c. (6)$\Rightarrow$(7). Lemma 2.2(2) and $C \cdot P=0$ imply $C \cdot S=0$. $C \cdot P=0$ and $C \cdot S=0$ together with Lemma 2.2(3) imply that $C \cdot R=0$. Theorem 4 of [4] then yields (7).

4. **Proof of theorem 2.** If M is Einstein, then clearly $P \cdot Q=0$. We will prove the converse. From $(P(e_i, e_j) \cdot Q)(e_i)=0$, we find that

$$(\mu_j-\mu_i)\left(\nu_{ij}-\frac{1}{2n-1}\mu_i\right)=0.$$

Interchanging i and j and adding we get $\mu_j=\mu_i$, i.e. $Q=\mu I$.

5. **Proof of theorem 3.** $R \cdot R=0$ and Lemma 2.2(1) imply $R \cdot S=0$. Together with Lemma 2.2(3) these imply $R \cdot P=0$. Conversely, $R \cdot P=0$ and Lemma 2.2(2) imply $R \cdot S=0$. Together with Lemma 2.2(3), then also $R \cdot R=0$.

6. **Proof of theorem 4.** The equivalence (3)$\Leftrightarrow$(4) is proved in [5]. As in Section 5, we can prove the equivalence between $B \cdot P=0$ and $B \cdot R=0$. Theorem 3 in [4] then implies (4).

We will now prove that (1)$\Rightarrow$(4). From $(P(e_i, e_{j*}) \cdot B)(e_i, e_{j*})e_i=0$ and $(P(e_i, e_j) \cdot B)(e_i, e_j)e_i=0$, with $i\neq j$, we get

$$\bar{\beta}_{ij}(\lambda_i^2-\lambda_j^2)=0 \quad (3.10) \qquad \text{and} \qquad \beta_{ij}(\lambda_i^2-\lambda_j^2)=0. \quad (3.11)$$

Substracting (3.10) and (3.11) yields

$$\lambda_i\lambda_j(\lambda_i^2-\lambda_j^2)=0. \tag{3.12}$$

(3.12) enables us to apply Lemma 2.1. If $0<k<n$, then Lemma 2.1 implies $\beta_{ax}\neq 0$ with $a\leqslant k<x$, and with (3.10) then $\lambda=0$. This is a contradiction.

Suppose next that $k=n$. Note that in this case $\lambda^2=c/4$. From $(P(e_i, e_{i*}) \cdot B)(e_i, e_j)e_i=0$, with $i\neq j$, we have $c=0$, which is again a contradiction. Therefore, we know that $k=0$, i.e. (4).

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A PSEUDO-RIEMANNIAN STRUCTURE IN LAGRANGE GEOMETRY

BY

V. OPROIU

Dedicated to Professor Gh. Gheorghiev on the occasion of his 80th birthday

In last years many efforts have been made to give a clear geometric meaning to the Finsler geometry. It seems that the natural frame is the differential geometry of the tangent bundle. In [9], the author has interpreted a Finsler connection as a linear connection on the vertical bundle over the tangent bundle. Next, using a nonlinear connection, he extended such a connection in the second order tangent bundle. Then the usual torsion and curvature tensor fields of the Finsler geometry are components of the torsion and curvature tensor fields of this extended connection. However the extension depends on the chosen nonlinear connection and the author was interested in the cases where there are some more or less natural possibilities to derive such a nonlinear connection from the given Finsler connection. Other aspects were discussed in [6, 7, 12, 14, 15].

In the case of the Lagrange manifolds (i. e. those manifolds carrying a regular Lagrangean defined on the tangent bundle) such a nonlinear connection is defined naturally by the Euler-Lagrange equations associated with the given Lagrangean. It appears that this nonlinear connection has some remarkable properties concerning the differential geometry of the tangent bundle. In [10] the author has considered a Riemannian metric defined on the tangent bundle of a Lagrange manifold similar to that derived by Sasaki from a Riemannian metric on the base manifold. Then the Levi Civita connection of this Riemannian metric defined on the tangent bundle induces on the vertical and horizontal distributions some linear connections which are closely related to the Cartan connection used in the differential geometry of the Finsler spaces. Thus, in the case of Lagrange manifolds, many properties from the Finsler geometry may be derived from the Riemannian geometry of the tangent bundle.

Recently, several physical applications of the Finsler geometry have been given in connection with Yukawa's bilocal theory, particle physics, high-energy cosmic ray phenomena, the generalized Einstein equations in relativity theory, etc. [3, 7, 13]. The natural symplectic structure defined on the tangent bundle of a Lagrangean manifold by the nonlinear connection determined by the Euler-Lagrange equations may lead to further applications in theoretical physics e.g. geometric quantization, geometric gauge theory, etc.

In this paper we study some aspects of the differential geometry of the tangent bundle of a Lagrange manifold when this tangent bundle is endowed with a pseudo-Riemannian metric obtained from the fundamental tensor field by a method similar to the obtaining of the complete lift of a pseudo-Riemannian metric on a differentiable manifold. A remarkable property of the nonlinear connection defined by the Euler-Lagrange equations is that this „complete lift" does coincide with the „horizontal lift" (see [8]). The Levi Civita connection associated with this pseudo-Riemannian metric is obtained, next the local coordinate expressions of its curvature tensor field are written down. Finally we get the corresponding Bianchi identities. Remark that some results of this paper are obtained firstly in the case of the so called generalized Lagrange manifolds (see [7]), next, the corresponding simpler results are given in the case of the Lagrange manifolds.

Functions, vector fields, tensor fields, and geometric objects we consider in this paper, are assumed to be differentiable of class C^∞. Throughout this paper $X, Y, Z, W,\ldots$ denote arbitrary vector fields over the tangent bundle TM of the manifold M and $\chi(TM)$ denotes the $\mathcal{F}(TM)$-module of vector fields over TM where $\mathcal{F}(TM)$ is the ring of differentiable functions over TM.

§1. **Lagrange manifolds.** Let M be a smooth n-dimensional manifold and denote by TM its tangent bundle. It is well known that TM can be endowed with a smooth manifolds structure obtained from that of smooth vector bundle structure of TM. By this, the natural projection $\tau: TM\to M$ becomes a smooth submersion and the coordinate transformations on TM preserve the vector space structure on the fibres of TM. If $(U, x^1,\ldots,x^n)$ is a local chart on M then the local chart $(\tau^{-1}(U), x^1\circ\tau,\ldots,x^n\circ\tau, y^1,\ldots,y^n)$ is defined on TM where $y^1,\ldots,y^n$ are the vector space coordinates with respect to the natural local frame $(\partial/\partial x^1,\ldots,\partial/\partial x^n)$ defined by the local chart $(U, x^1,\ldots, x^n)$. For convenience we should delete τ from the expressions $x^i\circ\tau$, so that the local coordinates on $\tau^{-1}(U)$ are written simply x^i, y^i; $i=1,\ldots,n$. For a coordinate transformation $\tilde{x}^i=\tilde{x}^i(x^1,\ldots,x^n)$ on M, the corresponding coordinate transformation on TM is

$$\tilde{x}^i=\tilde{x}^i(x^1,\ldots,x^n),\qquad \tilde{y}^i=\frac{\partial\tilde{x}^i}{\partial x^k}y^k. \tag{1}$$

The change of the natural frames on TM is given by

$$\frac{\partial}{\partial x^i}=\frac{\partial\tilde{x}^k}{\partial x^i}\frac{\partial}{\partial\tilde{x}^k}+\frac{\partial^2\tilde{x}^k}{\partial x^i\partial x^h}y^h\frac{\partial}{\partial\tilde{y}^k},\qquad \frac{\partial}{\partial y^i}=\frac{\partial\tilde{x}^k}{\partial x^i}\frac{\partial}{\partial\tilde{y}^k}. \tag{2}$$

Definition. *A regular Lagrangean on M is a smooth real valued function $L: TM\to\mathbf{R}$, defined on the tangent bundle TM such that the matrix with the entries*

$$g_{ik}=\frac{\partial^2L}{\partial y^i\partial y^k} \tag{3}$$

is everywhere nondegenerate.

From the second formula (2) it follows that the change rule for g_{ik} is that of a second order covariant tensor on M. Thus the nondegeneracy conditions is well defined.

Definition. *A Lagrange manifold is a smooth manifold M carrying a regular Lagrangean.*

Recall that in the case of Lagrange manifolds, for every smooth path in M, $\gamma: J\to M$, where $J=[a, b]\subset\mathbf{R}$ is a closed interval in $\mathbf{R}$, the action integral is defined by :

$$S[\gamma]=\int_a^b L\left(\gamma(t),\frac{d\gamma}{dt}\right)dt,$$

where $d\gamma/dt$ denotes the vector tangent to γ in $\gamma(t)$. The Euler-Lagrange equations satisfied by the critical paths (with fixed endpoints) of the considered action are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}^i}\right)-\frac{\partial L}{\partial x^i}=0,$$

where $x^i=x^i(t)$ is the local coordinate expression of γ and $\dot{x}^i=dx^i/dt$. Using the composite function differentiation, the Euler Lagrange equations become

$$g_{ik}\frac{d^2x^k}{dt^2}+\frac{\partial^2L}{\partial x^k\partial\dot{x}^i}\frac{dx^k}{dt}-\frac{\partial L}{\partial x^i}=0.$$

Next, transvecting by g^{ih}, the entries of the inverse of the nondegenerate matrix $[g_{ik}]$, we get

$$\frac{d^2x^h}{dt}+G^h\left(x^i,\frac{dx^i}{dt}\right)=0, \tag{4}$$

where

$$G^h(x^j, y^j)=g^{hi}\left(\frac{\partial^2L}{\partial x^k\partial y^i}y^k-\frac{\partial L}{\partial x^i}\right),\qquad h=1,\ldots,n. \tag{4'}$$

Following the general theory from the Finsler geometry, we get a nonlinear connection defined by :

$$N^h_k=\frac{1}{2}\frac{\partial G^h}{\partial y^k}. \tag{5}$$

Remark. The same nonlinear connection was obtained also in [3]. Thus, in the case of the Lagrange manifolds, we have defined a horizontal vector subbundle in $HTM\subset TTM$, complementary to the vertical vector subbundle VTM — the kernel of the differential of τ. Recall that a local frame $(\delta/\delta x^1,\ldots,\delta/\delta x^n)$ in HTM is defined by

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-N^k_i\frac{\partial}{\partial y^k}. \tag{6}$$

Next $(\partial/\partial y^1,\ldots,\partial/\partial y^n)$ is a local frame in VTM and we have the transformation formulae

$$\frac{\delta}{\delta x^i}=\frac{\partial\tilde{x}^k}{\partial x^i}\frac{\delta}{\delta\tilde{x}^k},\qquad \frac{\partial}{\partial y^i}=\frac{\partial\tilde{x}^k}{\partial x^i}\frac{\partial}{\partial\tilde{y}^k}, \tag{7}$$

The generalized Lagrange manifolds are defined as follows. Suppose that $g_{ik}(x, y)$, $i, k=1,\dots,n$ define a nondegenerate Finsler second order covariant tensor field on TM, i.e. the components g_{ik} change like the components of a tensor on M but they are not necessarily defined as in (3). Next suppose N^i_k define a nonlinear connection on TM but N^i_k are not necessarily obtained as in (5). A manifold M whose tangent bundle TM is endowed with Finsler tensor field g_{ik} and a nonlinear connection as above, is called a *generalized Lagrange manifold*. The geometry of the generalized Lagrange manifolds has been intensively studied in last years, see [7]. For a generalized Lagrange manifold we may define the vector fields (6) and the transformation formulae (7) remain still valid.

We have

$$TTM = HTM \oplus VTM$$

and $(\delta/\delta x^1,\dots,\delta/\delta x^n, \partial/\partial y^1,\dots,\partial/\partial y^n)$ is a local frame of TTM adapted to this direct sum decomposition. The local coframe of T^*TM, dual to this local frame is $(dx^1,\dots,dx^n, \delta y^1,\dots,\delta y^n)$ where

(6′)
$$\delta y^i = dy^i + N^i_k dx^k, \quad i=1,\dots,n.$$

According to the Frobenius theorem the complete integrability of the differential system defined by the vector subbundle HTM depends on the vanishing of the brackets $[\delta/\delta x^i, \delta/\delta x^k]$ modulo $\delta/\delta x^1,\dots,\delta/\delta x^n$. We have

(8)
$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^k}\right] = -R^h_{ik}\frac{\partial}{\partial y^h},$$

where

(9)
$$R^h_{ik} = \frac{\delta N^h_k}{\delta x^i} - \frac{\delta N^h_i}{\delta x^k} = \frac{\partial N^h_k}{\partial x^i} - \frac{\partial N^h_i}{\partial x^k} - N^j_i\frac{\partial N^h_k}{\partial y^j} + \frac{\partial N^h_i}{\partial y^j}N^j_k.$$

The tensor field R^h_{ik} measuring the nonholonomy of HTM corresponds to the torsion tensor field of a nonholonomic space as defined by G. V r ă n c e a n u or to the C h e r n - B e r n a r d invariant of the G-structure defined by a differential system. The tensor field R^h_{ik} will be called the torsion tensor field of the given nonlinear connection.

Remark. The exterior differential 2-forms obtained from (6′) by exterior differentiation are:

(8′)
$$d\delta y^i = \frac{1}{2} R^i_{kh} dx^k \wedge dx^h + \frac{\partial N^i_h}{\partial y^k}\delta y^k \wedge dx^h.$$

From the Jacobi identity or, equivalently, by taking the exterior differential in (8′) we get the following identity

(9)
$$\sum_{(i,j,k)} \left\{\frac{\delta}{\delta x^k} R^h_{ij} + R^l_{ij}\frac{\partial}{\partial y^k} N^h_l\right\} = 0,$$

where $\sum_{(i,j,k)}$ denotes the sum obtained by cyclic permutations of i, j, k.

In the case of the Lagrange manifolds the tensor field R^h_{ij} has some special properties obtained by considering the 1-form:

(10)
$$\theta = \frac{\partial L}{\partial y^i} dx^i.$$

Proposition 1. *The exterior differential of the 1-form* θ *is*

(11)
$$d\theta = g_{ik}\delta y^i \wedge dx^k.$$

Proof. We have

$$d\theta = \left(g_{ik}\delta y^i + \frac{\delta}{\delta x^i}\frac{\partial L}{\partial y^k} dx^i\right) \wedge dx^k.$$

Next, verify by a straightforward computation, using (5) and (6)

(12)
$$\frac{\delta}{\delta x^i}\frac{\partial L}{\partial y^k} - \frac{\delta}{\delta x^k}\frac{\partial L}{\partial y^i} = 0,$$

obtaining the expression (11) for $d\theta$.

Remark. It follows that the tangent bundle of a Lagrange manifold has a symplectic structure related to the given Lagrangean, defined by the exact, hence closed, 2-form $d\theta$ whose rank equals the dimension $2n$ of TM. In fact this symplectic structure does correspond to the canonical symplectic structure defined on the cotangent bundle T^*M through the inverse of the Legendre transformation associated with the Lagrangean L.

Proposition 2. *For a Lagrange manifold we have the following identities*

(13)
$$\sum_{(i,j,k)} g_{ih}R^h_{jk} = 0$$

(14)
$$\frac{\delta g_{ih}}{\delta x^k} - \frac{\delta g_{kh}}{\delta x^i} - g_{ij}\frac{\partial N^j_k}{\partial y^h} + g_{kj}\frac{\partial N^j_i}{\partial y^h} = 0.$$

Proof. Consider the exterior differential in (11) and use (8′)

$$0 = d^2\theta = \left(\frac{\delta g_{ik}}{\delta x^j}dx^j + \frac{\partial g_{ik}}{\partial y^j}\delta y^j\right) \wedge \delta y^i \wedge dx^k + g_{hk}\left(\frac{1}{2}R^h_{ij}dx^i \wedge dx^j + \right.$$
$$\left. + \frac{\partial N^h_j}{\partial y^i}\delta y^i \wedge dx^j\right) \wedge dx^k.$$

Next use the skew-symmetry of the exterior product and the property

$$\frac{\partial g_{ik}}{\partial y^j} = \frac{\partial g_{ij}}{\partial y^k},$$

valid when g_{ik} is defined by (3).

Remark. In the case of the Lagrange manifolds the nonlinear connection defined by (5) has the property

(15)
$$\frac{\partial N^h_i}{\partial y^k} = \frac{\partial N^h_k}{\partial y^i},$$

obtained from the definition.

Now, return to the case of the generalized Lagrange manifolds and remark that the transformation matrix involved in the formulae (7) is the same as the transformation matrix for the natural frames on TM. It follows that for every element $y \in T_pM$ an isomorphism between T_pM and V_yTM (between T_pM and H_yTM) is defined naturally. The vector z_y^v from V_yTM corresponding to $z_p = z^i \left(\frac{\partial}{\partial x^i}\right)_p \in T_pM$ is called the *vertical lift* of z_p and has the local expression

$$z_y^v = z^i \left(\frac{\partial}{\partial y^i}\right)_y.$$

Similarly, the vector z_y^h from H_yTM corresponding to $z_p \in T_pM$ is called the *horizontal lift* and has the local expression

$$z_y^h = z^i \left(\frac{\delta}{\delta x^i}\right)_y.$$

The vertical and horizontal lifts of vector fields on M are defined pointwise and are vector fields on TM.

Using the isomorphism between HTM and VTM obtained via the isomorphism of their fibres with the corresponding tangent spaces to M, we may define an almost complex structure F on TM by:

$$F(z_y^h) = z_y^v, \quad F(z_y^v) = -z_y^h, \quad y, z_p \in T_pM, \ p \in M.$$

Thus

$$F\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial y^i}, \qquad F\left(\frac{\partial}{\partial y^i}\right) = -\frac{\delta}{\delta x^i}.$$

Suppose that the quadratic form $g_{ik}(x, y)dx^i dx^k$ is positive definite. Then we may consider the Sasaki metric

$$G = g_{ik} dx^i dx^k + g_{ik} \delta y^i \delta y^k$$

and we have

$$G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^k}\right) = G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^k}\right) = g_{ik}, \quad G\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^k}\right) = G\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^k}\right) = 0,$$

thus HTM, VTM are orthogonal each other with respect to G and the scalar products induced from G on HTM and VTM are corresponding each other by the natural isomorphism between HTM and VTM.

Theorem 3. *The Sasaki metric G and the almost complex structure F, both defined on the tangent bundle of a generalized Lagrange manifold define on TM an almost Hermitian structure. If M is a Lagrange manifold this structure is almost Kählerian.*

Proof. It is a straightforward verification that

$$G(FX, FY) = G(X, Y), \ X, Y \in \chi(TM),$$

by taking for X, Y the local vector fields $\delta/\delta x^i$, $\partial/\partial y^k$. Thus TM is almost Hermitian. Then the associated 2-form Φ is defined by

$$\Phi(X, Y) = G(X, FY).$$

If M is a Lagrange manifold we have

$$\Phi\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^k}\right) = \Phi\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^k}\right) = 0, \qquad \Phi\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^k}\right) = -\Phi\left(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^i}\right) = g_{ik}$$

and comparing with (11) we get

$$\Phi = d\theta.$$

Thus the almost Hermitian structure is almost Kählerian.

§2. **The pseudo-Riemannian metric on** TM. In the differential geometry of the tangent bundle the complete lift is obtained as a natural mapping sending fields of geometric objects (usually tensor fields or connections) defined on the base manifold in fields of geometric objects of the same type defined on the tangent bundle. We have

$$f^c = \frac{\partial f}{\partial x^i} y^i, \qquad \left(\frac{\partial}{\partial x^i}\right)^c = \frac{\partial}{\partial x^i}, \qquad (dx^i)^c = dy^i,$$

where f is a function defined on the base manifold and the superscript c denotes just the complete lift. Next for S, T two tensor fields on the base manifold, we have

$$(S \otimes T)^c = S^c \otimes T^v + S^v \otimes T^c,$$

where S^v, T^v are the vertical lifts. Remark that the vertical lift for vector fields was defined in §1, thus $(\partial/\partial x^i)^v = (\partial/\partial y^i)$. Next the vertical lift is defined by $f^v = f \circ \tau$, $(dx^i)^v = dx^i$ and $(S \otimes T)^v = S^v \otimes T^v$.

Remark. We have

$$(x^i)^c = y^i.$$

As an example, the complete lift of a (pseudo-) Riemannian metric $g = g_{ik} dx^i dx^k$ defined on M is

$$g^c = \frac{\partial g_{ik}}{\partial x^h} y^h dx^i dx^k + 2 g_{ik} dy^i dx^k$$

and we can prove that it defines a pseudo-Riemannian metric on TM.

Now, let like in the Finsler geometry

$$g = g_{ik}(x, y) dx^i dx^k \tag{16}$$

be a nondegenerate symmetric 2-form, where the coefficients depend on the tangent vectors to M, i.e. M. is a generalized Lagrange manifold. In fact g can be given as a symmetric section in the vector bundle $V^\circ TM \otimes V^\circ TM$ over TM where $V^\circ TM \subset T^*TM$ is the vanishing vector bundle for VTM:

$$\omega \in V_y^0 TM \subset T_y^* TM \Leftrightarrow \omega(X) = 0, \quad \forall X \in V_y TM, \quad y \in TM.$$

We should like to obtain something like the complete lift for g considered in (16). We see that we know the complete lifts of the 1-forms dx^i and the vector fields $\partial/\partial x^i$ but we do not know the „complete lift" of a function $f(x, y)$. For $y \in TM$ let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth path in M such that

$\gamma(0)=\tau(y)$, $d\gamma/dt(0)=y$. Let us write locally

$$\frac{d\gamma}{dt}=\frac{dx^i}{dt}\frac{\partial}{\partial x^i}.$$

Since $dx^i/dt(0)=y^i$ then d^2x^i/dt^2 should correspond to the complete lifts of the coordinates y^i. In the case of the Lagrange manifolds we have a family of paths as above, namely the solutions of the Euler-Lagrange equations satisfying the corresponding initial conditions. Thus it is quite natural to define

$$(y^i)^c=-G^i(x,y),$$

and :

$$(f(x,y))^c=\frac{\partial f}{\partial x^i}y^i+\frac{\partial f}{\partial y^i}(-G^i(x,y)).$$

We call such a function the „complete lift“ of the function $f: TM\to\mathbf{R}$.

We can apply te above results to the expression (16)

$$g^c=\left(\frac{\partial g_{ik}}{\partial x^h}y^h-\frac{\partial g_{ik}}{\partial y^h}G^h\right)dx^idx^k+2g_{ik}dy^idx^k. \tag{17}$$

Proposition 4. *Let M be a Lagrange manifold. Then the pseudo-Riemannian metric g^c on TM defined by (17) has the following expression*

$$g^c=2g_{ik}\delta y^idx^k. \tag{17'}$$

Proof. Replace in (17) dy^i by

$$dy^i=\delta y^i-\frac{1}{2}\frac{\partial G^i}{\partial y^k}dx^k;$$

next use the formula (4′) and the symmetry properties of the derivatives of g_{ik} (see the proof of the proposition 3)

$$g^c=2g_{ik}\delta y^idx^k+\left[\frac{\partial g_{ik}}{\partial x^h}y^h-\frac{\partial}{\partial y^k}\left(\frac{\partial^2L}{\partial y^i\partial x^h}y^h-\frac{\partial L}{\partial x^i}\right)\right]dx^idx^k=$$

$$=2g_{ik}\delta y^idx^k+\left(\frac{\partial^2L}{\partial x^k\partial y^i}-\frac{\partial^2L}{\partial y^k\partial x^i}\right)dx^idx^k.$$

Next remark that the last term vanishes due to the symmetry of dx^idx^k.

Remark. If instead of the „complete lift“ we use the „horizontal lift“ with respect to the nonlinear connection (5) we obtain the same expression (17′). It seems that this is a remarkable property of the nonlinear connection defined in (5).

Remark. Recently Y. Ichijyo [2], has defined the complete lift of the function f on a generalized Lagrange manifold by taking

$$f^c=\left(\frac{\partial f}{\partial x^i}-N_i^k\frac{\partial f}{\partial y^k}\right)y^i$$

and studied the properties of some G-structures induced on TM from Finsler G-structures.

Remark. Since the Lagrangean L is regular, it follows that g^c is nondegenerate on TM and its signature is (n, n).

Let us write the values of g^c for the local vector fields of the local frame adapted to the direct sum decomposition $TTM=HTM\oplus VTM$.

$$g^c\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k}\right)=g^c\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^k}\right)=0,$$
$$g^c\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}\right)=g^c\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^k}\right)=g_{ik}. \tag{18}$$

Thus the distributions HTM, VTM are isotropic for the pseudo-Riemannian metric g^c.

Consider the following almost product structure P defined on TM by the distributions HTM, VTM, see §1 :

$$P(z_y^h)=z_y^h,\ P(z_y^v)=-z_y^v,\ y, z_p\in T_pM,\ p\in M.$$

Then we may verify by a straightforward computation that g^c and P define a hyperbolic almost Hermitian structure on TM, i.e.

$$g^c(PX, PY)=-g^c(X, Y),\ X, Y\in\chi(TM).$$

Then the 2-form Φ associated with this structure is given by

$$\Phi(X, Y)=g^c(X, PY).$$

Just like in the proof of the theorem 4 we may show that

$$\Phi=d\theta.$$

Thus we have a hyperbolic almost Kählerian structure defined on the tangent bundle TM.

Denote by ∇ the Levi Civita connection of the pseudo-Riemannian metric g^c defined on the tangent bundle TM of a generalized Lagrange manifold. We shall obtain the local coordinate expression of this connection. Denote, for convenience, $n+i=\hat{i}$, $y^i=x^{n+i}=x^{\hat{i}}$ whenever $i\in\{1,\ldots,n\}$. Next

$$\nabla_{\frac{\partial}{\partial y^i}}=\nabla_{\hat{i}},\quad \nabla_{\frac{\delta}{\delta x^i}}=\nabla_i.$$

Write

$$\nabla_{\hat{i}}\frac{\partial}{\partial y^j}=D_{ij}^k\frac{\partial}{\partial y^k}+b_{ij}^k\frac{\delta}{\delta x^k},\quad \nabla_{\hat{i}}\frac{\delta}{\delta x^j}=E_{ij}^k\frac{\delta}{\delta x^k}+P_{ij}^k\frac{\partial}{\partial y^k},$$

$$\nabla_i\frac{\partial}{\partial y^j}=\Gamma_{ij}^k\frac{\partial}{\partial y^k}+\tilde{E}_{ij}^k\frac{\delta}{\delta x^k},\quad \nabla_i\frac{\delta}{\delta x^j}=\Phi_{ij}^k\frac{\delta}{\delta x^k}+a_{ij}^k\frac{\partial}{\partial y^k},$$

for the general local coordinate expression of ∇. To obtain the expression of the coefficients involved, we put the condition that ∇ be torsion free and

$$\nabla g^c=0. \tag{19}$$

The condition for ∇ to be torsion free implies the following relations:

$$(20)\qquad \begin{aligned} &D^k_{ji}=D^k_{ij}, \quad b^k_{ji}=b^k_{ij}, \quad \tilde{E}^k_{ji}=E^k_{ij},\\ &P^k_{ij}-\Gamma^k_{ji}+\frac{\partial N^k_j}{\partial y^i}=0, \quad \Phi^k_{ji}=\Phi^k_{ij}, \quad a^k_{ij}-a^k_{ji}=-R^k_{ij}. \end{aligned}$$

Next, from (19) we get

$$(21)\qquad \begin{aligned} &g_{hj}b^h_{ki}+g_{hi}b^h_{kj}=0, && g_{hj}E^h_{ik}+g_{ih}E^h_{jk}=0,\\ &g_{hj}P^h_{ki}+g_{hi}P^h_{kj}=0, && g_{hj}a^h_{ki}+g_{ih}a^h_{kj}=0,\\ &g_{hj}E^h_{ki}+g_{ih}D^h_{kj}=\frac{\partial g_{ij}}{\partial y^k}, && g_{hj}\Phi^h_{ki}+g_{ih}\Gamma^h_{kj}=\frac{\delta g_{ij}}{\delta x^k}. \end{aligned}$$

From the systems (20), (21) we get by a straightforward computation the unique solution:

$$(22)\qquad \begin{aligned} &D^i_{jk}=\frac{1}{2}g^{ih}\left(\frac{\partial g_{hj}}{\partial y^k}+\frac{\partial g_{hk}}{\partial y^j}\right), \quad b^i_{jk}=0, \quad E^i_{jk}=\frac{1}{2}g^{ih}\left(\frac{\partial g_{kh}}{\partial y^j}-\frac{\partial g_{jk}}{\partial y^h}\right),\\ &P^i_{jk}=\frac{1}{2}g^{ih}\left(\frac{\delta g_{hj}}{\delta x^k}-\frac{\delta g_{jk}}{\delta x^h}+g_{kl}\frac{\partial N^l_h}{\partial y^j}-\frac{\partial N^l_k}{\partial y^j}g_{hl}\right),\\ &\Gamma^i_{jk}=\frac{1}{2}g^{ih}\left(\frac{\delta g_{hk}}{\delta x^j}-\frac{\delta g_{jk}}{\delta x^h}+g_{jl}\frac{\partial N^l_h}{\partial y^k}+g_{hl}\frac{\partial N^l_j}{\partial y^k}\right),\\ &\Phi^i_{jk}=\frac{1}{2}g^{ih}\left(\frac{\delta g_{hk}}{\delta x^j}+\frac{\delta g_{jh}}{\delta x^k}-g_{jl}\frac{\partial N^l_k}{\partial y^h}-g_{kl}\frac{\partial N^l_j}{\partial y^h}\right),\\ &a^i_{jk}=\frac{1}{2}g^{ih}(R_{jkh}-R_{hjk}-R_{khj}), \end{aligned}$$

where

$$R_{hjk}=g_{hi}R^i_{jk}.$$

In the case of the Lagrange manifolds the solution (22) may be written in a more simple form by using the identities from §1.

Theorem 5. *Let M be a Lagrange manifold. Then the local coordinate expression of the Levi-Civita connection of the pseudo-Riemannian metric g^c on TM is given by*

$$(23)\qquad \begin{aligned} &\nabla_{\hat{i}}\frac{\partial}{\partial y^j}=D^k_{ij}\frac{\partial}{\partial y^k}, && \nabla_{\hat{i}}\frac{\delta}{\delta x^j}=0,\\ &\nabla_i\frac{\partial}{\partial y^j}=\Gamma^k_{ij}\frac{\partial}{\partial y^k}, && \nabla_i\frac{\delta}{\delta x^j}=\Phi^k_{ij}\frac{\delta}{\delta x^k}+g^{kh}R_{ijh}\frac{\partial}{\partial y^k}, \end{aligned}$$

where

$$(24)\qquad D^k_{ij}=g^{kh}\frac{\partial g_{hj}}{\partial y^i}, \quad \Gamma^k_{ij}=\frac{\partial N^k_j}{\partial y^i}, \quad \Phi^k_{ij}=g^{kh}\left(\frac{\delta g_{hj}}{\delta x^i}-g_{jl}\frac{\partial N^l_i}{\partial y^h}\right).$$

Remark. The Levi Civita connection of g^c leaves invariant the vertical vector subbundle VTM. However HTM is not invariant under ∇ and the term $g^{ih}R_{ijh}$ is closely related to the nonholonomy of the horizontal vector subbundle HTM. This is quite natural since ∇ is torsion free and preserving VTM and HTM by ∇ would imply that these vector subbundles are integrable.

From the formulae (23) we obtain the covariant derivatives of the 1-forms $(dx^i, \delta y^i)$ defining the local coframe dual to the adapted local frame $(\delta/\delta x^i, \partial/\partial y^i)$,

$$(23')\qquad \begin{aligned} &\nabla_{\hat{i}}dx^k=0, \quad \nabla_i dx^k=-\Phi^k_{ij}dx^j, \quad \nabla_{\hat{i}}\delta y^k=-D^k_{ij}\delta y^j\\ &\nabla_i\delta y^k=-\Gamma^k_{ij}\delta y^j-R_{ijh}g^{hk}dx^j. \end{aligned}$$

Finally we should remark some interesting formulae concerning the condition $\nabla g^c=0$. Since $\nabla_{\hat{k}}g_{ij}=0$ we get:

$$(25)\qquad g_{ih}D^h_{kj}=g_{jh}D^h_{ki}=\frac{\partial g_{ij}}{\partial y^k}.$$

Next, since $\nabla_k g_{ij}=0$ we get

$$(26')\qquad g_{ih}\Gamma^h_{kj}+g_{hj}\Phi^h_{ki}=g_{ih}\Phi^h_{kj}+g_{hj}\Gamma^h_{ki}=\frac{\delta g_{ij}}{\delta x^k}.$$

Similarly, since $\nabla_{\hat{k}}g^{ij}=0$ we get

$$(25')\qquad g^{ih}D^j_{kh}=g^{hj}D^i_{kh}=-\frac{\partial g^{ij}}{\partial y^k},$$

and, since $\nabla_k g^{ij}=0$ we get

$$(26')\qquad g^{ih}\Gamma^j_{kh}+g^{hj}\Phi^i_{kh}=g^{ih}\Phi^j_{kh}+g^{hj}\Gamma^i_{kh}=-\frac{\delta g^{ij}}{\delta x^k}.$$

§3. **The curvature tensor field.** Denote by K the curvature tensor field of the Levi Civita connection ∇ of the pseudo-Riemannian metric g^c on TM where M is Lagrange.

$$K(X, Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z, \quad X, Y, Z\in\chi(TM).$$

Now we get the local coordinate expressions of K by taking for X, Y, Z, the local vector fields of the adapted local frame $(\delta/\delta x^i, \partial/\partial y^i)$. Recall the convention from §2 to denote $y^i=x^{n+i}=x^{\hat{i}}$, thus $\hat{i}=n+i$. Next, taking $X=\partial/\partial y^j$, $Y=\partial/\partial y^k$, $Z=\partial/\partial y^i$ we get:

$$\begin{aligned} K^{\hat{h}}_{\hat{i}\hat{j}\hat{k}}\frac{\partial}{\partial y^h}+K^h_{\hat{i}\hat{j}\hat{k}}\frac{\delta}{\delta x^h}&=\nabla_{\hat{j}}\nabla_{\hat{k}}\frac{\partial}{\partial y^i}-\nabla_{\hat{k}}\nabla_{\hat{j}}\frac{\partial}{\partial y^i}=\\ &=\left(\frac{\partial D^h_{ki}}{\partial y^j}-\frac{\partial D^h_{ji}}{\partial y^k}+D^h_{jl}D^l_{ki}-D^h_{kl}D^l_{ji}\right)\frac{\partial}{\partial y^h}. \end{aligned}$$

On the other hand we get by a direct computation

$$\frac{\partial D^h_{ki}}{\partial y^j}=g^{hl}\frac{\partial^2 g_{li}}{\partial y^j\partial y^k}-D^h_{jl}D^l_{ki}.$$

Thus

$$K^{\hat{h}}_{\hat{i}\hat{j}\hat{k}}=K^{h}_{\hat{i}\hat{j}\hat{k}}=0. \tag{27}$$

Next keeping the same values for the first two arguments and taking $Z=\delta/\delta x^i$ we get

$$K^{\hat{h}}_{\hat{i}\hat{j}k}\frac{\partial}{\partial y^h}+K^{h}_{\hat{i}\hat{j}k}\frac{\delta}{\delta x^h}=\nabla_{\hat{j}}\nabla_{\hat{k}}\frac{\delta}{\delta x^i}-\nabla_{\hat{k}}\nabla_{\hat{j}}\frac{\delta}{\delta x^i}=0,$$

whence

$$K^{\hat{h}}_{i\hat{j}\hat{k}}=K^{h}_{i\hat{j}\hat{k}}=0. \tag{28}$$

Similarly we have

$$K^{h}_{\hat{i}\hat{j}k}\frac{\partial}{\partial y^h}+K^{h}_{\hat{i}\hat{j}k}\frac{\delta}{\delta x^h}=\left(\nabla_{\hat{j}}\nabla_k-\nabla_k\nabla_{\hat{j}}+\frac{\partial N^h_k}{\partial y^j}\nabla\right)\frac{\partial}{\partial y^i}=$$

$$=\left(\frac{\partial\Gamma^h_{ki}}{\partial y^j}+D^h_{jl}\Gamma^l_{ki}+D^h_{il}\Gamma^l_{jk}-\Gamma^h_{kl}D^l_{ij}-\frac{\delta D^h_{ij}}{\delta x^k}\right)\frac{\partial}{\partial y^h}.$$

Thus

$$K^{h}_{\hat{i}\hat{j}k}=0, \tag{29}$$

and $K^{\hat{h}}_{\hat{i}\hat{j}k}$ is obtained as the coefficient of $\partial/\partial y^h$ in the last expression.

Further

$$K^{\hat{h}}_{i\hat{j}k}\frac{\partial}{\partial y^h}+K^{h}_{i\hat{j}k}\frac{\delta}{\delta x^h}=(\nabla_{\hat{j}}\nabla_k-\nabla_k\nabla_{\hat{j}}+\Gamma^h_{jk}\nabla_{\hat{h}})\frac{\delta}{\delta x^i}=$$

$$=\nabla\left(\Phi^h_{ki}\frac{\delta}{\delta x^h}+g^{hl}R_{kil}\frac{\delta}{\delta y^h}\right)=\frac{\partial\Phi^h_{ki}}{\partial y^j}\frac{\delta}{\delta x^h}+g^{hl}\frac{\partial R_{kil}}{\partial y^j}\frac{\partial}{\partial y^h}.$$

Thus

$$K^{\hat{h}}_{i\hat{j}k}=g^{hl}\frac{\partial R_{kil}}{\partial y^j},\qquad K^{h}_{i\hat{j}k}=\frac{\partial\Phi^h_{ki}}{\partial y^j}. \tag{30}$$

Next

$$K^{\hat{h}}_{\hat{i}jk}\frac{\partial}{\partial y^h}+K^{h}_{\hat{i}jk}\frac{\delta}{\delta x^h}=(\nabla_j\nabla_k-\nabla_k\nabla_j+R^h_{jk}\nabla_{\hat{h}})\frac{\partial}{\partial y^i}=$$

$$=\left(\frac{\delta\Gamma^h_{ki}}{\delta x^j}-\frac{\delta\Gamma^h_{ji}}{\delta x^k}+\Gamma^h_{jl}\Gamma^l_{ki}-\Gamma^h_{kl}\Gamma^l_{ji}+D^h_{il}R^l_{jk}\right)\frac{\partial}{\partial y^h}.$$

Thus

$$K^{h}_{\hat{i}jk}=0, \tag{31}$$

and $K^{\hat{h}}_{\hat{i}jk}$ is given as the coeficient of $\partial/\partial y^h$ in the last expression. Finally

$$K^{\hat{h}}_{ijk}\frac{\partial}{\partial y^h}+K^{h}_{ijk}\frac{\delta}{\delta x^h}=(\nabla_j\nabla_k-\nabla_k\nabla_j+R^h_{jk}\nabla_{\hat{h}})\frac{\delta}{\delta x^i}=$$

$$=\left(\frac{\delta(g^{hl}R_{kil})}{\delta x^j}-\frac{\delta(g^{hl}R_{jil})}{\delta x^k}+\Gamma^h_{jm}R_{kil}g^{ml}-\Gamma^h_{km}R_{jil}g^{ml}+R_{jlm}g^{hm}\Phi^l_{ki}-\right. \tag{32}$$

$$\left.-g^{hm}R_{klm}\Phi^l_{ji}\right)\frac{\partial}{\partial y^h}+\left(\frac{\delta\Phi^h_{ki}}{\delta x^j}-\frac{\delta\Phi^h_{ji}}{\delta x^k}+\Phi^h_{jl}\Phi^l_{ki}-\Phi^h_{kl}\Phi^l_{ji}\right)\frac{\delta}{\delta x^h},$$

from which we get the corresponding expressions of $K^{\hat{h}}_{ijk}$, K^{h}_{ijk}.

Some of the local coordinate expressions of K may be written simpler if we use the following identity

$$g^c(K(X,Y)Z,W)+g^c(Z,K(X,Y)W)=0, \tag{33}$$

obtained from the property $\nabla g^c=0$ of ∇.

Taking

$$X=\frac{\partial}{\partial y^j},\qquad Y=\frac{\delta}{\delta x^k},\qquad Z=\frac{\partial}{\partial y^i},\qquad W=\frac{\delta}{\delta x^l},$$

in (33) we get

$$K^{\hat{h}}_{\hat{i}\hat{j}k}=-g_{il}g^{hm}\frac{\partial\Phi^l_{km}}{\partial y^j}. \tag{29'}$$

On the other hand use the expression (24) of Γ^k_{ij} and the commutation formula

$$\left[\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^i}\right]=\Gamma^h_{ji}\frac{\partial}{\partial y^h}$$

in the expression of $K^{\hat{h}}_{\hat{i}jk}$ to obtain

$$K^{\hat{h}}_{\hat{i}jk}=\frac{\partial R^h_{jk}}{\partial y^i}+D^h_{il}R^l_{jk}=g^{hl}\frac{\partial R_{ljk}}{\partial y_i}. \tag{31'}$$

Remark. The formula (31′) will be obtained again in the next section from the first formula (30), the identity (13) and the first Bianchi identity satisfied by K.

Next, let us take

$$X=\frac{\delta}{\delta x^j},\qquad Y=\frac{\delta}{\delta x^k},\qquad Z=\frac{\delta}{\delta x^i},\qquad W=\frac{\partial}{\partial y^l}$$

in (33) to get the following expression for the coefficient of $\delta/\delta x^h$ in (32)

$$K^{h}_{ijk}=-g_{il}g^{hm}K^{\hat{l}}_{\hat{m}jk}=-D^h_{il}R^l_{jk}-g_{il}g^{hm}\frac{\partial R^l_{jk}}{\partial y^m}=-g^{hl}\frac{\partial R_{ijk}}{\partial y^l} \tag{32'}$$

Finally we shall use the formula (26′) to obtain the following expression for the coefficient of $\partial/\partial y^h$ in (32)

$$
(32'') \qquad K_{\hat{i}\hat{j}k}^{\hat{h}} = g^{hl}\left(\frac{\delta R_{ktl}}{\delta x^j} - \frac{\delta R_{jtl}}{\delta x^k} - R_{kml}\Phi^m_{ji} - R^k_{kim}\Phi^m_{ji} + \right.
$$

$$
\left. + R_{jml}\Phi^m_{ki} + R_{jim}\Phi^m_{kl}\right) = g^{hl}(\nabla_j R_{kil} - \nabla_k R_{jil}),
$$

where $\nabla_j R_{kil}$ is defined by

$$
\nabla_j R_{kil} dx^k \otimes dx^i \otimes dx^l = \nabla_j (R_{kil} dx^k \otimes dx^i \otimes dx^l).
$$

The Riemann-Christoffell tensor field is defined as usual and the corresponding local coordinate components are

$$
K_{\hat{i}\hat{j}\hat{k}l} = K_{ij\hat{k}l} = K_{\hat{i}jkl} = K_{ij\hat{k}l} = K_{\hat{i}jkl} = K_{\hat{i}jkl} = 0,
$$

$$
K_{\hat{i}j\hat{h}l} = -K_{\hat{j}i\hat{h}l} = g_{jh}\frac{\partial \Phi^h_{li}}{\partial y^k}, \qquad K_{ij\hat{k}l} = -K_{l\hat{k}ij} = \frac{\partial R_{lij}}{\partial y^k},
$$

$$
K_{ijkl} = \nabla_k R_{lij} - \nabla_l R_{kij}.
$$

§ 4. **Bianchi identities.** We shall write down the essential local coordinate expressions derived from the first Biachi identity satisfied by the curvature tensor field K of ∇

$$
K(X, Y)Z + K(Y, Z)X + K(Z, X)Y = 0, \quad X, Y, Z \in \chi(TM),
$$

$$
(34) \qquad K^{\hat{h}}_{\hat{j}\hat{k}i} = K^{\hat{h}}_{\hat{k}\hat{j}i}, \qquad X = \frac{\partial}{\partial y^j}, \qquad Y = \frac{\partial}{\partial y^k}, \qquad Z = \frac{\delta}{\delta x^i},
$$

$$
(35) \qquad K^{h}_{\hat{j}ki} = K^{h}_{\hat{k}ji} - K^{h}_{\hat{i}jk}, \qquad K^{h}_{\hat{i}jk} = K^{h}_{\hat{k}ji}, \qquad X = \frac{\partial}{\partial y^j}, \qquad Y = \frac{\delta}{\delta x^k}, \qquad Z = \frac{\delta}{\delta x^i}.
$$

The first identity (35) gives by a straightforward computation the expression (31′) of $K^{\hat{h}}_{\hat{i}jk}$.

$$
(36) \qquad \sum_{(i,\,j,\,k)} K^{\hat{h}}_{ijk} = 0, \qquad \sum_{(i,\,j,\,k)} K^{h}_{ijk} = 0, \qquad X = \frac{\delta}{\delta x^i}, \qquad Y = \frac{\delta}{\delta x^j}, \qquad Z = \frac{\delta}{\delta x^k}
$$

The first identity from (36) is equivalent to the identity (9).

The second Bianchi identity satisfied by K is

$$
(\nabla_X K)(Y, Z)W + (\nabla_Y K)(Z, X)W + (\nabla_Z K)(X, Y)W = 0.
$$

In order to get the local coordinate expressions of this identity we have to write down the local coordinate expressions of the covariant derivatives of K. Let us write

$$
K = (K^{\hat{h}}_{\hat{i}\hat{j}k}\delta y^i \otimes \delta y^j \otimes dx^k - K^{\hat{h}}_{\hat{i}k\hat{j}}\delta y^i \otimes dx^j \otimes \delta y^k + K^{\hat{h}}_{i\hat{j}k} dx^i \otimes \delta y^j \otimes dx^k -
$$

$$
- K^{\hat{h}}_{i k\hat{j}} dx^i \otimes dx^j \otimes \delta y^k + K^{\hat{h}}_{\hat{i}jk}\delta y^i \otimes dx^j \otimes dx^k + K^{\hat{h}}_{ijk} dx^i \otimes dx^j \otimes dx^k) \otimes \frac{\partial}{\partial y^h} +
$$

$$
+ (K^{h}_{i\hat{j}k} dx^i \otimes \delta y^j \otimes dx^k - K^{h}_{ik\hat{j}} dx^i \otimes dx^j \otimes \delta y^k + K^{h}_{ijk} dx^i \otimes dx^j \otimes dx^k) \otimes \frac{\delta}{\delta x^h},
$$

and use the formulae (23), (23′) to obtain

$$
\nabla_l K = (\nabla_l K^{\hat{h}}_{\hat{i}\hat{j}k}\delta y^i \otimes \delta y^j \otimes dx^k - \nabla_l K^{\hat{h}}_{\hat{i}k\hat{j}}\delta y^i \otimes dx^j \otimes \delta y^k +
$$

$$
+ \nabla_l K^{\hat{h}}_{i\hat{j}k} dx^i \otimes \delta y^j \otimes dx^k - \nabla_l K^{\hat{h}}_{ik\hat{j}} dx^i \otimes dx^j \otimes \delta y^k + \nabla_l K^{\hat{h}}_{\hat{i}jk}\delta y^i \otimes dx^j \otimes dx^k +
$$

$$
+ \nabla_l K^{\hat{h}}_{ijk} dx^i \otimes dx^j \otimes dx^k) \otimes \frac{\partial}{\partial y^h} + (\nabla_l K^{h}_{i\hat{j}k} dx^i \otimes \delta y^j \otimes dx^k -
$$

$$
- \nabla_l K^{h}_{ik\hat{j}} dx^i \otimes dx^j \otimes \delta y^k + \nabla_l K^{h}_{ijk} dx^i \otimes dx^j \otimes dx^k) \otimes \frac{\delta}{\delta x^h}.
$$

where

$$
(37) \qquad \nabla_l K^{\hat{h}}_{\hat{i}\hat{j}k} = \frac{\delta K^{\hat{h}}_{\hat{i}\hat{j}k}}{\delta x^l} - \Gamma^m_{li}K^{\hat{h}}_{\hat{m}\hat{j}k} - \Gamma^m_{lj}K^{\hat{h}}_{\hat{i}\hat{m}k} - \Phi^m_{lk}K^{\hat{h}}_{\hat{i}\hat{j}m} + \Gamma^h_{lm}K^{\hat{m}}_{\hat{i}\hat{j}k},
$$

$$
(38) \qquad \nabla_l K^{\hat{h}}_{i\hat{j}k} = \frac{\delta K^{\hat{h}}_{i\hat{j}k}}{\delta x^l} - \Phi^m_{li}K^{\hat{h}}_{m\hat{j}k} - \Gamma^m_{lj}K^{\hat{h}}_{i\hat{m}k} - \Phi^m_{lk}K^{\hat{h}}_{i\hat{j}m} +
$$

$$
+ \Gamma^h_{lm}K^{\hat{m}}_{i\hat{j}k} - R_{lia}g^{am}K^{\hat{h}}_{\hat{m}\hat{j}k} + R_{lma}g^{ah}K^{m}_{i\hat{j}k},
$$

$$
(39) \qquad \nabla_l K^{h}_{i\hat{j}k} = \frac{\delta K^{h}_{i\hat{j}k}}{\delta x^l} - \Phi^m_{li}K^{h}_{m\hat{j}k} - \Gamma^m_{lj}K^{h}_{i\hat{m}k} - \Phi^m_{lk}K^{h}_{i\hat{j}m} + \Phi^h_{lm}K^{m}_{i\hat{j}k},
$$

$$
(40) \qquad \nabla_l K^{\hat{h}}_{\hat{i}jk} = \frac{\delta K^{\hat{h}}_{\hat{i}jk}}{\delta x^l} - \Gamma^m_{li}K^{\hat{h}}_{\hat{m}jk} - \Phi^m_{lj}K^{\hat{h}}_{\hat{i}mk} - \Phi^m_{lk}K^{\hat{h}}_{\hat{i}jm} + \Gamma^h_{lm}K^{\hat{m}}_{\hat{i}jk} -
$$

$$
- R_{lja}g^{am}K^{\hat{h}}_{\hat{i}\hat{m}k} + R_{lka}g^{am}K^{\hat{h}}_{\hat{i}\hat{m}j},
$$

$$
(41) \qquad \nabla_l K^{h}_{ijk} = \frac{\delta K^{h}_{ijk}}{\delta x^l} - \Phi^m_{li}K^{h}_{mjk} - \Phi^m_{lj}K^{h}_{imk} - \Phi^m_{lk}K^{h}_{ijm} + \Phi^h_{lm}K^{m}_{ijk} -
$$

$$
- R_{lja}g^{am}K^{h}_{i\hat{m}k} + R_{lka}g^{am}K^{h}_{i\hat{m}j},
$$

$$
(42) \qquad \nabla_l K^{\hat{h}}_{ijk} = \frac{\delta K^{\hat{h}}_{ijk}}{\delta x^l} - \Phi^m_{li}K^{\hat{h}}_{mjk} - \Phi^m_{lj}K^{\hat{h}}_{imk} - \Phi^m_{lk}K^{\hat{h}}_{ijm} + \Gamma^h_{lm}K^{\hat{m}}_{ijk} -
$$

$$
- R_{lja}g^{am}K^{\hat{h}}_{i\hat{m}k} + R_{lka}g^{am}K^{\hat{h}}_{i\hat{m}j} - R_{lia}g^{am}K^{\hat{h}}_{\hat{m}jk} + R_{lma}g^{ah}K^{m}_{ijk}.
$$

Similar formulae are obtained for $\nabla_{\hat{i}} K$ with Γ^i_{jk} replaced by D^i_{jk} and Φ^i_{jk}, R_{ijk} replaced by 0.

Then the essential local coordinate expressions of the second Bianchi identity are

$$
(43) \qquad \nabla_{\hat{l}} K^{\hat{h}}_{\hat{i}\hat{j}k} = \nabla_{\hat{j}} K^{\hat{h}}_{\hat{i}\hat{l}k},
$$

$$
(44) \qquad \nabla_{\hat{l}} K^{\hat{h}}_{i\hat{j}k} = \nabla_{\hat{j}} K^{\hat{h}}_{i\hat{l}k}, \qquad \nabla_{\hat{l}} K^{h}_{i\hat{j}k} = \nabla_{\hat{j}} K^{h}_{i\hat{l}k},
$$

$$
(45) \qquad \nabla_{\hat{j}} K^{\hat{h}}_{\hat{i}kl} = \nabla_k K^{\hat{h}}_{\hat{i}\hat{j}l} - \nabla_l K^{\hat{h}}_{\hat{i}\hat{j}k},
$$

$$
(46) \qquad \nabla_{\hat{j}} K^{\hat{h}}_{ikl} = \nabla_k K^{\hat{h}}_{i\hat{j}l} - \nabla_l K^{\hat{h}}_{i\hat{j}k}, \qquad \nabla_{\hat{j}} K^{h}_{ikl} = \nabla_k K^{h}_{i\hat{j}l} - \nabla_l K^{h}_{i\hat{j}k},
$$

(47) $$\nabla_l K^{h}_{ijk} + \nabla_j K^{\hat{h}}_{ikl} + \nabla_k K^{\hat{h}}_{ilj} = 0,$$

(48) $$\nabla_l K^{\hat{h}}_{ijk} + \nabla_j K^{\hat{h}}_{ikl} + \nabla_k K^{\hat{h}}_{\ \ i\,l} = 0,$$

(49) $$\nabla_l K^{h}_{ijk} + \nabla_j K^{h}_{ikl} + \nabla_k K^{h}_{ilj} = 0.$$

The Ricci tensor field is obtained easily according to its definition

(50) $$K_{i\hat{j}} = K^{\hat{h}}_{i\hat{h}\hat{j}} - K^{h}_{\hat{i}\hat{j}h} = 0,$$

(51) $$K_{i\hat{j}} = K^{h}_{i\hat{h}\hat{j}} + K^{h} = -K^{h}_{ijh} = -\frac{\partial \Phi^{h}_{hi}}{\partial y^{j}},$$

(52) $$K_{ij} = K^{\hat{h}}_{i\hat{h}j} + K^{h}_{ihj} = -\frac{\partial g^{hl}}{\partial y^{i}}\left(\frac{\delta g_{lh}}{\delta x^{j}} - g_{hm}\Gamma^{m}_{lj}\right) -$$
$$-g^{hl}\frac{\partial}{\partial y^{h}}\left(\frac{\delta g_{il}}{\delta x^{j}} - g_{lm}\Gamma^{m}_{ij}\right) = -\frac{\partial \Phi^{h}_{hj}}{\partial y^{i}},$$

(53) $$K_{ij} = K^{\hat{h}}_{i\hat{h}j} + K^{h}_{ihr} = g^{hl}\frac{\partial}{\partial y^{h}}(R_{ijl} + R_{jil}).$$

Finally the scalar curvature is

(54) $$K = 2K_{ij}g^{ij} = -2g^{ij}\frac{\partial \Phi^{h}_{hj}}{\partial y^{i}}.$$

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A GEOMETRIC PROOF OF HUREWICZ'S THEOREM

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1. **Introduction.** If we consider the homology groups as the bordism groups constructed with pseudomanifolds we obtain an easy geometric interpretation of the singular cycles. This fact permits of more intuitive proofs, from the geometric point of view, of many results of Algebraic Topology.

The aim of this note is, following the algebraic ideas developed in Spanier's book [7], to give a geometric proof of the well-known Hurewicz's Theorem that relates the basic algebraic invariants of a topological space: its homotopy and homology groups.

The references for the results of Polyhedral Topology and Homotopy Theory will be [6] and [7] respectively. A polyhedron is an Euclidean subspace covered by finite many simplices. Given the interval $I=[0, 1]$, we denote by I^n the ordinary n-cube with its canonical PL-structure and orientation. S^n is the quotient $I^n/\partial I^n$ with its canonical PL-structure and orientation. As usually, $\pi_n(X, x_0)$ is the n-th homotopy group of the pointed topological space (X, x_0), defined by the

$$f : (I^n, \partial I^n) \to (X, x_0),$$

and $\pi_n'(X, x_0)$ is the quotient of $\pi_n(X, x_0)$ by its commutator subgroup. It is known that

$$\pi_n(X, x_0) = \pi_n'(X, x_0) \text{ if } n \geqslant 2.$$

In general, we regard $\pi_n'(X, x_0)$ as $\pi_n(X, x_0)$ with its group operation made commutative.

A closed n-pseudomanifold is an n-dimensional polyhedron P which admits a subpolyhedron SP, called singular set of P, such that:

a) $\dim SP \leqslant n-2$,

b) $P-SP$ is an oriented PL-manifold without boundary.

An n-pseudomanifold P with boundary ∂P is an n-dimensional polyhedron verifying a) and the following conditions:

a') $P-SP$ is an oriented PL-manifold with boundary $\partial P - SP$.

b') P is a closed $(n-1)$-pseudomanifold such that $S(\partial P) = \partial P \cap SP$.

A singular n-pseudomanifold on a topological pair (X, A) is a pair (P, f) where P is an n-pseudomanifold with boundary ∂P, possibly empty, and

$$f : (P, \partial P) \to (X, A)$$

is a continuos mapping. If (P_1, f_1) and (P_2, f_2) are two singular n-pseudomanifolds on (X, A) we say that they are pseudobordant if there exists a pair (Q, F) such that: