Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXIII, s. I a. Matematică, 1987, f. 3

ON WEYL'S PROJECTIVE CURVATURE TENSOR OF KAEHLERIAN HYPERSURFACES

BY

J. DEPREZ, F. DILLEN, M. PETROVIC-TORGASEV and L. VERSTRAELEN

1. Introduction. In this paper we study Kaehler hypersurfaces of a complex space form which satisfy some condition on the projective curvature tensor P of Weyl. Hypersurfaces of a Euclidean space satisfying similar conditions on P were studied in [3].

In the following, for the submanifolds under consideration, the Riemann-Christoffel curvature tensor, the Weyl conformal curvature tensor and the Bochner curvature tensor will be denoted by R, C and B, respectively, and the scalar curvature will be denoted by r. Further, Q will denote the (1,1) tensor field defined by $<QX, Y>=S(X, Y)$ for all vector fields X and Y tangent to the manifold where S denotes the Ricci tensor field and $<,>$ denotes the metric.

We will prove the following theorems.

Theorem 1. *For a Kaehler hypersurface M^n in a complex space form $\tilde{M}^{n+1}(c)$ of curvature c the following conditions are equivalent:*

(1) $P=0$,	(4) $P \cdot P=0$,
(2) $P \cdot C=0$,	(5) $Q \cdot P=0$,
(3) $P \cdot R=0$,	(6) $C \cdot P=0$,

(7) $c=0$ *and* M^n *is totally geodesic in* $\tilde{M}^{n+1}(0)$.

Theorem 2. *The only Kaehler hypersurfaces of a complex space form which satisfy $P \,.\, Q=0$ are the Einstein Kaehler hypersurfaces.*

Chern proved in [2] that the only Einstein hypersurfaces of a complex space form are the totally geodesic hypersurfaces and the hypersurfaces in a complex projective space $\mathbf{C}P^{n+1}(c)$, $c>0$, which are locally isometric to the complex hypersphere Q^n in $\mathbf{C}P^{n+1}(c)$; Q^n is given by the equation $z_0^2+z_1^2+\ldots+z_{n+1}^2=0$, where $z_0, z_1,\ldots,z_{n+1}$ are homogeneous coordinates of $\mathbf{C}P^{n+1}(c)$.

Theorem 3. *The only Kaehler hypersurfaces of a complex space form which satisfy $R \cdot P = 0$ are the semi-symmetric hypersurfaces.*

Semi-symmetric spaces are manifolds satisfying $R \,.\, R=0$. Let A denote the second fundamental tensor of a Kaehler hypersurface M^n of a complex space form M^{n+1} (c). If $c=0$, then $R \cdot R=0$ is equivalent with rank $A \leqslant 2$; if $c \neq 0$, then $R \cdot R=0$ is equivalent with M is Einstein ([6]).

According to a result of Abe ([1]), a complete Kaehler hypersurface in a complex Euclidean space for which rank $A \leqslant 2$ is a complex hypercylinder, i.e. the product of an $(n-1)$-dimensional complex hyperplane $\mathbf{C}^{n-1}$ in $\mathbf{C}^n \subset \mathbf{C}^{n+1}$ with a complex curve in a 2-dimensional complex plane orthogonal to $\mathbf{C}^{n-1}$ in $\mathbf{C}^{n+1}$.

Theorem 4. *For a Kaehler hypersurface M^n of a complex space form $\tilde{M}^{n+1}(c)$ the following conditions are equivalent:*

(1) $P \cdot B = 0$, (3) $B = 0$,

(2) $B \cdot P = 0$, (4) M^n *is totally geodesic.*

2. **Basic formulas.** Let $\tilde{M}^{n+1}(c)$ be a complex space form of complex dimension $n+1$ and of holomorphic sectional curvature $c \in \mathbf{R}$, with metric $<,>$ complex structure J and Levi-Civita connection $\tilde{\nabla}$. Then the curvature tensor $\tilde{R}$ of $\tilde{M}^{n+1}(c)$ takes the following form:

$$\tilde{R}(X, Y) = (c/4)\,(X \wedge Y + JX \wedge JY + 2 <X, JY> J),$$

for X and Y belonging to $T_p\tilde{M}$, $p \in \tilde{M}$, and where $X \wedge Y$ is the endomorphism of T_pM defined by

$$(X \wedge Y)Z = <Y, Z>X - <X, Z>Y.$$

It is known that a complete simply connected complex space form $\tilde{M}^{n+1}(c)$ is holomorphically isometric with $\mathbf{C}P^{n+1}(c)$, $\mathbf{C}^{n+1}$ or $\mathbf{D}^{n+1}(c)$, according to c being positive, zero or negative. $\mathbf{C}P^{n+1}(c)$ is the complex projective space with the Study-Fubini metric of holomorphic sectional curvature c, $\mathbf{C}^{n+1}$ the complex Euclidean space, and $\mathbf{D}^{n+1}(c)$ the unit ball in $\mathbf{C}^{n+1}$ with the Bergmann metric of holomorphic sectional curvature c.

Let M^n be a Kaehler hypersurface of $\tilde{M}^{n+1}(c)$. We denote the induced metric, complex structure and connection respectively by $<,>$, J and ∇. At each point p of M we can choose an orthonormal frame $e_1, e_2, \ldots, e_n$, $e_{1*} = Je_1$, $e_{2*} = Je_2, \ldots, e_{n*} = Je_n$, ξ, $J\xi$, such that $e_1, \ldots, e_{n*}$ are tangent to M^n and $J\xi$ are normal to M^n, and such that the second fundamental tensor $A = A_\xi$ in the direction of ξ satisfies

$$Ae_i = \lambda_i e_i, \quad Ae_{i*} = -\lambda_i e_{i*},$$

with $\lambda_i \in \mathbf{R}^+$, $i \in \{1, \ldots, n\}$.

Note that if M^n is locally isometric with the complex hypersphere Q^n in $\mathbf{C}P^{n+1}(c)$, then $A^2 = (c/4)\,I$.

The Weyl conformal tensor C, the Weyl projective tensor P, and the Bochner tensor B of M^n are defined by

$$C(X, Y) = R(X, Y) - [1/(2(n-1))]\,(QX \wedge Y + X \wedge QY) + [r/(2(n-1)(2n-1))]\,X \wedge Y,$$

$$P(X, Y) = R(X, Y) - [1/(2n-1)]\,(X \wedge Y) \circ Q$$

$$B(X, Y) = R(X, Y) - [1/(2(n+2))]\,(QX \wedge Y + X \wedge QY + QJX \wedge JY + JX \wedge QJY - 2 <QJX, Y> J - 2 <JX, Y> Q) + [r/(4(n+1)(n+2))]\,(X \wedge Y + JX \wedge JY + 2 <X, JY> J).$$

These curvature tensors satisfy the following relations:

$$R(X, Y) = R(JX, JY), \qquad B(JX, JY) = B(X, Y),$$
$$R(X, Y)J = JR(X, Y), \qquad B(X, Y)J = JB(X, Y),$$
$$QJ = JQ, \qquad P(JX, JY) = -JP(X, Y)J.$$
$$C(JX, JY) = -JC(X, Y)J,$$

The following formulas can be verified by straightforward computations.

$$\begin{cases} R(e_i, e_j) = \nu_{ij}(e_i \wedge e_j + e_{i*} \wedge e_{j*}), \\ R(e_i, e_{j*}) = \bar{\nu}_{ij}(e_i \wedge e_{j*} - e_{i*} \wedge e_j) - (c/2)\delta_{ij} J, \end{cases}$$

where

$$\nu_{ij} = c/4 + \lambda_i\lambda_j \text{ and } \bar{\nu}_{ij} = c/4 - \lambda_i\lambda_j;$$

$$Qe_i = \mu_i e_i, \quad Qe_{i*} = \mu_i e_{i*},$$

where

$$\mu_i = ((n+1)/2)c - 2\lambda_i^2;$$

$$\begin{cases} C(e_i, e_j) = (\nu_{ij} + \alpha_{ij})e_i \wedge e_j + \nu_{ij} e_{i*} \wedge e_{j*}, \\ C(e_i, e_{j*}) = (\bar{\nu}_{ij} + \alpha_{ij})e_i \wedge e_{j*} - \bar{\nu}_{ij} e_{i*} \wedge e_j - \dfrac{c}{2}\,\delta_{ij} J, \end{cases}$$

where

$$\alpha_{ij} = \frac{r}{2(n-1)(2n-1)} - \frac{1}{2(n-1)}\,(\mu_i + \mu_j);$$

$$P(e_i, e_j)e_k = \left(\nu_{ij} - \frac{1}{2n-1}\,\mu_k\right)(\delta_{kj} e_i - \delta_{ki} e_j),$$

$$P(e_i, e_j)e_{k*} = \nu_{ij}(\delta_{kj} e_{i*} - \delta_{ki} e_{j*}),$$

$$P(e_i, e_{j*})e_k = -\left(\bar{\nu}_{ij} - \frac{1}{2n-1}\,\mu_k\right)\delta_{ik} e_{j*} - \bar{\nu}_{ij}\delta_{jk} e_{i*} - \frac{c}{2}\,\delta_{ij} e_{k*},$$

$$P(e_i, e_{j*})e_{k*} = \left(\bar{\nu}_{ij} - \frac{1}{2n-1}\,\mu_k\right)\delta_{jk} e_i + \bar{\nu}_{ij}\delta_{ik} e_j + \frac{c}{2}\delta_{ij} e_k;$$

$$B(e_i, e_j) = \beta_{ij}(e_i \wedge e_j + e_{i*} \wedge e_{j*}),$$

$$B(e_i, e_j) = \bar{\beta}_{ij}(e_i \wedge e_{j*} - e_{i*} \wedge e_j) + \delta_{ij}\left(k_i + \frac{1}{n+2}\,Q\right)J,$$

where

$$\beta_{ij} = \nu_{ij} - 1/[1/(2(n+2))]\,(\mu_i + \mu_j) + [r/(4(n+1)(n+2))],$$

$$\bar{\beta}_{ij} = \bar{\nu}_{ij} - [1/(2(n+2))]\,(\mu_i + \mu_j) + [r/(4(n+1)(n+2))],$$

and

$$k_i = -(c/2) - [r/(2(n+1)(n+2))] + [1/(n+2)]\mu_i,$$

for all $i, j, k \in \{1, \ldots, n\}$.

From [4] we quote the following result.

Lemma 2.1. *Let M^n be a Kaehler hypersurface of $\tilde{M}^{n+1}(c)$. If for all indices $i, j \in \{1, \ldots, n\}$ we have*

$$\lambda_i\lambda_j(\lambda_i^2 - \lambda_j^2) = 0,$$

then there exists a number k in $\{0, \ldots, n\}$ such that

$$\lambda_1 = \ldots = \lambda_k = \lambda \in \mathbf{R}_0^+, \quad \lambda_{k+1} = \ldots = \lambda_n = 0.$$

If $0 < k < n$, then $\beta_{ax} = \bar{\beta}_{ax} \neq 0$ with $1 \leqslant a \leqslant k < x \leqslant n$, and if $0 < k < n$, then $\alpha_{ax} \neq 0$ with $1 \leqslant a \leqslant k < x \leqslant n$.

For a (1, 1) tensor field K on M^n and, for instance, a (1, 3) tensor field T, we define the (1, 3) tensor field $K \,.\, T$ by

$$(K \,.\, T)(X, Y)Z = K(T(X, Y)Z) - T(KX, Y)Z - T(X, KY)Z - T(X, Y)KZ.$$

Similar definitions can be given if T is an arbitrary tensor field. If X and Y are vector fields on M, then $R(X, Y)$ is a (1, 1) tensor field. Thus, for instance, $R(X, Y) \,.\, C$ is well defined, and one can easily prove that

$$R(X, Y) \,.\, C = \nabla_X\nabla_Y C - \nabla_Y\nabla_X C - \nabla_{[X, Y]}C.$$

The condition $R(X, Y) \,.\, C=0$ for all vector fields X and Y will be abbreviated as $R \,.\, C=0$.

For a (1, 3) tensor field T on M^n we define (0, 2) tensor fields $C_{1,3}T$ and $C_{1,4}T$ by

$$(C_{1,3}T)(Y, Z) = \sum_{i=1}^{n} \langle T(E_i, Y)E_i, Z\rangle \quad \text{and}$$

$$(C_{1,4}T)(Y, Z) = \sum_{i=1}^{n} \langle T(E_i, Y)Z, E_i\rangle,$$

where $\{E_i\}_{i=1}^n$ is an orthonormal frame field on M. This definition is independent of the choice of $\{E_i\}_{i=1}^n$. For instance

$$C_{1,3}P = -\frac{n}{n-1}S + \frac{r}{n-1}\langle\, ,\rangle.$$

We have the following lemma ([3]).

Lemma 2.2. *If K is a (1, 1) tensor field and T a (1, 3) tensor field on M^n, then*

(1) $C_{1,4}(K \cdot T) = K \,.\, (C_{1,4}T)$,

(2) $C_{1,3}(K \cdot T) = K \,.\, (C_{1,3}T)$ *if K is antisymmetric,*

(3) $(K \cdot P)(X, Y)Z = (K \cdot R)(X, Y)Z - (1/(n-1))[(K \cdot S)(Z, Y)X - (K \cdot S)(Z, X)Y]$

for all vector fields X, Y and Z.

Note that $R(X, Y)$, $C(X, Y)$ and $B(X, Y)$ are antisymmetric for all vector fields X and Y; but Q and $P(X, Y)$ are in general not antisymmetric. Moreover $P(X, Y)$ is antisymmetric for all X and Y if and only if M^n is Einstein, and Q is antisymmetric if and only if $Q=0$, i.e. if M^n is Ricci flat.

3. **Proof of Theorem 1.** The equivalence (1)⇒(7) is well-known. It is also clear that (1) implies all other statements. So it remains to prove that (2)⇒(7), (3)⇒(4), (4)⇒(7), (5)⇒(7) and (6)⇒(7).

a. (2)⇒(7). Since $(P(e_i, e_{j*}) \,.\, C)(e_i, e_{j*})e_i=0$ and $(P(e_i, e_j) \,.\, C)(e_i, e_j)e_i=0$, it follows for different i and j that

$$(\bar{\nu}_{ij}+\alpha_{ij})(\lambda_j^2-\lambda_i^2)=0, \quad \text{and} \quad (\nu_{ij}+\lambda_{ij})(\lambda_j^2-\lambda_i^2)=0.$$

Adding, respectively subtracting these equations gives

$$\lambda_i\lambda_j(\lambda_j^2-\lambda_i^2)=0 \tag{3.1}$$

and

$$\left(\frac{c}{4}+\alpha_{ij}\right)(\lambda_j^2-\lambda_i^2)=0. \tag{3.2}$$

Also for distinct i and j $(P(e_i, e_j) \cdot C)(e_i, e_{i*})e_i=0$ implies that

$$\nu_{ij}\left(2\bar{\nu}_{ii}-\bar{\nu}_{ij}+\alpha_{ii}-\alpha_{ij}+\frac{c}{2}\right)-\left(\nu_{ij}-\frac{1}{2n-1}\mu_i\right)\left(\frac{c}{2}+\bar{\nu}_{ij}\right)=0. \tag{3.3}$$

Interchanging i and j and substracting, we get, making use also of (3.1), that

$$c(\lambda_j^2-\lambda_i^2)=0. \tag{3.4}$$

(3.2) and (3.4) yield

$$\alpha_{ij}(\lambda_j^2-\lambda_i^2)=0. \tag{3.5}$$

Because (3.1) holds, we can apply Lemma 2.1. If $c=0$ and $0<k<n$, then Lemma 2.1 implies $\alpha_{ar}\neq 0$ with $a\leq k<n$, and (3.5) gives a contradiction. Thus if $c=0$, then $k=0$ (because $k=n$ cannot occur). This implies (7).

If $c\neq 0$, then (3.4) learns that M is Einstein, such that $k=0$ or $k=n$. Both imply with (3.3) that $c=0$, which is a contradiction.

b. (3)⇒(4). Lemma 2.2 (1) says that $P \cdot R=0$ implies that $P \cdot S=0$. Lemma 2.2 (3) therefore yelds that $P \cdot P=0$.

c. (4)⇒(7). $(P(e_i, e_{i*}) \cdot P(e_i, e_j)e_i=0$ yields for $i\neq j$ that

$$(2\bar{\nu}_{ii}-[1/(2n-1)]\mu_i)(\bar{\nu}_{ij}-\bar{\nu}_{ji})=0. \tag{3.6}$$

Interchanging i and j in (3.6) and substracting we obtain

$$\lambda_i\lambda_j(\lambda_i^2-\lambda_j^2)=0. \tag{3.7}$$

$(P(e_i, e_i) \cdot P)(e_i, e_{i*})e_i=0$ implies for $i\neq j$ that

$$\nu_{ii}\left(2\bar{\nu}_{ii}+\frac{c}{2}-\bar{\nu}_{ii}\right)-\left(\nu_{ij}-\frac{1}{2n-1}\mu_i\right)\left(\bar{\nu}_{ij}+\frac{c}{2}\right)=0. \tag{3.8}$$

Interchanging i and j in (3.8) and substracting, we have

$$c(\lambda_i^2-\lambda_j^2)=0. \tag{3.9}$$

Because of (3.7), again Lemma 2.1 can be applied. If $c\neq 0$, then (3.9) implies that M is Einstein.

If $k=n$, then (3.6) implies $c=0$; and if $k=0$, then (3.8) implies also $c=0$. These are contradictions.

Now, suppose $c=0$. Then (3.6) implies that $\lambda_i\lambda_j=0$, such that $k=0$ or $k=1$. If $k=1$, then $(P(e_2, e_{1*}) \cdot P)(e_1, e_{1*})e_1=-\frac{4}{2n-1}\lambda^2 e_2$, which contradicts (4). So we have $k=0$ and $c=0$, i.e. (7).

d. (5)⇒(7). For distinct i and j, $(Q \,.\, P)(e_i, e_{i*})e_j$ implies that $\mu_i c=0$. If $c\neq 0$, then $\mu_i=0$ for all i, thus $\lambda_i^2=\frac{n+1}{4}c$. This contradicts the result of Chern mentioned in Section 1.

Thus we have $c=0$. Then $(Q \,.\, P)(e_i, e_{i*})e_i=0$ implies $\lambda_i=0$ for all i, i.e. (7).

c. (6)⇒(7). Lemma 2.2(2) and $C \cdot P=0$ imply $C \cdot S=0$. $C \cdot P=0$ and $C \cdot S=0$ together with Lemma 2.2(3) imply that $C \cdot R=0$. Theorem 4 of [4] then yields (7).

4. **Proof of theorem 2.** If M is Einstein, then clearly $P \cdot Q=0$. We will prove the converse. From $(P(e_i, e_j) \cdot Q)(e_i)=0$, we find that

$$(\mu_j-\mu_i)\left(\nu_{ij}-\frac{1}{2n-1}\mu_i\right)=0.$$

Interchanging i and j and adding we get $\mu_j=\mu_i$, i.e. $Q=\mu I$.

5. **Proof of theorem 3.** $R \cdot R=0$ and Lemma 2.2(1) imply $R \cdot S=0$. Together with Lemma 2.2(3) these imply $R \cdot P=0$. Conversely, $R \cdot P=0$ and Lemma 2.2(2) imply $R \cdot S=0$. Together with Lemma 2.2(3), then also $R \cdot R=0$.

6. **Proof of theorem 4.** The equivalence (3)⇔(4) is proved in [5]. As in Section 5, we can prove the equivalence between $B \cdot P=0$ and $B \cdot R=0$. Theorem 3 in [4] then implies (4).

We will now prove that (1)⇒(4). From $(P(e_i, e_{j*}) \cdot B)(e_i, e_{j*})e_i=0$ and $(P(e_i, e_j) \,.\, B)(e_i, e_j)e_i=0$, with $i \neq j$, we get

(3.10) $\bar{\beta}_{ij}(\lambda_i^2-\lambda_j^2)=0$ and (3.11) $\beta_{ij}(\lambda_i^2-\lambda_j^2)=0$.

Substracting (3.10) and (3.11) yields

(3.12) $$\lambda_i\lambda_j(\lambda_i^2-\lambda_j^2)=0.$$

(3.12) enables us to apply Lemma 2.1. If $0<k<n$, then Lemma 2.1 implies $\beta_{ax}\neq 0$ with $a \leqslant k < x$, and with (3.10) then $\lambda=0$. This is a contradiction.

Suppose next that $k=n$. Note that in this case $\lambda^2=c/4$. From $(P(e_i, e_{i*}) \cdot B)(e_i, e_j)e_i=0$, with $i \neq j$, we have $c=0$, which is again a contradiction. Therefore, we know that $k=0$, i.e. (4).

REFERENCES

1. Abe, K. — *The complex version of Hartman-Nirenberg cylinder theorem*, J. Diff. Geom. 7 (1972), 453 — 460.
2. Chern, S. S. — *Einstein hypersurfaces in a Kaehlerian manifold of constant holomorphic curvature*, J. Diff. Geom. 1 (1967), 21 — 31.
3. Deprez, J., Dillen, F., Verheyen, P. and Verstraelen, L. — *Conditions on the projective curvature tensor of hypersurfaces in Euclidean space*, to appear.
4. Deprez, J., Verheyen, P. and Verstraelen, L. — *Intrinsic characterizations for complex hypercylinders and complex hyperspheres*, Geometriae Dedicata 16 (1984), 217 — 229.
5. Kon, M. — *Kaehler immersions with vanishing Bochner curvature tensors*, Kodai Math. Sem. Rep. 27 (1976), 329 — 333.
6. Ryan, P. J. — *A class of complex hypersurfaces*, Colloq. Math. XXVI (1972), 175 — 182.

Received 10 I 1986

Katolieke Universiteit Leuven
Faculteit der Wetenschappen
Departement Wiskunde
Celestijnenlaan 200 B
B — 3 030 Leuven (Heverlee)
Belgie

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXIII, s. I a. Matematică, 1987, f. 3

A PSEUDO-RIEMANNIAN STRUCTURE IN LAGRANGE GEOMETRY

BY

V. OPROIU

Dedicated to Professor Gh. Gheorghiev on the occasion of his 80th birthday

In last years many efforts have been made to give a clear geometric meaning to the Finsler geometry. It seems that the natural frame is the differential geometry of the tangent bundle. In [9], the author has interpreted a Finsler connection as a linear connection on the vertical bundle over the tangent bundle. Next, using a nonlinear connection, he extended such a connection in the second order tangent bundle. Then the usual torsion and curvature tensor fields of the Finsler geometry are components of the torsion and curvature tensor fields of this extended connection. However the extension depends on the chosen nonlinear connection and the author was interested in the cases where there are some more or less natural possibilities to derive such a nonlinear connection from the given Finsler connection. Other aspects were discussed in [6, 7, 12, 14, 15].

In the case of the Lagrange manifolds (i. e. those manifolds carrying a regular Lagrangean defined on the tangent bundle) such a nonlinear connection is defined naturally by the Euler-Lagrange equations associated with the given Lagrangean. It appears that this nonlinear connection has some remarkable properties concerning the differential geometry of the tangent bundle. In [10] the author has considered a Riemannian metric defined on the tangent bundle of a Lagrange manifold similar to that derived by Sasaki from a Riemannian metric on the base manifold. Then the Levi Civita connection of this Riemannian metric defined on the tangent bundle induces on the vertical and horizontal distributions some linear connections which are closely related to the Cartan connection used in the differential geometry of the Finsler spaces. Thus, in the case of Lagrange manifolds, many properties from the Finsler geometry may be derived from the Riemannian geometry of the tangent bundle.

Recently, several physical applications of the Finsler geometry have been given in connection with Yukawa's bilocal theory, particle physics, high-energy cosmic ray phenomena, the generalized Einstein equations in relativity theory, etc. [3, 7, 13]. The natural symplectic structure defined on the tangent bundle of a Lagrangean manifold by the nonlinear connection determined by the Euler-Lagrange equations may lead to further applications in theoretical physics e.g. geometric quantization, geometric gauge theory, etc.

In this paper we study some aspects of the differential geometry of the tangent bundle of a Lagrange manifold when this tangent bundle is