_{s-1}^{0}(\tau_s, \tau_{s-1}, x_{s-1}^+))$ is nonnegative or negative, $\Delta_s^{(0)}(t) = \Delta(t, x_s^{(0)}(t, \tau_s, x_s^+))$.

$$x_s^+ = x_{s-1}^{(0)}(\tau_s, \tau_{s-1}, x_{s-1}^+) + \varepsilon I_s(x_{s-1}^{(0)}(\tau_s, \tau_{s-1}, x_{s-1}^+)) =$$

$$= x_0 + \varepsilon \sum_{i=1}^{s} \int_{\tau_{i-1}}^{\tau_i} X_{i-1}^{(K_{i-1})}(0, x_0, x_0, 0)\, d\theta + \varepsilon \sum_{i=1}^{s} I_i^{(0)} + \sum_{i=1}^{s} R_{i-1}^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^+, \varepsilon), \quad k_0 = 1,$$

$$I_s^{(0)} \equiv I_s(x_{s-1}^{(0)}(\tau_s, \tau_{s-1}, x_{s-1}^+)), \quad x_0^+ \equiv x_0.$$

The solution of (11) coincides with the solution of the system (1) in the interval $[-\delta, \tau_{s+1}]$.

We consider the function

$$\tilde{x}_s^{(0)}(t, \tau_s, x_s^+) = x_s^+ + \varepsilon \int_{\tau_s}^{t} X_s^{(k)}(0, x_0, x_0, 0)\, d\theta.$$

One can show, as it was done in the case $s = 0$, that, for the difference

$$R_s^{(0)}(t, \tau_s, x_s^+, \varepsilon) = x_s^{(0)}(t, \tau_s, x_s^+) - \tilde{x}_s^{(0)}(t, \tau_s, x_s^+)$$

for $t \in (\tau_s, T]$ the following inequality holds

$$\| R_s^{(0)}(t, \tau_s, x_s^+, \varepsilon) \| \leq \varepsilon^2 KM[(T+s)^2 + T] +$$

$$+ \varepsilon\gamma(\varepsilon)(\delta\sqrt{n}+1)KT + 2\varepsilon KT \sum_{i=1}^{s} \omega_{i-1}^{(0)}(\varepsilon^2, T) \equiv \omega_s^{(0)}(\varepsilon^2, T).$$

Hence, the function $\tilde{x}_s^{(0)}(t, \tau_s, x_s^+)$ approximates the solution of (11) in the interval $(\tau_s, T]$ with accuracy of order ε^2.

The instant of time τ_{s+1} $(s = \overline{1, d_1})$ at which the trajectory $(t, x(t))$ meets the hypersurface σ_{s+1} appears to be a solution of the equation $t = t_{s+1}(x_s^{(0)}(t, \tau_s, x_s^+))$. Since for $0 < \tau_s < t \leq T$ the following inequalities hold

$$| t_{s+1}(x_s^{(0)}(t, \tau_s, x_s^+)) - t_{s+1}(x_0) | \leq N \| x_s^{(0)}(t, \tau_s, x_s^+) - x_0 \| \leq \varepsilon MN(T+s) +$$

$$+ N \sum_{i=1}^{s} \omega_{i-1}^{(0)}(\varepsilon^2, T),$$

there exists a constant $\lambda_{s+1}^{(0)} \in (-1, 1)$ such that $\tau_{s+1} = t_{s+1}^{(0)} + \varepsilon\lambda_{s+1}^{(0)}\Theta_{s+1}^{(0)}$, where

$$\Theta_{s+1}^{(0)} = MN(T+s) + \varepsilon^{-1} N \sum_{i=1}^{s} \omega_{i-1}^{(0)}(\varepsilon^2, T).$$

From $t_{s+1}^{(0)} > t_s^{(0)}$ it follows that $\tau_{s+1} > \tau_s$ when ε is sufficiently small. Combining the cases $s = 0$ and $s = \overline{1, d_1}$ and using the relation (4), we see that $x(t)$ can be written in the form

$$x(t) = x_s^{(0)}(t, \tau_s, x_s^+) = x_0 + \varepsilon \sum_{i=0}^{s} \int_{\tau_{i-1}}^{\tau_i} X_{i-1}^{(k_{i-1})}(\theta, x_0, x_0, 0)\, d\theta +$$

$$(12)\quad + \varepsilon \sum_{i=0}^{s} I_i^{(0)} + \sum_{i=0}^{s} R_{i-1}^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^+, \varepsilon) + \varepsilon \int_{\tau_s}^{s} X_s^{(k_s)}(0, x_0, x_0, 0)\, d\theta +$$

$$+ R_s^{(0)}(t, \tau_s, x_s^+, \varepsilon) = x_0 + \varepsilon \int_0^t X(0, x_0, x_0, 0)\, d\theta + \varepsilon \sum_{i=0}^{s} I_i^{(0)} +$$

$$+ \sum_{i=0}^{s} R_{i-1}^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^+, \varepsilon) + R_s^{(0)}(t, \tau_s, x_s^+, \varepsilon)$$

for

$$t_s^{(0)} + \varepsilon\lambda_s^{(0)}\Theta_s^{(0)} = \tau_s < t \leq \tau_{s+1} = t_{s+1}^{(0)} + \varepsilon\lambda_{s+1}^{(0)}\Theta_{s+1}^{(0)}, \qquad s = \overline{0, (d_1 - 1)},$$

as well as for

$$t_{d_1}^{(0)} + \varepsilon\lambda_{d_1}^{(0)}\Theta_{d_1}^{(0)} = \tau_{d_1} < t \leq T, \quad s = d_1.$$

where

$$X_{-1}^{(k_{-1})}(t, x, y, z) \equiv 0, \quad I_0^{(0)} \equiv R_{-1}^{(0)}(\tau_0, \tau_{-1}, x_{-1}^+, \varepsilon) \equiv 0, \quad k_0 = 1,$$

$$t_0^{(0)} = \Theta_0^{(0)} = \tau_0 = 0, \quad \lambda_s^{(0)} \in (-1, 1), \quad s = \overline{0, d_1},$$

$$\Theta_s^{(0)} = MN(T + (s-1)) + \varepsilon^{-1} N \sum_{i=0}^{s-1} \omega_{i-1}^{(0)}(\varepsilon^2, T), \quad s = \overline{1, (d_1 - 1)}.$$

Hence

$$x(T) = x_{d_1}^{(0)}(T, \tau_{d_1}, x_{d_1}^+) = x_0 + \varepsilon \int_0^T X(0, x_0, x_0, 0)\, d\theta +$$

$$+ \varepsilon \sum_{i=0}^{d_1} I_i^{(0)} + \sum_{i=0}^{d_1} R_{i-1}^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^+, \varepsilon) + R_{d_1}^{(0)}(T, \tau_{d_1}, x_{d_1}^+, \varepsilon).$$

Let $\bar{x}(t)$ be a solution of the averaged system (6) with initial condition (7). Then for $t \geq 0$ the following equality holds

$$\bar{x}(t) = x_0 + \varepsilon \int_0^t [X_0(\bar{x}(\theta)) + I_0(\bar{x}(\theta))]\, d\theta;$$

from this we find that

$$\bar{x}(T) = x_0 + \varepsilon \int_0^T [X_0(\bar{x}(\theta)) + I_0(\bar{x}(\theta))]\, d\theta.$$

Further on we would like to estimate the norm of the difference $x(T) - \bar{x}(T)$. For that purpose, taking into account (9) we write $x(T)$ in the form

$$x(T) = x_0 + \varepsilon[X_0(x_0) + I_0(x_0)]\, T + \varepsilon \int_0^T [X(\theta, x_0, x_0, 0) - X_0(x_0)]\, d\theta +$$

$$(13)\quad + \varepsilon\left[\sum_{i=0}^{d_1} I_i^{(0)} - I_0(x_0)\, T\right] + \sum_{i=0}^{d_1} R_{i-1}^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^+, \varepsilon) + R_{d_1}^{(0)}(T, \tau_{d_1}, x_{d_1}^+, \varepsilon).$$

Define the operator A_0

$$A_0 x = x + \varepsilon[X_0(x) + I_0(x)]\, T, \quad x \in D.$$

From (13), according to (9) and the conditions of the Theorem 1, we get

$$\| x(T) - A_0 x_0 \| \leqslant \varepsilon \left\| \int_0^T [X(\theta,\ x_0,\ x_0,\ 0) - X_0(x_0)]\, d\theta \right\| +$$

$$(14)\quad + \varepsilon \left\| \sum_{i=0}^{d_1} I_i^{(0)} - I_0(x_0)\, T \right\| + \sum_{i=0}^{d_1+1} \omega_{i-1}^{(0)}(\varepsilon^2,\ T) \leqslant \varepsilon\alpha(T)\, T/2 + \varepsilon \left\| \sum_{i=1}^{d_1} I_i(x_0) - \right.$$

$$\left. I_0(x_0)\, T \right\| + \varepsilon \left\| \sum_{i=1}^{d_1} (I_i^{(0)} - I_i(x_0)) \right\| + \sum_{i=0}^{d_1+1} \omega_{i-1}^{(0)}(\varepsilon^2,\ T) \leqslant \varepsilon\alpha(T)\, T +$$

$$\varepsilon^2 KMTd_1 + \varepsilon^2 K \sum_{i=1}^{d_1} \sum_{l=0}^{i-1} \| I_l^{(0)} \| + \varepsilon K \sum_{i=1}^{d_1} \sum_{l=0}^{i} \| R_{l-1}^{(0)}(\tau_l,\ \tau_{l-1},\ x_{l-1}^{+},\ \varepsilon) \| +$$

$$+ \sum_{i=0}^{d_1+1} \omega_{i-1}^{(0)}(\varepsilon^2,\ T) \leqslant \varepsilon\alpha(T)\, T + \varepsilon^2 M_1,$$

where $\omega_{-1}^{(0)}(\varepsilon^2,\ T) \equiv 0$ and $M_1 = M_1(T,\ d_1)$ is a constant.

For $t \geqslant 0$, $\tau \in [0,\ T)$ and $x \in D$ we have

$$\| X_0(x) \| \leqslant M, \quad \| I_0(x) \| \leqslant Md, \quad \| \bar{x}(\tau) - x_0 \| \leqslant \varepsilon(1+d)\, MT,$$

$$\| X_0(\bar{x}(\tau)) - X_0(x_0) \| \leqslant 2\varepsilon(1+d)\, KMT, \quad \| I_0(\bar{x}(\tau)) - I_0(x_0) \| \leqslant \varepsilon d(1+d)\, KMT.$$

Using these estimates we obtain

$$\| \bar{x}(T) - A_0 x_0 \| \leqslant \varepsilon \int_0^T \{ \| X_0(\bar{x}(\theta)) - X_0(x_0) \| + \| I_0(\bar{x}(\theta)) - I_0(x_0) \| \}\, d\theta \leqslant$$

$$(15)\qquad \leqslant \varepsilon^2(1+d)\,(2+d)\, KMT^2.$$

From (14) and (15) it follows immediately that

$$(16)\quad \| x(T) - \bar{x}(T) \| \leqslant \| x(T) - A_0 x_0 \| + \| \bar{x}(T) - A_0(x_0) \| \leqslant \varepsilon\alpha(T)\, T + \varepsilon^2 \bar{M},$$

where $\bar{M} = (1+d)\,(2+d)\, KMT^2 + M_1$.

Thus, we have estimated the norm of the difference $x(T) - \bar{x}(T)$ and have shown the closedness of the sets of points $x(T)$ and $\bar{x}(T)$. Moreover, since $\bar{x}(T)$ belongs to the domain D with its neighbourhood of radius ρ, then from (15) and (16) it follows that the points $A_0 x_0$ and $x(T)$ belong to D as well. Hence, the inequality (8) holds for $p=1$.

We introduce the notation

$$d(j-1\,;\ i) = d_0 + d_1 + \ldots + d_{j-1} + i, \quad d_0 \equiv 0, \quad \tau_i^{(j-1)} \equiv \tau_{d(j-1;i)}, \quad x_i^{(j-1)+} \equiv x_{d(j-1;i)}^{+},$$

$$i = \overline{1,\ d_j}. \qquad \tau_0^{(j-1)} \equiv (j-1)\, T, \quad \tau_{d_j+1}^{(j-1)} \equiv jT,$$

$$x_0^{(j-1)+} \equiv x((j-1)\, T), \quad x_{d_j+1}^{(j-1)+} \equiv x(jT), \quad j = 1,\ 2,\ \ldots\ .$$

We note that, according to the introduced notation,

$$\tau_0^{(j-1)} \equiv \tau_{d_{j-1}+1}^{(j-2)}, \quad x_0^{(j-1)+} \equiv x_{d_{j-1}+1}^{(j-2)+}, \quad j = 2,\ 3,\ \ldots\ .$$

Assume that for $p=j$, $j \geqslant 2$, the inequality (8) and the results of type (12) and (14)—(16) hold. We prove the validity of (8) for $p=j+1$. Let d_{j+1} points $t_{d(j;1)}(\bar{x}(jT)), \ldots, t_{d(j;d_{j+1})}(\bar{x}(jT))$ lie in the interval $(jT,\ (j+1)\, T)$ for which the following inequalities hold

$$t_{d(j;i)}(\bar{x}(jT)) < t_{d(j;i+1)}(\bar{x}(jT)) \quad \text{for} \quad i = \overline{1,\ (d_{j+1}-1)}.$$

Then, from the continuity of the functions $t_i(x)$, $i = 1,\ 2,\ \ldots$ and the assumption that (8) holds for $p=j$, it follows that if ε is sufficiently small, the points

$$(17)\qquad t_{d(j;1)}(x(jT)) = t_1^{(j)}, \ \ldots,\ t_{d(j;d_{j+1})}(x(jT)) = t_{d_{j+1}}^{(j)}$$

lie in the interval $(jT,\ (j+1)\, T)$ and $t_i^{(j)} < t_{i+1}^{(j)}$ for $i = \overline{1,\ (d_{j+1}-1)}$.

The solution $x(t)$ of the system (1), which is assumed to be constructed in the intervals $((p-1)\, T,\ pT]$, $p = \overline{1,\ j}$ is extended in the next interval $(jT,\ (j+1)\, T]$. For simplicity, we denote $x(jT)$ as x_{jT}. Let $x_0^{(j)}(t,\ jT,\ x_{jT})$ be a solution of the system

$$(18)\quad x_0^{(j)}(t,\ jT,\ x_{jT}) = \begin{cases} x_{jT} + \varepsilon \int_{jT}^{t} X_{d(j;0)}^{(K_{d(j;0)})}(\theta,\ x_0^{(j)}(\theta,\ jT,\ x_{jT}), \\ x_0^{(j)}(\Delta_0^{(j)}(\theta),\ jT,\ x_{jT}),\ \dot{x}_0^{(j)}(\Delta_0^{(j)}(\theta),\ jT,\ x_{jT}))\, d\theta, \quad t > jT, \\ x_{d_j}^{(j-1)}(t,\ \tau_{d_j}^{(j-1)},\ x_{d_j}^{(j-1)+}), \quad -\delta \leqslant t \leqslant jT, \end{cases}$$

$$\dot{x}_0^{(j)}(t,\ jT,\ x_{jT}) = \dot{x}_{d_j}^{(j-1)}(t,\ \tau_{d_j}^{(j-1)},\ x_{d_j}^{(j-1)}), \quad -\delta \leqslant t \leqslant jT,$$

where $k_{d(j;0)}$ is equal to 1 or to 2, $\Delta_0^{(j)}(t) = \Delta(t,\ x_0^{(j)}(t,\ jT,\ x_{jT}))$.

The solution of (18) coincides with the solution of the system (1) till to the moment $\tau_1^{(j)}$ at which the trajectory meets the hypersurface $\tau_{d(j;1)}$ i.e. $x(t) = x_0^{(j)}(t,\ jT,\ x_{jT})$ for $t \in [-\delta,\ \tau_1^{(j)}]$.

Consider the function

$$\tilde{x}_0^{(j)}(t,\ jT,\ x_{jT}) = x_{jT} + \varepsilon \int_{jT}^{t} X_{d(j;0)}^{(K_{d(j;0)})}(\theta,\ x_{jT},\ x_{jT},\ 0)\, d\theta.$$

For $jT < t \leqslant (j+1)\, T$ we have

$$\| R_0^{(j)}(t,\ jT,\ x_{jT},\ \varepsilon) \| = \| x_0^{(j)}(t,\ jT,\ x_{jT}) - \tilde{x}_0^{(j)}(t,\ jT,\ x_{jT}) \| \leqslant$$

$$\leqslant \varepsilon \int_{jT}^{t} \| X_{d(j;0)}^{(K_{d(j;0)})}(\theta,\ x_0^{(j)}(\theta,\ jT,\ x_{jT}),\ x_0^{(j)}(\Delta_0^{(j)}(\theta),\ jT,\ x_{jT}),$$

$$\dot{x}_0^{(j)}(\Delta_0^{(j)}(\theta),\ jT,\ x_{jT})) - X_{d(j;0)}^{(K_{d(j;0)})}(\theta,\ x_{jT},\ x_{jT},\ 0) \| \, d\theta \leqslant$$

$$\leqslant \varepsilon K \int_{jT}^{t} \{ \| x_0^{(j)}(\theta, jT, x_{jT}) - x_{jT} \| + \| x_0^{(j)}(\Delta_0^{(j)}(\theta), jT, x_{jT}) - x_{jT} \| +$$

$$+ \| \dot{x}_0^{(j)}(\Delta_0^{(j)}(\theta), jT, x_{jT}) \| \} \, d\theta \leqslant \varepsilon^2 KM \int_{jT}^{t} d\theta \int_{jT}^{\theta} dl +$$

$$+ \varepsilon K \{ \sum_{i=0}^{dj} \int_{J_{jT,t}^{i}} [\| x_i^{(j-1)}(\Delta_0^{(j)}(\theta), \tau_i^{(j-1)}, x_i^{(j-1)+}) - x_{(j-1)T} \| +$$

$$+ \| x_{jT} - x_{(j-1)T} \|] \, d\theta + \varepsilon M \int_{\tilde{J}_{jT,t}^{+}} d\theta \int_{jT}^{\Delta_0^{(j)}} dl \} +$$

$$+ \varepsilon K \{ \sum_{i=0}^{dj} \int_{J_{jT,t}^{i}} \| \dot{x}_i^{(j-1)}(\Delta_0^{(j)}(\theta), \tau_i^{(j-1)}, x_i^{(j-1)+}) \| \, d\theta +$$

$$+ \int_{\tilde{J}_{jT,t}^{+}} \| \dot{x}_0^{(j)}(\Delta_0^{(j)}(\theta), jT, x_{jT}) \| \, d\theta \} \leqslant \frac{1}{2} \varepsilon^2 KMT^2 +$$

$$\varepsilon^2 KM \int_{jT}^{t} [\theta - (j-1) T + d_j] \, d\theta + \varepsilon KT \sum_{l=0}^{dj} \omega_{l-1}^{(j-1)}(\varepsilon^2, T) +$$

$$+ \varepsilon^2 KT [\alpha(T) T + \varepsilon M_j + M(1+d)] + \varepsilon^2 KM \int_{jT}^{t} (\theta - jT) \, d\theta +$$

$$+ \varepsilon^2 KMT \leqslant \varepsilon^2 KM [(T+\theta)^2 + T] + \varepsilon^2 KMT(2T + d_j) + \varepsilon^2 KMT(1+d) +$$

$$+ \varepsilon^2 KT [\alpha(T) T + \varepsilon M_j] + \varepsilon kT \sum_{l=0}^{dj} \omega_{l-1}^{(j-1)}(\varepsilon^2, T) \equiv \omega_0^{(j)}(\varepsilon^2, T),$$

where

$$(\bigcup_{i=0}^{dj} J_{jT,t}^{i}) \cup \tilde{J}_{jT,t}^{+} = (jT, t],$$

$$J_{jT,t}^{i} = \{ \theta : \theta \in (jT, t] \wedge \Delta_0^{(j)}(\theta) \in (\tau_i^{(j-1)}, \tau_{i+1}^{(j-1)}] \}, \quad i = \overline{0, d_j},$$

$$\tilde{J}_{jT,t}^{+} = (jT, t] \setminus (\bigcup_{i=0}^{dj} J_{jT,t}^{i})$$

$$(\tau_i^{(0)} \equiv \tau_i, \ i = \overline{0, d_1}, \ \tau_{d_1+1}^{(0)} \equiv T, \ \omega_{-1}^{(j-1)}(\varepsilon^2, T) \equiv 0).$$

Hence, the function $\bar{x}_0^{(j)}(t, jT, x_{jT})$ approximates the solution of (18) in the interval $(jT, (j+1) T]$ with accuracy of order ε^2.

From the inequalities

$$| t_{d(j;1)}(x_0^{(j)}(t, jT, x_{jT})) - t_{d(j;1)}(x_{jT}) | \leqslant N \| x_0^{(j)}(t, jT, x_{jT}) -$$

$$- x_{jT} \| \leqslant \varepsilon MNT,$$

which hold for $jT < t \leqslant (j+1) T$ it follows that there exists a constant $\lambda_1^{(j)} \in (-1, 1)$ such that $\tau_1^{(j)} = t_1^{(j)} + \varepsilon \lambda_1^{(j)} \Theta_1^{(j)}$,

where $\Theta_1^{(j)} = MNT$. From $t_1^{(j)} > jT$ it follows that $\tau_1^{(j)} > jT$, when ε is sufficiently small.

Thus

$$x(t) = x_0^{(j)}(t, jT, x_{jT}) = \bar{x}_0^{(j)}(t, jT, x_{jT}) + R_0^{(j)}(t, jT, x_{jT}, \varepsilon)$$

for $jT < t \leqslant \tau_1^{(j)}$.

Further on we obtain

$$x_1^{(j)+} = x_0^{(j)}(\tau_1^{(j)}, jT, x_{jT}) + \varepsilon I_{d(j;1)}(x_0^{(j)}(\tau_1^{(j)}, jT, x_{jT})) = x_{jT} +$$

$$+ \varepsilon \int_{jT}^{\tau_1^{(j)}} X_{d(j;0)}^{(K_{d(j;0)})}(\theta, x_{jT}, x_{jT}, 0) \, d\theta + \varepsilon I_1^{(j)} + R_0^{(j)}(\tau_1^{(j)}, jT, x_{jT}, \varepsilon),$$

where $I_1^{(j)}$.

In the general case $(s = 1, d_{j-1})$ we denote as $x_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+})$ the solution of the system

$$x_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+}) = \begin{cases} x_s^{(j)+} + \varepsilon \int_{\tau_s^{(j)}}^{t} X_{d(j;s)}^{(K_{d(j;s)})}(\theta, x_s^{(j)}(\theta, x_s^{(j)}(\theta, \tau_s^{(j)}, x_s^{(j)+}), \\ x_s^{(j)}(\Delta_s^{(j)}(\theta), \tau_s^{(j)}, x_s^{(j)+}), \dot{x}_s^{(j)}(\Delta_s^{(j)}(\theta), \tau_s^{(j)}, x_s^{(j)+})) \, d\theta, \ t > \tau_s^{(j)}, \\ x_{s-1}^{(j)}(t, \tau_{s-1}^{(j)}, x_{s-1}^{(j)+}), \ -\delta \leqslant t \leqslant \tau_s^{(j)}, \end{cases} \tag{19}$$

$$\dot{x}_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+}) = \dot{x}_{s-1}^{(j)}(t, \tau_{s-1}^{(j)}, x_{s-1}^{(j)+}), \quad -\delta \leqslant t \leqslant \tau_s^{(j)},$$

where $\Delta_s^{(j)}(t) = \Delta(t, x_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+}))$ and

$$x_s^{(j)} = x_{s-1}^{(j)}(\tau_s^{(j)}, \tau_{s-1}^{(j)}, x_{s-1}^{(j)+}) + \varepsilon I_{d(j;s)}(x_{s-1}^{(j)}(\tau_s^{(j)}, \tau_{s-1}^{(j)}, x_{s-1}^{(j)+})) =$$

$$= x_{jT} + \varepsilon \sum_{i=1}^{s} \int_{\tau_{i-1}^{(j)}}^{\tau_i^{(j)}} X_{d(j;i-1)}^{(K_{d(j;i-1)})}(\theta, x_{jT}, x_{jT}, 0) \, d\theta + \varepsilon \sum_{i=1}^{s} I_i^{(j)} +$$

$$+ \sum_{i=1}^{s} R_{i-1}^{(j)}(\tau_i^{(j)}, \tau_{i-1}^{(j)}, x_{i-1}^{(j)+}, \varepsilon), \quad I_s^{(j)} \equiv I_{d(j;s)}(x_{s-1}^{(j)}(\tau_s^{(j)}, \tau_{s-1}^{(j)}, x_{s-1}^{(j)+})).$$

The solution of (19) coincides with the solution of the system (1) in the interval $[-\delta, \tau_{s+1}^{(j)}]$.

We consider the function

$$\bar{x}_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+}) = x_s^{(j)+} + \varepsilon \int_{\tau_s^{(j)}}^{t} X_{d(j;s)}^{(K_{d(j;s)})}(\theta, x_{jT}, x_{jT}, 0) \, d\theta.$$

One can show that for $jT < \tau_s^{(j)} < t \leqslant (j+1) T$ the following inqauality holds

$$\| R_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+}, \varepsilon) \| = \| x_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+}) - \bar{x}_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+}) \| \leqslant$$

$$\leqslant \varepsilon^2 KM [(T+s)^2 + T] + \varepsilon^2 KMT(2T + d_j) + \varepsilon^2 KMT(1+d) +$$

$$+ \varepsilon^2 KT [\alpha(T) T + \varepsilon M_j] + \varepsilon KT \sum_{l=0}^{dj} \omega_{l-1}^{(j-1)}(\varepsilon^2, T) +$$

$$+ 2\varepsilon KT \sum_{i=1}^{s} \omega_{i-1}^{(j)}(\varepsilon^2, T) \equiv \omega_s^{(j)}(\varepsilon^2, T)$$

etc. Combining the cases $s=0$ and $s=\overline{1, d_{j+1}}$ and using (4) we can write $x(t)$ in the form

$$
\begin{aligned}
x(t)=x_s^{(j)}(t,\ \tau_s^{(j)},\ x_s^{(j)+}) &= x_{jT}+\varepsilon \sum_{i=0}^{s} \int_{\tau_{i-1}^{(j)}}^{\tau_i^{(j)}} X_{d(j;i-1)}^{(K_{d(j;i-1)})}(\theta,\ x_{jT},\ x_{jT},\ 0)\, d\theta + \\
&+\varepsilon \sum_{i=0}^{s} I_i^{(j)} + \sum_{i=0}^{s} R_{i-1}^{(j)}(\tau_i^{(j)},\ \tau_{i-1}^{(j)},\ x_{i-1}^{(j)+},\ \varepsilon) + \\
&+\varepsilon \int_{\tau_s^{(j)}}^{t} X_{d(j;s)}^{(K_{d(j;s)})}(\theta,\ x_{jT},\ x_{jT},\ 0)\, d\theta + R_s^{(j)}(t,\ \tau_s^{(j)},\ x_s^{(j)+},\ \varepsilon) = \\
&= x_{jT}+\varepsilon \int_{jT}^{t} X(\theta,\ x_{jT},\ x_{jT},\ 0)\, d\theta+\varepsilon \sum_{i=0}^{s} I_i^{(j)} + \\
&+\sum_{i=0}^{s} R_{i-1}^{(j)}(\tau_i^{(j)},\ \tau_{i-1}^{(j)},\ x_{i-1}^{(j)+},\ \varepsilon)+R_s^{(j)}(t,\ \tau_s^{(j)},\ x_s^{(j)+},\ \varepsilon).
\end{aligned}
\tag{20}
$$

The expression (20) holds for

$$t_s^{(j)}+\varepsilon\lambda_s^{(j)}\Theta_s^{(j)}=\tau_s^{(j)}<t\leqslant \tau_{s+1}^{(j)}+\varepsilon\lambda_{s+1}^{(j)}\Theta_{s+1}^{(j)},\quad s=\overline{0,\ (d_{j+1}-1)},$$

as well as for

$$t_{dj+1}^{(j)}+\varepsilon\lambda_{dj+1}^{(j)}\Theta_{dj+1}^{(j)}=\tau_{dj+1}^{(j)}<t\leqslant (j+1)\ T,\quad s=d_{j+1},$$

where

$$X_{d(j;i-1)}^{(K_{d(j;i-1)})}(t,\ x,\ y,\ z)\equiv 0,\quad I_0^{(j)}\equiv R_{-1}^{(j)}(\tau_0^{(j)},\ \tau_{-1}^{(j)},\ x_{-1}^{(j)+},\ \varepsilon)\equiv 0,$$

$$\Theta_s^{(j)}=MN(T+(s-1))+\varepsilon^{-1}N\sum_{i=1}^{s}\omega_{i-1}^{(j)}(\varepsilon^2,\ T),\quad s=\overline{1,\ (d_{j+1}-1)}$$

$$\lambda_s^{(j)}\in(-1,\ 1),\quad s=\overline{0,\ d_{j+1}},\quad t_0^{(j)}=jT,\quad \Theta_0^{(j)}=0.$$

We derive $x((j+1)\ T)$ and $\overline{x}((j+1)\ T)$.

$$
\begin{aligned}
x((j+1)\ T)&=x_{dj+1}^{(j)}((j+1)\ T,\ \tau_{dj+1}^{(j)},\ x_{dj+1}^{(j)+})= \\
&= x_{jT}+\varepsilon\int_{jT}^{(j+1)T} X(\theta,\ x_{jT},\ x_{jT},\ 0)\ d\theta+\varepsilon\sum_{i=0}^{d_{j+1}} I_i^{(j)}+ \\
&+\sum_{i=0}^{d_{j+1}+1} R_{i-1}^{(j)}(\tau_i^{(j)},\ \tau_{i-1}^{(j)},\ x_{i-1}^{(j)+},\ \varepsilon)=x_{jT}+\varepsilon[X_0(x_{jT})+I_0(x_{jT})]+ \\
&+\varepsilon\int_{jT}^{(j+1)T}[X(\theta,\ x_{jT},\ x_{jT},\ 0)-X_0(x_{jT})]\ d\theta+\varepsilon\Big[\sum_{i=0}^{d_{j+1}} I_i^{(j)}-I_0(x_{jT})\Big]+ \\
&+\sum_{i=0}^{d_{j+1}+1} R_{i-1}^{(j)}(\tau_i^{(j)},\ \tau_{i-1}^{(j)},\ x_{i-1}^{(j)},\ \varepsilon),
\end{aligned}
$$

$$\overline{x}((j+1)\ T)=\overline{x}(jT)+\varepsilon\int_{jT}^{(j+1)T}[X_0(\overline{x}(\theta))+I_0(\overline{x}(\theta))]\ d\theta.$$

In order to estimate the norm of the difference $x((j+1)\ T)-\overline{x}((j+1)T)$ we use the inequality

$$
\begin{aligned}
\|\ x((j+1)\ T)-\overline{x}((j+1)\ T)\ \| &\leqslant \|\ x((j+1)\ T)-A_0x_{jT}\ \|+ \\
&+\|\ A_0x_{jT}-A_0\overline{x}(jT)\ \|+\|\ A_0\overline{x}(jT)-\overline{x}((j+1)\ T)\ \|.
\end{aligned}
\tag{21}
$$

By repeating the arguments in (14) we find the following estimate for the first summand in the right hand side of (21):

$$
\begin{aligned}
&\|\ x((j+1)\ T)-A_0x_{jT}\ \|\leqslant \\
&\leqslant \varepsilon\alpha(T)\ T+\varepsilon^2 KMd_{j+1}\ (2MT+d_{j+1}-1)/2+ \\
&+\varepsilon K\sum_{i=1}^{d_{j+1}}\sum_{l=0}^{i}\omega_{l-1}^{(j)}(\varepsilon^2,\ T)+\sum_{i=0}^{d_{j+1}+1}\omega_{i-1}^{(j)}(\varepsilon^2,\ T)\leqslant\varepsilon\alpha(T)\ T+\varepsilon^2 M_{j+1},
\end{aligned}
\tag{22}
$$

where $\omega_{-1}^{(j)}(\varepsilon^2,\ T)\equiv 0$, $M_{j+1}=M_{j+1}(T,\ d_1,\ \ldots,\ d_{j+1})$.

For the second summand in the right hand side of (21) we have

$$
\begin{aligned}
&\|\ A_0x_{jT}-A_0\overline{x}(jT)\ \|\leqslant\|\ x_{jT}-\overline{x}(jT)\ \|\ +\varepsilon T\ \|\ X_0(x_{jT})-X_0(\overline{x}(jT))\ \|\ + \\
&+\varepsilon T\ \|\ I_0(x_{jT})-I_0(\overline{x}(jT))\ \|\leqslant[1+\varepsilon(2+d)\ KT]\ \|\ x_{jT}-\overline{x}(jT)\ \|\leqslant \\
&\leqslant[1+\varepsilon(2+d)\ KT]\sum_{i=0}^{j-1}[1+\varepsilon(2+d)\ KT]^i\ [\varepsilon\alpha(T)\ T+\varepsilon^2\overline{M}],
\end{aligned}
\tag{23}
$$

where $\overline{M}=(1+d)\ (2+d)\ KMT^2+\max\limits_{i=\overline{1,j}} M_i$.

Since for $t\in(jT,\ (j+1)\ T]$ the following inequality holds

$$\|\ \overline{x}(t)-\overline{x}(jT)\ \|\leqslant\varepsilon\int_{jT}^{t}[\|X_0(\overline{x}(\theta))\ \|+\|\ I_0(\overline{x}(\theta))\ \|]\ d\theta\leqslant\varepsilon(1+d)\ MT,$$

then for the third summand in the right hand side of (21) we get

$$
\begin{aligned}
&\|\ A_0\overline{x}(jT)-\overline{x}((j+1)\ T)\ \|=\|\overline{x}(jT)+\varepsilon[X_0(\overline{x}(jT))+I_0(\overline{x}(jT))]\ T- \\
&-\overline{x}(jT)-\varepsilon\int_{jT}^{(j+1)T}[X_0(\overline{x}(\theta))+I_0(\overline{x}(\theta))]\ d\theta\ \|\leqslant\varepsilon\int_{jT}^{(j+1)T}[\|\ X_0(\overline{x}(jT))- \\
&-X_0(\overline{x}(\theta))\ \|+\|\ I_0(\overline{x}(jT))-I_0(\overline{x}(\theta))\ \|]\ d\theta\leqslant\varepsilon^2(1+d)\ (2+d)\ KMT^2.
\end{aligned}
$$

From (21)—(24) it follows that

$$\|\ x((j+1)\ T)-\overline{x}((j+1)\ T)\ \|\leqslant\sum_{i=0}^{j}[1+\varepsilon(2+d)\ KT]^i\ [\varepsilon\alpha(T)\ T+\varepsilon^2\overline{M}],$$

where $\overline{M}=(1+d)\ (2+d)\ KMT^2+\max\limits_{i=\overline{1,(j+1)}} M_i$.

The last inequality implies that (8) holds for $p=j+1$ and $x((j+1)\ T)$ belongs to the domain D.

This proves Lemma 1.

Proof of Theorem 1. According to the condition 3 of the Theorem 1 there exists a constant $C(T)<\infty$ such that for every $i=1, 2, \ldots$ the inequality $d_i<C(T)$ holds. Hence, there exists a constant $M_0(T)<\infty$ as well, such that

$$\overline{M}=(1+d)(2+d)KMT^2+\max_{i=1,2,\ldots} M_i\leqslant M_0(T). \tag{25}$$

Let q be the integer part of the number $L/\varepsilon T$. Then for every $p\in\overline{1, q}$, according to (25) and Lemma 1, we have

$$\|x(pT)-\overline{x}(pT)\|\leqslant\varepsilon\sum_{i=0}^{p-1}[1+\varepsilon(2+d)KT]^i[\alpha(T)T+\varepsilon M_0(T)]\leqslant$$

$$\leqslant[\alpha(T)T+\varepsilon M_0(T)][e^{(2+d)KL}+O(\varepsilon)]/(2+d)KT.$$

Choose T sufficiently large, such that $e^{(2+d)KL}\alpha(T)/(2+d)K<\eta/4$, and then choose ε sufficiently small such that

$$O(\varepsilon)\frac{\alpha(T)}{(2+d)K}+\varepsilon[e^{(2+d)KL}+O(\varepsilon)]\frac{M_0(T)}{(2+d)KT}<\frac{\eta}{4}.$$

Then, according to the choise of T and ε, for every $p\in\overline{1, q}$ the following inequality holds

$$\|x(pT)-\overline{x}(pT)\|<\eta/2. \tag{26}$$

Now we estimate $\|\overline{x}(t)-\overline{x}((p-1)T)\|$ and $\|x(t)-x((p-1)T)\|$ in the interval $(p-1)T\leqslant t\leqslant pT$.

We have

$$\|\overline{x}(t)-\overline{x}((p-1)T)\|\leqslant\varepsilon\int_{(p-1)T}^{t}[\|X_0(\overline{x}(\theta))\|+\|I_0(\overline{x}(\theta))\|]\,d\theta\leqslant\varepsilon(1+d)MT.$$

From (20) we get

$$\|x(t)-x((p-1)T)\|=\|x_s^{(p-1)}(t, \tau_s^{(p-1)}, x_s^{(p-1)+})-x((p-1)T)\|\leqslant$$

$$\leqslant\varepsilon\int_{(p-1)T}^{t}\|X(\theta, x_{(p-1)T}, x_{(p-1)T}, 0)\|\,d\theta+\varepsilon\sum_{i=0}^{s}\|I_i^{(p-1)}\|+$$

$$+\sum_{i=0}^{s}\|R_{i-1}^{(p-1)}(\tau_i^{(p-1)}, \tau_{i-1}^{(p-1)}, x_{i-1}^{(p-1)}, \varepsilon)\|+\|R_s^{(p-1)}(t, \tau_s^{(p-1)}x_s^{(p-1)+}, \varepsilon)\|\leqslant \tag{28}$$

$$\leqslant\varepsilon M(T+s)+\sum_{i=0}^{s-1}\omega_{i-1}^{(p-1)}(\varepsilon^2, T)\leqslant\varepsilon M[T+C(T)]+\varepsilon^2 M_0(T)\equiv\Psi(\varepsilon, T).$$

Hence, according to the choise of T, if ε is sufficiently small, we shall have

$$\Psi(\varepsilon, T)<\eta/2. \tag{29}$$

It follows from (26)—(29) that, according to the choise of T, if ε is sufficiently small, for $p=1, 2, \ldots, q$ the following inequality will be fulfilled in the interval $(p-1)T\leqslant t\leqslant pT$

$$\|x(t)-\overline{x}(t)\|<\eta.$$

Hence, according to the choise of T, if ε is sufficiently small $(0<\varepsilon\leqslant\leqslant\varepsilon_0\leqslant\mathscr{E})$, in the whole interval $0\leqslant t\leqslant L\varepsilon^{-1}$ the inequality $\|x(t)-\overline{x}(t)\|<\eta$ holds.

The proof of Theorem 1 is complete.

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Paisii Hilendarski University of Plovdiv Bulgaria

Higher Institute of Mechanical and Electrical Engeneering Sofia, Bulgaria