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BOUNDEDNESS OF APPROXIMATIONS IN SUMMATION OF ACCRETIVE OPERATORS

BY

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1. **Introduction.** A well known result on summing maximal monotone operators in a Hilbert space is as follows [2, Th. 2.4]. Let A and B be maximal monotone, with Yosida approximation $B_\lambda=\lambda^{-1}(I-(I+\lambda B)^{-1})$, $\lambda>0$. For $y\in H$, $y\in R(I+A+B)$ if and only if $B_\lambda u_\lambda$ is bounded as $\lambda\to 0$, where $y\in(I+A+B_\lambda)u_\lambda$.

This general result, which is important in considering the maximality of $A+B$, has appeared in the setting of monotone maps from a space to its dual, in a paper by Brezis, Crandall and Pazy [3]. Like many such results it extends partially to accretive operators. The fact that boundedness of $B_\lambda u_\lambda$ gives $y\in R(I+A+B)$ has been used by Kato [6], Konishi [7] and others, in the setting of accretive operators in a Banach space with uniformly Fréchet differentiable norm.

The goal of this paper is to investigate the converse.

We let $J: X\to 2^{X^*}$ be the duality map, i.e. $f\in Jx$ iff $\|f\|=\|x\|$ and $(x,f)=\|x\|^2$. We recall $B: X\to 2^X$ is accretive iff for $B'x\in Bx$ and $B'y\in By$, there is $J'(x-y)\in J(x-y)$ such that

$$(B'x-B'y,\ J'(x-y))\geq 0.$$

We let J_λ denote the resolvent $(I+\lambda B)^{-1}$ and let B_λ denote the Yosida approximation $\lambda^{-1}(I-J_\lambda)$, $\lambda>0$. We call B m-accretive when $R(I+\lambda B)=X$.

Suppose A and B are m-accretive in X, and suppose $y\in(I+A+B)u$ and $y\in(I+A+B_\lambda)u_\lambda$. We give two conditions on X which imply $B_\lambda u_\lambda$, bounded as $\lambda\to 0$. The first, studied in Section 2, is a strong condition on J which roughly says J can only twist a line segment through less than a right angle. The second condition on X, dealt with in Section 3, is that X be two dimensional.

It is very much an open question whether there exist spaces X and operators A and B giving a counterexample, i.e. $B_\lambda u_\lambda$ unbounded. Even in a three dimensional space with L_p norm it is not known if $B_\lambda u_\lambda$ would have to be bounded for general m-accretive A and B.

An obvious related question is the convergence of u_λ to u. This is treated in Section 4.

Throughout this paper we let $y=u+A'u+B'u$ where $A'u\in Au$ and $B'u\in Bu$, and likewise $y=u_\lambda+A'u_\lambda+B_\lambda u_\lambda$, where $A'u_\lambda\in Au_\lambda$. In general the dash denotes an element of the relevant set, as in $J'a \in Ja$ in (2.1). For $x \neq 0$ we write Rx for $\|x\|^{-1}x$.

2. **A Strong Condition on J.** In this section we look at two conditions, here denoted P and Q, closely related to J being bilipschitzian. These conditions are sufficient for the simple Hilbert space proof of the boundedness of the approximations $B_\lambda u_\lambda$ to go through. We show they do not hold in L_p. It would be surprising if Q, had not appeared before but the authors cannot recall any mention of it.

Definition 1.1. *Let X be a Banach space over* $\mathbf{R}$ *and suppose for $c \in X$ there is $K(c) \in \mathbf{R}$ such that for all a and b in X, if $J'a \in Ja$, $J'b \in Jb$, and*

$$(2.1) \qquad (a-b,\ J'a-J'b) \leqslant (c,\ J'a-J'b)$$

then $\|a-b\| \leqslant K(c)$. We say X (or J) satisfies condition P.

Theorem 2.1. *Let X be a Banach space satisfying condition P. Let A and B be m-accretive. Let $y \in R(I+A+B)$ and $y \in (I+A+B_\lambda)u_\lambda$. Then $B_\lambda u_\lambda$ are bounded.*

Proof. Let

$$(2.2) \qquad y = u + A'u + B'u, \qquad \text{and} \qquad (2.3)\quad y = u_\lambda + A'u_\lambda + B_\lambda u_\lambda,$$

where $A'u \in Au$, $B'u \in Bu$. Take $J'(u-u_\lambda)$ with $(A'u-A'u_\lambda,\ J'(u-u_\lambda)) \geqslant 0$. Then subtracting (2.3) from (2.2) and pairing with $J'(u-u_\lambda)$,

$$0 \geqslant (u-u_\lambda,\ J(u-u_\lambda)) + (B'u-B_\lambda u_\lambda,\ J'(u-u_\lambda)) \geqslant$$

$$\geqslant (B'u-B_\lambda u_\lambda,\ J'(u-J_\lambda u_\lambda)) + (B'u-B_\lambda u_\lambda,\ J'(u-u_\lambda)-J'(u-J_\lambda u_\lambda)).$$

where $J'(u-J_\lambda u_\lambda) \in J(u-J_\lambda u_\lambda)$ is chosen with

$$(B'u-B_\lambda u_\lambda,\ J'(u-J_\lambda u_\lambda)) \geqslant 0.$$

Hence,

$$(\lambda^{-1}(u-u_\lambda)-\lambda^{-1}(u-J_\lambda u_\lambda)+B'u,\ J'(u-u_\lambda)/\lambda-J'(u-J_\lambda u_\lambda)/\lambda) \leqslant 0.$$

By (2.1), $\|\lambda^{-1}(u_\lambda-J_\lambda u_\lambda)\| \leqslant K(B'u)$. q.e.d.

We now investigate P by relating it to two other conditions.

Definition 2.2 *Let X be a Banach space over* $\mathbf{R}$ *and suppose there is $\varepsilon > 0$ such that for a and a in X, $J'a \in Ja$, and $J'b \in Jb$,*

$$(2.4) \qquad (a-b,\ J'a-J'b) \geqslant \varepsilon\|a-b\|\ \|J'a-J'b\|.$$

We say X (or J) satisfies condition Q.

Proposition 2.1. *Q implies P. When X is finite dimensional, P implies Q.*

Proof. $Q \Rightarrow P$. Given $c \in X$, and a and b in X, with $J'a \in Ja$, $J'b \in Jb$, if (2.1) holds, then by (2.4)

$$\varepsilon\|a-b\|\ \|J'a-J'b\| \leqslant (a-b,\ J'a-J'b) \leqslant (c,\ J'a-J'b) \leqslant \|c\|\ \|J'a-J'b\|.$$

and $\|a-b\| \leqslant \varepsilon^{-1}\|c\|$.

$P \Rightarrow Q$, with X finite dimensional. If Q does not hold, then for all n there exist a_n, $J'a_n$, b_n and $J'b_n$ with

$$(2.5) \qquad (a_n-b_n,\ J'a_n-J'b_n) < n^{-1}\|a_n-b_n\|\ \|J'a_n-J'b_n\|.$$

If a_n and b_n are multiplied by a constant (2.5) still holds, so we assume $\|a_n-b_n\| = n$. By (2.5), $\|J'a_n-J'b_n\| \neq 0$, and we assume by compactness, taking a subsequence, that $(c,\ (J'a_n-J'b_n)\|J'a_n-J'b_n\|^{-1}) \to 1$ where $\|c\| = 1$. (In infinite dimensions we might have weak star convergence to 0). For n large, $\|J'a_n-J'b_n\| < 2(c,\ J'a_n-J'b_n)$. By (2.5),

$$(a_n-b_n,\ J'a_n-J'b_n) < (2c,\ J'a_n-J'b_n)$$

for n large, while $\|a_n-b_n\| = n$ and P does not hold. q.e.d.

Definition 2.3. *We say J is bilipschitzian if J and J^{-1} are Lipschitz and everywhere defined*

Proposition 2.2. *If J is bilipschitzian then Q holds. If X has C^2 norm away from 0 and is reflexive, then condition Q implies J^{-1} is Lipschitz.*

Proof. Suppose J bilipschitzian. By [4], for all a and b,

$$\|a-b\|^2 \leqslant 4\|J^{-1}\|_{\mathrm{Lip}}(a-b,\ Ja-Jb).$$

Hence

$$\|Ja-Jb\|\ \|a-b\| \leqslant \|J\|_{\mathrm{Lip}}\|a-b\|^2 \leqslant 4\|J\|_{\mathrm{Lip}}\|J^{-1}\|_{\mathrm{Lip}}(a-b,\ Ja-Jb).$$

Suppose that Q holds and the norm is C^2 away from 0. For $a \neq 0$, taking limits in (2.4) gives, with DJ the derivative of J,

$$(2.6) \qquad (x,\ DJ(a)x) \geqslant \varepsilon\|x\|\ \|DJ(a)x\|.$$

Now $(y,\ DJ(a)x) = (x,\ DJ(a)y)$ for all x and y, and for all a, $\inf\limits_{\|x\|=1}\|DJ(a)x\| = f(a) \geqslant 0$. Let $\lambda = \inf\limits_{\|a\|=1} f(a)$.

Suppose $\lambda = 0$. Given $\varepsilon > 0$ there exists a with norm 1, and x depending on a, with norm 1, and $\|DJ(a)x\| \leqslant \varepsilon$. Note that $DJ(a)a = \lim\limits_{t \to 0}(J(a+ta)-J(a))/t = Ja$. Also $(x,\ DJ(a)a) = (a,\ DJ(a)x) \leqslant \|DJ(a)x\| \leqslant \varepsilon$.

Take $z = x + \delta a$, $\delta > \varepsilon$ to be specified, giving $\|z\| \geqslant 1-\delta$. Since $DJ(a)z = DJ(a)x + \delta DJ(a)a = DJ(a)x + \delta Ja$ we have $\|DJ(a)z\| \geqslant \delta - \varepsilon$.

$$(z,\ DJ(a)z) = (x,\ DJ(a)x) + \delta(x,\ DJ(a)a) + \delta(a,\ DJ(a)x) + \delta^2(a,\ DJ(a)a) \leqslant$$

$$\leqslant \varepsilon + 2\delta\varepsilon + \delta^2 \leqslant (1-\delta)^{-1}(\delta-\varepsilon)^{-1}(\varepsilon+2\delta\varepsilon+\delta^2)\|z\|\ \|DJ(a)z\|.$$

The constant on the right hand side may be made arbitrarily small by taking ε, δ small and ε/δ small, contradicting Q.

Hence $\lambda > 0$, and for all a with $a \neq 0$ and all x, since $DJ(a) = DJ(a/\|a\|)$.

$$(2.7) \qquad \|DJ(a)x\| \geqslant \lambda\|x\|.$$

By (2.7), $R(DJ(a))$ is closed, and if $R(DJ(a)) \neq X^*$, take $y \neq 0$ in X with $(y,\ DJ(a)x) = 0$ for all x, by reflexivity of X. But by (2.6) and (2.7), $y = 0$, a contradiction giving $DJ(a)$ onto. By (2.7) $DJ(a)$ is an isomorphism and the inverse function theorem shows J^{-1} is C^1 near $J(a)$, with $D(J^{-1})(J(a)) = DJ(a)^{-1}$. So $\|D(J^{-1})(J(a))\| \leqslant \lambda^{-1}$ which gives J^{-1} Lipschitz with constant λ^{-1}. q.e.d.

Corollary 2.1. *Suppose X is finite dimensional and has C^2 norm away from 0. J is bilipschitzian iff Q holds iff P holds.*

Proof. By Propositions 2.1 and 2.2.

The following result was first established by direct methods, but it follows readily from Proposition 2.3.

Theorem 2.2. *Let $X = L_p(X, B, \mu)$ for $p \neq 2$, with dimension > 1. Then condition P does not hold.*

Proof. Let Y be a two dimensional space spanned by the characteristic functions of two disjoint sets A and B.

Let $p > 2$. J^{-1} is not Lipschitzian, but Y has C^2 norm away from 0, so by Corollary 2.1, P does not hold in Y, and hence not in X.

Suppose $p < 2$. By the case $p > 2$ Q does not hold in Y^* and by symmetry of condition (2.4), Q does not hold in Y. By Corollary 2.1 P fails in Y and hence in X.

Remark. We can see that we have a local version of the conditions P etc., implying that in $l_p^{(3)}$ for example if $\|B_\lambda u_\lambda\| \to \infty$, then J^{-1} is not Lipschitz near the direction of $u_\lambda - u$, so that $u_\lambda - u$ is near a co-ordinate plane.

3. **X two dimensional.** In this section we show that in a two dimensional space $B_\lambda u_\lambda$ is bounded without any assumption of smoothness for X or X^*.

Theorem 3.1. *Let X be a two dimensional Banach space. Let A and B be m-accretive in X. Let $y \in (I+A+B)u$, $y \in (I+A+B_\lambda)u_\lambda$, $\lambda > 0$. Then $B_\lambda u_\lambda$ is bounded as $\lambda \to 0$.*

Proof. Suppose for some sequence $\lambda(n) \to 0$, $\|B_\lambda u_\lambda\| \to \infty$ writing λ instead of $\lambda(n)$ in this proof. We know by the proof of [1, Lemma 3.2] that u_λ is bounded. We may by translation suppose $u = 0$, since if $B^u x = B(u+x)$ and $A^u x = A(u+x)$ then $y - u \in (I + A^u + B^u)(0)$ and $y - u \in x_\lambda + A^u(x_\lambda) + (B^u)_\lambda(x_\lambda)$ where $x_\lambda = u_\lambda - u$, and $B_\lambda u_\lambda = (B^u)_\lambda x_\lambda$.

Suppose $RB_\lambda u_\lambda \to w$.

$u_\lambda \to u$: Suppose $u_\lambda \to v \neq 0$; we will obtain a contradiction.

Suppose first v not a multiple of w. Take w on the negative x axis and v in the upper half plane.

Since $R(u_\lambda - J_\lambda u_\lambda) \to w$ we have $J_\lambda u_\lambda$ in a cone $\{u_\lambda + (x, y) : |y| \leq \varepsilon x\}$ where $\varepsilon \to 0$ as $\lambda \to 0$. Since $J_\lambda u_\lambda \neq u_\lambda$, $J_\lambda u_\lambda$ is strictly clockwise from u_λ, for λ small.

We have by the defining equations for $B_\lambda u_\lambda$

$$(B_\lambda u_\lambda - B'(0),\ J'(u_\lambda)) \leq -\|u_\lambda\|^2, \tag{3.1}$$

and by accretivity of B

$$(B_\lambda u_\lambda - B'(0),\ J'(J_\lambda u_\lambda)) \geq 0. \tag{3.2}$$

These imply since $R(B_\lambda u_\lambda - B'u) \to w$, that $J_\lambda u_\lambda$ is anticlockwise from u_λ, a contradiction.

Suppose next $v = kw$, $k > 0$. By (3.1), $(w, J'w) \leq 0$, a contradiction.

Suppose $v = -kw$, $k > 0$. Since $R(u_\lambda - J_\lambda u_\lambda) \to w$ we have $J_\lambda u_\lambda \to v - \lambda w$ some $\lambda > 0$, taking a subsequence. Taking limits in (3.2) gives $(w, J'(-k - \lambda)w) \geq 0$, a contradiction. Hence $u_\lambda \to 0$.

$u_\lambda \neq u \neq J_\lambda u_\lambda$: Suppose $u_\lambda = u$; then $\|B_\lambda u_\lambda\| = \|B_\lambda u\| \leq \|B'u\|$. Suppose $J_\lambda u_\lambda = u$; by (3.1) $(B_\lambda u_\lambda - B'u,\ J'(u_\lambda - u)) \leq 0$, giving $(B_\lambda u_\lambda - B'u, J'(B_\lambda u_\lambda)) \leq 0$, and $\|B_\lambda u_\lambda\| \leq \|B'u\|$. Since $\|B_\lambda u_\lambda\| \to \infty$ $u_\lambda \neq u \neq J_\lambda u_\lambda$.

Convergence of $RJ'''(J_\lambda u_\lambda)$: Let $R(u_\lambda - u) \to a$, $RJ'(u_\lambda - u) \to f$ with $J'(u_\lambda - u)$ satisfying (3.1) and $R(J_\lambda u_\lambda - u) \to b$, taking a subsequence.

Since $u \in D(B)$, $(B_\lambda u_\lambda - B_\lambda u,\ J'(u_\lambda - u)) \geq 0$ with the same $J'(u_\lambda - u)$ since B_λ is accretive and Lipschitz. Hence $(w, f) \geq 0$. By (3.1), $(w, f) \leq 0$ giving $(w, f) = 0$. Now $f \in J(a)$, and let us without loss of generality take f on the positive y axis, the y axis in X^* being $N(w) = \{g : (w, g) = 0\}$.

For λ small, $N(B_\lambda u_\lambda - B'u)$ is near $N(w)$, and by (3.2) there is $J'(J_\lambda u_\lambda)$ to the left of $N(B_\lambda u_\lambda - B'u)$. And for $s \in D(B)$, $(B_\lambda u_\lambda - B's,\ J(J_\lambda u_\lambda - s) \geq 0$, and hence $(w, J''(-s)) \geq 0$ for some $J''(-s)$. In particular with $s = J_\lambda u_\lambda$, there is $J''(J_\lambda u_\lambda) \in J(J_\lambda u_\lambda)$ on the right of $N(w)$. Since $J(J_\lambda u_\lambda)$ is convex there is $J'''(J_\lambda u_\lambda)$ in the narrow set between $N(w)$ and $N(B_\lambda u_\lambda - B'u)$ as boundaries. Hence $RJ'''(J_\lambda u_\lambda)$ converges to f or $-f$, taking a subsequence.

Two cases : Suppose $RJ'''(J_\lambda u_\lambda) \to f$. So $b \in J^{-1}(f)$, $(a, f) = (b, f) = 1$ and $(w, f) = 0$, so a and b are in the upper half plane. As in the case $u_\lambda \to v$, $J_\lambda u_\lambda$ is anticlockwise from u_λ by (3.1) and (3.2), a contradiction.

Suppose now $RJ'''(J_\lambda u_\lambda) \to -f$. Thus $a \in J^{-1}(f)$ and $b \in J^{-1}(-f)$. Since $Ru_\lambda \to a$ and $RJ_\lambda u_\lambda \to b$, we have $R(u_\lambda - J_\lambda u_\lambda)$ converging to an element of the cone spanned by $J^{-1}f$. Hence $(w, f) = 1$, a contradiction, giving $B_\lambda u_\lambda$ bounded. q.e.d.

Corollary 3.1. *Let X be a two dimensional Banach space, A and B m-accretive. For $y \in X$, $y \in R(I+A+B)$ iff $B_\lambda u_\lambda$ is bounded where $y \in (I+A+B_\lambda)u_\lambda$.*

4. **Convergence of u_λ.** In this section we show that $u_\lambda \to u$ with only the assumption of a *UG* norm. This shows that we have, if $A+B$ is m-accretive, pointwise convergence of $(I+A+B_\lambda)^{-1}$ to $(I+A+B)^{-1}$. One might think this would follow from general results on convergence as in Theorem 4.1 of [5], but these seem to require more smoothness on X and X^*, at least.

Theorem 4.1. *Let X be a Banach space with uniformly Gateaux differentiable norm. Let A, B be m-accretive with $y \in (I+A+B)u$ and $y \in (I+A+B_\lambda)u_\lambda$. Then $u_\lambda \to u$ as $\lambda \to 0^+$.*

Proof. Let $y = u_\lambda + A'u_\lambda + B_\lambda u_\lambda$. Then $\lambda(y - u_\lambda - A'u_\lambda) = u_\lambda - J_\lambda u_\lambda$. So $y - u_\lambda - A'u_\lambda \in B(J_\lambda u_\lambda)$ and $J_\lambda u_\lambda = (u_\lambda + \lambda u_\lambda + \lambda A'u_\lambda - \lambda y)$.

Since $y = u + A'u + B'u$, we have $(y - u_\lambda - A'u_\lambda - B'u,\ J(J_\lambda u_\lambda - u)) \geq 0$. Now

$$\|J_\lambda u_\lambda - u\|^2 = (u_\lambda + \lambda u_\lambda + \lambda A'u_\lambda - \lambda y - u,\ J(J_\lambda u_\lambda - u)) =$$
$$= ((u_\lambda - u - \lambda B'u) - \lambda(y - u_\lambda - A'u_\lambda - B'u),\ J(J_\lambda u_\lambda - u)) \leq$$
$$\leq (u_\lambda - u - \lambda B'u,\ J(J_\lambda u_\lambda - u)) \leq \|u_\lambda - u - \lambda B'u_\lambda\| - \|J_\lambda u_\lambda - u\|.$$

Hence

$$\frac{1}{2}\|u_\lambda - u + \lambda u_\lambda + \lambda A'u_\lambda - \lambda y\|^2 - \frac{1}{2}\|u_\lambda - u\|^2 \leq \frac{1}{2}\|u_\lambda - u - \lambda B'u\|^2 - \frac{1}{2}\|u_\lambda - u\|^2,$$

and

$$(u_\lambda + A'u_\lambda - y,\ J(u_\lambda - u)) \leq (2\lambda)^{-1}(\|u_\lambda - u - \lambda B'u\|^2 - \|u_\lambda - u\|^2).$$

So

$$(u_\lambda + A'u_\lambda - u - A'u - B'u,\ J(u_\lambda - u)) \leqslant (2\lambda)^{-1}(\|u_\lambda - u - \lambda B'u\|^2 - \|u_\lambda - u\|^2),$$

giving

$$\|u_\lambda - u\|^2 \leqslant (B'u,\ J(u_\lambda - u)) + (2\lambda)^{-1}(\|u_\lambda - u - \lambda B'u\|^2 - \|u_\lambda - u\|^2) \to 0$$ as $\lambda \to 0^+$ since X is (UG) and u_λ is bounded. q.e.d.

REFERENCES

1. Barbu V. — *Semigrupuri de contracţii neliniare în spaţii Banach*, Editura Academiei, Bucureşti, 1974.
2. Brezis H. — *Opérateurs maximaux monotones*, North Holland, Amsterdam, 1973.
3. Brezis H., Crandall M. and Pazy A. — *Perturbations of nonlinear maximal monotone sets*, Comm. Pure Appl. Math. 23 (1970), 123—144.
4. Brezis H. and Browder F. — *Nonlinear integral equations and systems of Hammerstein type*, Adv. in Math. 18 (1975), 115—147.
5. Brezis H. and Pazy A. — *Convergence and approximation of non-linear semigroups in Banach spaces*, J. Funct. Anal. (1971), 63—74.
6. Kato T. — *Accretive operaters and nonlinear evolution equations in Banach spaces*, Proc. Symp. in Pure Maths., (1970), 138—161.
7. Konishi Y. — *A remark on perturbation of m-accretive operators in Banach spaces*, Proc. Japan Acad. 47 (1971), 452—455.

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THE SEMI-INVARIANT SUBMANIFOLDS WITH SEMI-FLAT NORMAL CONNECTION OF A SASAKIAN SPACE FORM

BY

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1. Preliminaries. The semi-invariant submanifolds of a Sasakian manifold correspond to the CR-submanifolds of a Kaehler manifold and they have been introduced and studied by A. Bejancu and the present author in [2], [3], [4], [1], [9].

In [6], B. Y. Chen gives a classification theorem for CR-submanifolds with semi-flat normal connection in a complex space form $\tilde M(c)$, $c \neq 0$.

The main purpose of the present paper is to classify the semi-invariant submanifolds with semi-flat normal connection of a Sasakian space form $\tilde M(c)$. Our classification theorem for semi-invariant submanifolds with semi-flat normal connection of a Sasakian space form (Theorem 2.2) is simpler than the classification theorem of B. Y. Chen (Theorem 5.11 in [6]) because in our case, the f-structure defined on the normal bundle of a semi-invariant submanifold is parallel (Theorem 2.1). The existence of semi-invariant products with semi-flat normal connection is also studied (Theorem 2.3).

First, we recall the definitions and some fundamental results for later use.

Let $\tilde M$ be a Sasakian manifold of dimension $2n+1$ with the Sasakian structure (φ, ξ, η, g), where φ is a tensor field of type (1, 1), ξ is a vector field, η is a 1-form and g is a Riemannian metric on $\tilde M$ which are related by:

(1.1) $$\varphi^2 = -I + \eta \otimes \xi\ ;\ \varphi(\xi) = 0\ ;\ \eta(\xi) = 1\ ;\ \eta \circ \varphi = 0 \quad \text{and}$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y tangent to $\tilde M$, where I denotes the identity tensor on $\tilde M$. We denote by $\tilde\nabla$ the Levi-Civita connection on $\tilde M$. Then, we also have ([5], p. 73)

(1.2) $(\tilde\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X\ ;\ \tilde\nabla_X \xi = -\varphi X$, for any vector fields X, Y tangent to $\tilde M$.

Now, let M be an m-dimensional Riemannian manifold isometrically immersed in $\tilde M$ such that the structure vector field ξ of $\tilde M$ is tangent to M. Denote by TM and $TM^\perp$ the tangent bundle of M and respectively the normal bundle to M and by $\{\xi\}$ the 1-dimensional distribution defined by ξ on M.