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MORPHISMS OF H-CONES AND SPECIFIC ORDER

BY

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Introduction. Let S and T be H-cones. All the notions and notations about H-cones will be as in [2]. Hom(S, T) will denote the set of all morphisms of H-cones from S to T (see [6]). We define two order relations on Hom(S, T). The specific order, denoted by $\precsim$, will be defined as follows: $\varphi \precsim \psi$ iff there exists $\varphi_1 \in$ Hom(S, T) such that: $\varphi + \varphi_1 = \psi$. The natural order, denoted by $\leqslant$, will be defined as follows: $\varphi \leqslant \psi$ iff: $\varphi(s) \leqslant \psi(s)$, $\forall s \in S$. (Hom$(S, T), \leqslant$) is a complete lattice [6].

$H(S, T)$ will denote the set of all semi-finite morphisms from S to T (see [5] and [6] for a comparison of various notions of morphisms of H-cones, proposed in [2] and [4]). $H(S, T)$ becomes an ordered convex cone, which satisfies all the properties of a H-cone, except maybe the Riesz decomposition property. However:

Theorem [2]. *$H(S, (T, \precsim))$ is an H-cone.*

In the proof of this theorem, a key role is played by the following:

Lemma 1. *If $\varphi, \psi \in H(S, T)$, then:*

$$\rho(s) = \bigvee \{R(\varphi(t) - \psi(t)) \mid t \leqslant s\}$$

defines a subadditive, increasing and continuous map.

Lemma 2. *Let $\psi : S \to \tilde{T}$ be a subadditive, increasing and continuous map. Then:*

$$\tilde{\psi}(s) = \bigvee \left\{ \sum_{i \in I} \psi(s_i) \,\middle|\, I \text{ finite}, \sum_{i \in I} s_i \leqslant s \right\}$$

defines the smallest morphism of H-cones, such that $\tilde{\psi}(s) \leqslant \psi(s)$ $\forall s \in S$, if S has the specific order.

The next result will also be used in the sequel:

Lemma 3. *Let $\varphi, \psi \in H(S, T)$. Then $\varphi \precsim \psi$ if and only if: $\varphi(s) \precsim \psi(s)$, $\forall s \in S$ and $s \leqslant t \Rightarrow \psi(s) + \varphi(t) \leqslant \psi(t) + \varphi(s)$.*

Proof. The necessity is clear. Conversely, let us define:

$$\psi'(s) = \psi(s) - \varphi(s).$$

ψ' is well defined for any $s \in D(\psi)$ and is clearly additive.

ψ' is increasing by the second condition. Since φ and ψ are continuous, it follows that $\psi' \in H(S, T)$.

The Main Result. It is known that $H(R_+, T) \simeq T$, hence it is an H-cone. More generally:

Theorem. *$H((S, \lesssim), T)$ is an H-cone.*

Proof. By [2] it remains to prove the Riesz decomposition property. Let $\varphi, \psi \in H((S, \lesssim), T)$. For $s \in D = D(\varphi) \cap D(\psi)$ we define:

$$\rho(s) = \bigvee \{R(\varphi(t) - \psi(t)) \mid t \leqslant s\},$$

ρ is subadditive, increasing and continuous on D, by lemma 1. Moreover, for any $s \in D$ and $t \lesssim s$ we have:

$$R(\varphi(t) - \psi(t)) \lesssim \varphi(t) \lesssim \varphi(s)$$

hence $\rho(s) \lesssim \varphi(s)$. For any finite set I, from $\sum_{i \in I} s_i \lesssim s$ it follows that:

$$\sum_{i \in I} \rho(s_i) \lesssim \sum_{i \in I} \varphi(s_i) = \varphi\Big(\sum_{i \in I} s_i\Big) \lesssim \varphi(s)$$

hence:

$$\tilde{\rho}(s) = \bigvee \Big\{\sum_{i \in I} \rho(s_i) \mid I \text{ finite}, \sum_{i \in I} s_i \lesssim s\Big\}$$

defines a $\tilde{\rho} \in H((S, \lesssim), T)$ such that: $\tilde{\rho}(s) \lesssim \varphi(s)$, $\forall s \in D$. We prove next that: $\tilde{\rho} = R(\varphi - \psi)$ in $H((S, \lesssim), T)$. Indeed, for each $s \in D$ we have:

$$\tilde{\rho}(s) + \psi(s) \gtrsim \rho(s) + \psi(s) \gtrsim R(\varphi(s) - \psi(s)) + \psi(s) \geqslant (\varphi(s) - \psi(s)) + \psi(s) \geqslant \varphi(s).$$

Let now $u \in H((S, \lesssim), T)$ be such that: $u \geqslant \varphi - \psi$. For any $s \in D$ and $t \lesssim s$ we have: $u(s) \geqslant R(\varphi(t) - \psi(t))$. Hence $u(s) \geqslant \rho(s)$, $\forall s \in D$. Moreover, if I is a finite set and $\sum_{i \in I} s_i \lesssim s$, then: $u(s) \gtrsim \sum_{i \in I} u(s_i) \gtrsim \sum_{i \in I} \rho(s_i)$, and $u(s) \geqslant \tilde{\rho}(s)$, $\forall s \in D$.

In order to prove that $\tilde{\rho} \lesssim \varphi$, we verify the two conditions from lemma 3. The first one follows using twice prop. 2.1.4 from [2]. The second condition is verified since S has the specific order: indeed, if $s \lesssim t$, then there exists $s_1 \in S$ such that $s + s_1 = t$. Then:

$$\varphi(s) + \tilde{\rho}(t) = \varphi(s) + \tilde{\rho}(s) + \tilde{\rho}(s_1) \leqslant \varphi(s) + \tilde{\rho}(s) + \varphi(s_1) = \varphi(t) + \tilde{\rho}(s).$$

We prove next that the characterisation given in [2] for the extreme rays in S^*, extends to $H(S, T)$, provided S or T has the specific order.

Lemma 4. *Suppose that S or T has the specific order. For any $\varphi \in H(S, T)$ and $s, t \in D(\varphi)$, there exists $\varphi_1, \varphi_2 \in H(S, T)$ such that: $\varphi = \varphi_1 + \varphi_2$ and $\varphi(s \wedge t) = \varphi_1(s) + \varphi_2(t)$.*

Proof. Let us denote: $f = t - s \wedge t$ and define:

$$\varphi_1(u) = \bigvee_{n \in N} \varphi(u \wedge n \cdot f), \quad u \in D(\varphi).$$

This definition is correct, since the family $\{(u \wedge n \cdot f)\}_{n \in N}$ is increasing and dominated by $\varphi(u) \in T$. φ_1 is clearly increasing. Since: $(u + v) \wedge n \cdot f \leqslant u \wedge n \cdot f + v \wedge n \cdot f \leqslant (u + v) \wedge 2n \cdot f$, φ_1 results additive. Let $F \subseteq D(\varphi)$ be increasing and $u = \bigvee_{v \in F} v$. Then:

$$\varphi_1(u) \geqslant \bigvee_{v \in F} \varphi_1(v) \geqslant \bigvee_{v \in F} \bigvee_{n \in N} \varphi(v \wedge n \cdot f) = \bigvee_{n \in N} \varphi(u \wedge n \cdot f) = \varphi_1(u).$$

Hence $\varphi_1 \in H(S, T)$.

If T has the specific order, then $\varphi_1(u) \lesssim \varphi(u)$, $\forall u \in D(\varphi)$. The same holds true if S has the specific order. Indeed, in this case $f \in S$, hence $\{u \wedge n \cdot f\}_{n \in N}$ is specifically increasing in S and $\{\varphi(u \wedge n \cdot f)\}_{n \in N}$ is also specifically increasing in T. Being specifically dominated by $\varphi(u)$, we get:

$$\varphi(u) \gtrsim \curlyvee_{n \in N} \varphi(u \wedge n \cdot f) = \bigvee_{n \in N} \varphi(u \wedge n \cdot f) = \varphi_1(u).$$

Let us define now: $\varphi_2(u) = \varphi(u) - \varphi_1(u)$. φ_2 is well defined on $D(\varphi)$ with values in T, and is additive. Let $u, v \in D(\varphi)$ be such that $u \leqslant v$. Then:

$$u + v \wedge n \cdot f \leqslant v + u \wedge n \cdot f, \quad \varphi(u) + \varphi(v \wedge n \cdot f) \leqslant \varphi(v) + \varphi(u \wedge n \cdot f),$$

$$\varphi(u) + \varphi_1(v) \leqslant \varphi(v) + \varphi_1(u), \quad \varphi_2(u) \leqslant \varphi_2(v).$$

It follows easily that $\varphi_2 \in H(S, T)$ and $\varphi = \varphi_1 + \varphi_2$. Now, since: $(s - s \wedge t) \wedge n.(t - s \wedge t) = 0$, we get: $s \wedge n \cdot f \leqslant (s \wedge t) \wedge n \cdot f$ which proves that: $\varphi_1(s) = \varphi_1(s \wedge t)$. From:

$$t \wedge n \cdot f \leqslant f + (s \wedge t) \wedge n \cdot f \leqslant (f + s \wedge t) \wedge 2n \cdot f,$$

it follows that: $\varphi_1(t) = \varphi(f) + \varphi_1(s \wedge t)$. From these relations, we obtain: $\varphi(s \wedge t) = \varphi_1(s) + \varphi_2(t)$.

With the same proof as in [2, cor. 2.3.5] we have:

Proposition 5. *Suppose that S or T has the specific order. Then $\varphi \in H(S, T)$ belongs to an extreme ray if and only if:*

$$\forall s, t \in D(\varphi) : \varphi(s \wedge t) = \varphi(s) + \varphi(t).$$

The Specific Order in H(S, T). We begin with some preliminary results concerning morphisms commuting with $\wedge$.

Definition. *Let S be a H-cone. $s \in S$ is called nearly continuous if, for any increasing family $(s_i)_{i \in I}$, with $s = \bigvee_{i \in I} s_i$, we have: $\bigwedge_{i \in I} R(s - s_i) = 0$.*

We denote S_n the set of nearly continuous elements of S.

Proposition 6. *S_n is a specifically solid, convex subcone.*

Proof. Let $s, t \in S_n$ and $s + t = \bigvee_{i \in I} u_i$, with $(u_i)_{i \in I}$ increasing. Defining: $u_{i,s} = R(u_i - t)$; $u_{i,t} = R(u_i - s)$ we get, by [2, prop. 2.2.3] that $(u_{i,s})_{i \in I}$, resp. $(u_{i,t})_{i \in I}$ is increasing to s, resp. t and: $u_i \geqslant u_{i,s} + u_{i,t}$. It follows that $R(s + t - u_i) \leqslant R(s - u_{i,s}) + R(t - u_{i,t})$, hence $s + t \in S_n$. If $t \lesssim s$ and $s \in S_n$, then, for any increasing family $(t_i)_{i \in I}$ to t, we have: $s = \bigvee_{i \in I} (t' + t_i)$, hence: $\bigwedge_{i \in I} R(t - s_i) = \bigwedge_{i \in I} R(s - t' - t_i) = 0$ (where $t' \in S$ is such that $t + t' = s$).

Proposition 7. *Let $\varphi : S_n \to T$ be an additive, increasing map, with the property:*

$\forall (u_i)_{i \in I}$ decreasing to 0: $\bigwedge_{i \in I} \varphi(u_i) = 0$.

Then, for any increasing family $(s_i)_{i \in I}$ to $s \in S$, we have: $\varphi(s) = \bigvee_{i \in I} \varphi(s_i)$.

Proof. Since $s \leqslant s_i + R(s - s_i)$, we get $\varphi(s) \leqslant \varphi(s_i) + \varphi(R(s - s_i))$ whence:

$$\varphi(s) \leqslant \bigvee_{i \in I} \varphi(s_i) + \bigwedge_{i \in I} \varphi(R(s - s_i)) = \bigvee_{i \in I} \varphi(s_i).$$

The converse inequality is obvious.

Proposition 8. *Let $\varphi : S_n \to T$ be a map as in prop. 7. If S_n is naturally solid and dense in S, then φ has a unique extention to an element of $H(S, T)$.*

Proof. Let us define:

$$\bar{\varphi}(s)=\bigvee\{\varphi(u)\mid u\in S_a,\ u\leqslant s\}$$

when the second member exists. Prop. 7 shows that $\bar{\varphi}|_{S_a}=\varphi\cdot\bar{\varphi}$ is clearly additive and increasing. Since S_a is naturally solid, it follows that $\bar{\varphi}$ is continuous: indeed, let $(t_j)_{j\in I}$ be an increasing family to $t\in S$. Let $u\in S_a$ be such that $u\leqslant t$. Since $u\wedge t_j\in S_a$, from prop. 7 we get: $\varphi(u)=\bigvee_{j\in I}\varphi(u\wedge t_j)\leqslant\bigvee_{j\in I}\bar{\varphi}(t_j)$. Since u is arbitrary, it follows that $\bar{\varphi}(t)\leqslant\bigvee_{j\in I}\bar{\varphi}(t_j)$. The converse inequality is obvious. Since S_a is dense in S, we have $\bar{\varphi}\in H(S,T)$ and also the fact that $\bar{\varphi}$ is unique.

Remarks. If S has the specific order, then $S_a=S$. If S is a stonian cone [1], then any regular element is nearly continuous. Hence S_a is dense in S, when S is a regular stonian cone. If S is a standard H-cone, then S_a naturally solid means that the axiom of nearly continuity holds in S. The fact that any contraction commutes with decreasing $\wedge$ means the axiom of convergence.

Lemma 9. *Let $\varphi: S\to T$ be an additive map. Then φ is a morphism from $(S,\preccurlyeq)$ to $(T,\preccurlyeq)$ if and only if φ commutes with decreasing $\curlywedge$.*

The proof is straightforward.

Proposition 10. a) *Let $(\varphi_i)_{i\in I}$ be a specifically increasing and dominated family from $H(S,T)$. Then $\curlyvee_{i\in I}\varphi_i$ exists in $H(S,T)$ and, for any $\varphi\in H(S,T)$ we have: $\varphi+\curlyvee_{i\in I}\varphi_i=\curlyvee_{i\in I}(\varphi+\varphi_i)$.*

b) *Suppose that the following property holds*

$$\forall t\in T,\ t\neq 0\ \exists \mu\in T^*\ \text{such that}\ \mu(t)\neq 0.$$

Let $(\varphi_i)_{i\in I}$ be a specifically decreasing family from $H(S,T)$. Then $\curlywedge_{i\in I}\varphi_i$ exists in $H(S,T)$ and: $(\curlywedge_{i\in I}\varphi_i)(s)=\curlywedge_{i\in I}\varphi_i(s)$, $\forall s\in D(\varphi_{i_0})$.

Proof. a) We prove that: $\varphi(s)=\curlyvee_{i\in I}\varphi_i(s)$, $\forall s\in D(\psi)$ defines a morphism, where $\psi\in H(S,T)$ is such that: $\psi\succcurlyeq\varphi_i$, $\forall i\in I$. Indeed:

$$\varphi(s+t)=\curlyvee_{i\in I}\varphi_i(s+t)=\curlyvee_{i\in I}(\varphi_i(s)+\varphi_i(t))=\curlyvee_{i\in I}\varphi_i(s)+\curlyvee_{i\in I}\varphi_i(t)=\varphi(s)+\varphi(t).$$

If $s\leqslant t$ then:

$$\varphi(s)=\curlyvee_{i\in I}\varphi_i(s)=\bigvee_{i\in I}\varphi_i(s)\leqslant\bigvee_{i\in I}\varphi_i(t)=\curlyvee_{i\in I}\varphi_i(t)=\varphi(t).$$

Let $(s_j)_{j\in J}$ be an increasing family to $s\in S$. Then:

$$\varphi(s)=\varphi(\bigvee_{j\in J}s_j)=\curlyvee_{i\in I}\varphi_i(\bigvee_{j\in J}s_j)=\bigvee_{i\in I}\bigvee_{j\in J}\varphi_i(s_j)=\bigvee_{j\in J}\varphi(s_j).$$

In order to prove that $\varphi\succcurlyeq\varphi_i$, we use lemma 3. The first condition is clearly satisfied. Let $s\leqslant t$. Then:

$$\varphi(s)+\varphi_i(t)=\curlyvee_{j\in J}\varphi_j(s)+\varphi_i(t)=\curlyvee_{j\in J}[\varphi_j(s)+\varphi_i(t)]=\curlyvee_{j\geqslant i}[\varphi_j(s)+\varphi_i(t)].$$

But, for $j\geqslant i$ we have $\varphi_j\succcurlyeq\varphi_i$, hence:

$$\varphi_j(s)+\varphi_i(t)\leqslant\varphi_i(s)+\varphi_j(t),$$

so that:

$$\varphi(s)+\varphi_i(t)\leqslant\bigvee_{j\in I}[\varphi_j(t)+\varphi_i(s)]=\varphi(t)+\varphi_i(s).$$

b) If $\varphi(s)=\curlywedge_{i\in I}\varphi_i(s)$, $\forall s\in D(\varphi_{i_0})$, then:

$$\varphi(s+t)=\curlywedge_{i\in I}\varphi_i(s+t)=\curlywedge_{i\in I}(\varphi_i(s)+\varphi_i(t))=\curlywedge_{i\in I}\varphi_i(s)+\curlywedge_{i\in I}\varphi_i(t)=\varphi(s)+\varphi(t).$$

If $s\leqslant t$ then:

$$\varphi(s)=\curlywedge_{i\in I}\varphi_i(s)=\bigwedge_{i\in I}\varphi_i(s)\leqslant\curlywedge_{i\in I}\varphi_i(t)=\curlywedge_{i\in I}\varphi_i(t)=\varphi(t).$$

Let $(s_j)_{j\in J}$ be increasing to $s\in S$, and let $\mu\in T^*$ be such that $\mu(\varphi_{i_0}(s))<\infty$. For $\varepsilon>0$, let $j_\varepsilon\in J$ be such that:

$$\mu(\varphi_{i_0}(s_j))+\varepsilon\geqslant\mu(\varphi_{i_0}(s)),\ \forall j\geqslant j_\varepsilon.$$ For $i\geqslant i_0$ we have: $\varphi_{i_0}=\varphi_i+\psi_i$, hence:

$$\mu(\varphi_i(s_j))+\mu(\psi_i(s))+\varepsilon\geqslant\mu(\varphi_i(s_j))+\mu(\psi_i(s_j))+\varepsilon=\mu(\varphi_{i_0}(s_j))+\varepsilon\geqslant\mu(\varphi_{i_0}(s))=$$
$$=\mu(\varphi_i(s))+\mu(\psi_i(s))$$

and:

$$\mu(\varphi_i(s_j))+\varepsilon\geqslant\mu(\varphi_i(s)),\ \forall i\geqslant i_0,\ j\geqslant j_\varepsilon.$$

We deduce:

$$\inf_{i\in I}\mu(\varphi_i(s_j))+\varepsilon\geqslant\inf_{i\in I}\mu(\varphi_i(s)),\ \forall j\geqslant j_\varepsilon,$$

and:

$$\mu[\curlywedge_{i\in I}\varphi_i(s_j)]+\varepsilon\geqslant\mu[\curlywedge_{i\in I}\varphi_i(s)],\ \forall j\geqslant j_\varepsilon,\ \bigvee_{j\in J}\curlywedge_{i\in I}\mu(\varphi_i(s_j))+\varepsilon\geqslant\curlywedge_{i\in I}\mu(\varphi_i(s)).$$

As $\varepsilon>0$ was arbitrary, we get finally:

$$\bigvee_{j\in I}\curlywedge_{i\in I}\varphi_i(s_j)\geqslant\curlywedge_{i\in I}\bigvee_{j\in I}\varphi_i(s_j).$$

The converse inequality is abvious. The fact that $\varphi\preccurlyeq\varphi_i$, $\forall i\in I$ results as above, using lemma 3.

Remark. The hypothesis in b) holds, for exemple when T is a standard H-cone.

On a result of Dixmier. We adapt to the case of H-cones with specific order, a result of Dixmier [3].

Let S be a stonian cone on the compact X. We consider the following condition:

(T) $\forall X_1\subseteq X$ open-closed, $\forall s\in S$ we have $s_1\in S$, where:

$$s_1=s\ \text{on}\ X_1\ \text{and}\ s_1=0\ \text{on}\ X/X_1.$$

Remark. (T) holds in any regular stonian cone, such that $\mathrm{Ch}_T X$ is connected.

Proposition 11. *The following properties are equivalent:*

1) *S has the specific order.*

2) *$\bar{S}=C+(X)$.*

3) *Any basic set is open and (T) holds.*

4) *X is a stonian space and (T) holds.*

Proof. $1 \Rightarrow 2$. Let $f \in C_+(X)$ be such that $f \geq \alpha > 0$ on X. There exists $s_n, t_n \in S$, such that $s_n \geq t_n$ and $s_n - t_n \to f$ uniformly on X. Hence $u_n = s_n - t_n \in S$ and $u_n \to f$ uniformly on X, hence $f \in S$. For arbitrary f, we use: $f = \inf_{n \in N}(f + 1/n)$.

$2 \Rightarrow 3$. Let $A \subseteq X$ be a basic set. For any $x \notin A$ there exists $s \in S$ such that $s = 1$ on A and $s(x) = 0$. Hence $[B^A 1 - 1 = 0] = X \setminus A$, and A is open. Clearly (T) holds.

$3 \Rightarrow 4$. Let $A \subseteq X$ be open. A is then a basic set, hence also open.

$4 \Rightarrow 1$. Let $s \leqslant t$. For any balayage B on S we have: $B(t-s) = t-s$ on $b(B)$; and (T) shows that $B(t-s) = 0 \leqslant t-s$ on $X \setminus b(B)$. Hence $t-s \in S$ and $s \leq t$.

Using [3, prop. 7] it follows:

Proposition 12. *The followings are equivalent:*

1) S has the specific order and there exists $\mu_0 \in S^*$ such that $\mu_0(1) < \infty$ and: $s < t \Rightarrow \mu_0(s) < \mu_0(t)$.

2) X is hyperstonian of countable type and (T) holds.

Also are equivalent:

1') S has the specific order and there exists $\mu_0 \in S^*$ such that $s \neq 0 \Rightarrow \mu_0(s) \neq 0$.

2') X is hyperstonian and (T) holds.

Proposition 13. *The followings are equivalent*

1) There exists $\mu_0 \in S^*$ such that $\mu_0(1) < \infty$ and $s < t \Rightarrow \mu_0(s) < \mu_0(t)$.

2) There exists a positive Radon measure on X, having X as support.

3) The union of the interiors of the supports of the finite H-integrals is dense in X and any family of non-void, open, disjoint sets in X is at most countable.

We can now prove:

Theorem 14. *Let S be a stonian cone, having the specific order. If there exists $\mu_0 \in S^*$ such that $s \neq 0 \Rightarrow \mu_0(s) \neq 0$, then S is an auto-dual H-cone. If, moreover, $\mu_0(1) < \infty$, then the associated Dirichlet space is $L^2(\mu_0)$.*

Proof. Using the above results, [3, prop. 8] shows that $\varphi(s)(t) = \mu_0(s.t)$ defines a morphism $\varphi : S \to S^*$, which is bimonotone and surjective. For any balayage B on S, we have: $\varphi(Bs)(t) = \mu_0(t.Bs)$; $B^*(\varphi(s))(t) = \varphi(s)(Bt) = \mu_0(s.Bt)$. Since $Bs = s$ on $b(B)$ and $Bs = 0$ on $X \setminus b(B)$, it follows that $s.Bt = t.Bs$ on X, hence S is auto-dual. The last part is a direct consequence of [2, prop. 7.4.5.] and [3, prop. 6].

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SOME REMARKS ON M-CONNECTIONS

BY

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Affine M-collineations as well as induced M-connections on submanifolds are considered and some of their properties are pointed out. The second bilinear fundamental form associated to the triplet (submanifold, normalization, M-structure) and the Gauss-Weingarten formulae are generalized. Relations between the M-torsion and M-curvature tensor fields associated respectively to an M-derivation and to the induced M-derivation are given.

1. Quasi-affine collineations. Let V be a finite dimensional smooth manifold. A tensor field M on V, of type (1, 1), defines a quasi-structure on V. A quasi-derivation D^M associated to the field M, is a mapping $D^M : (X, Y) \in \mathcal{X} \times \mathcal{X} \to D^M_X Y \in \mathcal{X}$, $\mathcal{F}$-linear with respect to the first argument, $\mathbf{R}$-linear with respect to the second one and which satisfies the relation $D^M_X(fY) = fD^M_X(Y) + MX(f) \cdot Y$ [6]. A M-structure (resp. M-derivation) is a quasi-structure (resp. quasi-derivation) with the property that the tensor field M is proper.

Let V and V' be smooth manifolds, with the quasi-structures M and M' respectively, D^M and $D^{M'}$ the corresponding quasi-derivations, $h : V \to V'$ a smooth diffeomorphism and $h_* : TV \to TV'$ its corresponding derivative (i.e. its tangent map).

Definition 1. *The quasi-structures M and M' are called h-related (h-correspondant) if and only if:*

$$M' h_* = h_* M. \tag{1}$$

The relation (1) shows that if a quasi-structure M and a smooth diffeomorphism h are given on a manifold V, there exists in an unique way a quasi-structure M' on V' such that

$$M' = h_* M h_*^{-1}. \tag{1'}$$

We shall denote this quasi-structure M' by $h(M)$ and we shall say that M' is the image of M with respect to h.

Proposition 1. *The relation of h-correspondence which is defined on the space of quasi-structures is an equivalence one.*

If $h_1, h_2 : V \to V'$ are smooth diffeomorphism, then the operator $N' = h_{2*} h_{1*}^{-1}$, has the property: : $M_2' N' = N' M_1'$, where $M_i' = h_{i*} M h_{i*}^{-1}$.

Definition 2. *The diffeomorphism $h : V \to V'$ is called a quasi-affine collineation with respect to the quasi-derivatives D^M and $D^{M'}$ if the following relations hold:*