Proof. $1\Rightarrow 2$. Let $f\in C_+(X)$ be such that $f\geqslant \alpha>0$ on X. There exists $s_n, t_n\in S$, such that $s_n\geqslant t_n$ and $s_n-t_n\to f$ uniformly on X. Hence $u_n=s_n-t_n\in S$ and $u_n\to f$ uniformly on X, hence $f\in S$. For arbitrary f, we use: $f=\inf_{n\in\mathbf{N}}(f+1/n)$.

$2\Rightarrow 3$. Let $A\subseteq X$ be a basic set. For any $x\notin A$ there exists $s\in S$ such that $s=1$ on A and $s(x)=0$. Hence $[B^A1=0]=X\setminus A$, and A is open. Clearly (T) holds.

$3\Rightarrow 4$. Let $A\subseteq X$ be open. A is then a basic set, hence also open.

$4\Rightarrow 1$. Let $s\leqslant t$. For any balayage B on S we have: $B(t-s)=t-s$ on $b(B)$; and (T) shows that $B(t-s)=0\leqslant t-s$ on $X\setminus b(B)$. Hence $t-s\in S$ and $s\leqslant t$.

Using [3, prop. 7] it follows:

Proposition 12. *The followings are equivalent:*

1) *S has the specific order and there exists $\mu_0\in S^*$ such that $\mu_0(1)<\infty$ and: $s<t\Rightarrow\mu_0(s)<\mu_0(t)$.*

2) *X is hyperstonian of countable type and (T) holds.*

Also are equivalent:

1') *S has the specific order and there exists $\mu_0\in S^*$ such that $s\neq 0\Rightarrow\mu_0(s)\neq 0$.*

2') *X is hyperstonian and (T) holds.*

Proposition 13. *The followings are equivalent*

1) *There exists $\mu_0\in S^*$ such that $\mu_0(1)<\infty$ and $s<t\Rightarrow\mu_0(s)<\mu_0(t)$.*

2) *There exists a positive Radon measure on X, having X as support.*

3) *The union of the interiors of the supports of the finite H-integrals is dense in X and any family of non-void, open, disjoint sets in X is at most countable.*

We can now prove:

Theorem 14. *Let S be a stonian cone, having the specific order. If there exists $\mu_0\in S^*$ such that $s\neq 0\Rightarrow\mu_0(s)\neq 0$, then S is an auto-dual H-cone. If, moreover, $\mu_0(1)<\infty$, then the associated Dirichlet space is $L^2(\mu_0)$.*

Proof. Using the above results, [3, prop. 8] shows that $\varphi(s)(t)=\mu_0(s.t)$ defines a morphism $\varphi: S\to S^*$, which is bimonotone and surjective. For any balayage B on S, we have: $\varphi(Bs)(t)=\mu_0(t.Bs)$; $B^*(\varphi(s))(t)=\varphi(s)(Bt)=\mu_0(s.Bt)$. Since $Bs=s$ on $b(B)$ and $Bs=0$ on $X\setminus b(B)$, it follows that $s.Bt=t.Bs$ on X, hence S is auto-dual. The last part is a direct consequence of [2, prop. 7.4.5.] and [3, prop. 6].

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Seminarul Matematic
Universitatea „Al. I. Cuza"
6600 Iaşi, Romania

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SOME REMARKS ON M-CONNECTIONS

BY

V. OBĂDEANU

Affine M-collineations as well as induced M-connections on submanifolds are considered and some of their properties are pointed out. The second bilinear fundamental form associated to the triplet (submanifold, normalization, M-structure) and the Gauss-Weingarten formulae are generalized. Relations between the M-torsion and M-curvature tensor fields associated respectively to an M-derivation and to the induced M-derivation are given.

1. Quasi-affine collineations. Let V be a finite dimensional smooth manifold. A tensor field M on V, of type (1, 1), defines a quasi-structure on V. A quasi-derivation D^M associated to the field M, is a mapping $D^M:(X, Y)\in\mathcal{X}\times\mathcal{X}\to D^M_XY\in\mathcal{X}$, $\mathcal{F}$-linear with respect to the first argument, **R**-linear with respect to the second one and which satisfies the relation $D^M_X(fY)=fD^M_X(Y)+MX(f)\cdot Y$ [6]. A M-structure (resp. M-derivation) is a quasi-structure (resp. quasi-derivation) with the property that the tensor field M is proper.

Let V and V' be smooth manifolds, with the quasi-structures M and M' respectively, D^M and $D^{M'}$ the corresponding quasi-derivations, $h: V\to V'$ a smooth diffeomorphism and $h_*: TV\to TV'$ its corresponding derivative (i.e. its tangent map).

Definition 1. *The quasi-structures M and M' are called h-related (h-correspondant) if and only if:*

$$M'h_*=h_*M. \tag{1}$$

The relation (1) shows that if a quasi-structure M and a smooth diffeomorphism h are given on a manifold V, there exists in an unique way a quasi-structure M' on V' such that

$$M'=h_*Mh_*^{-1}. \tag{1'}$$

We shall denote this quasi-structure M' by $h(M)$ and we shall say that M' is the image of M with respect to h.

Proposition 1. *The relation of h-correspondence which is defined on the space of quasi-structures is an equivalence one.*

If $h_1, h_2: V\to V'$ are smooth diffeomorphism, then the operator $N'=h_{2*}h_{1*}^{-1}$, has the property: : $M_2'N'=N'M_1'$, where $M_i'=h_{i*}Mh_{i*}^{-1}$.

Definition 2. *The diffeomorphism $h: V\to V'$ is called a quasi-affine collineation with respect to the quasi-derivatives D^M and $D^{M'}$ if the following relations hold:*

a) *The quasi-structures M and M' are h-related (i.e. they satisfy (1)).*
b) *We have*

$$(2)\qquad D^M_X(h^*f') = h^*(D^{M'}_{h_*X}f'), \qquad h_*(D^M_X Y) = D^{M'}_{h_*X}h_*Y,$$

for each $X, Y \in \mathcal{V}(V)$, $f' \in \Lambda^0(V')$.

If we suppose that on the manifold V are given a quasi-structure M and a quasi-derivation D^M, then each diffeomorphism $h: V \to V'$ determines in an unique way a quasi-derivation $D^{M'}$ on V' given by:

$$(2')\qquad \begin{aligned} D^{M'}_{X'}f' &= D^M_{h_*^{-1}X'}(h^*f').h^{-1}, \quad \forall X' \in \mathcal{V}(V'), \quad f' \in \Lambda^0(V'), \\ D^{M'}_{X'}Y' &= h_*(D^M_{h_*^{-1}X'}, h_*^{-1}Y'), \quad \forall X', Y' \in \mathcal{V}(V'). \end{aligned}$$

We shall denote $D^{M'} = h(D^M)$ and we shall say that $D^{M'}$ is the image of D^M with respect to h.

Proposition 2. *The relation of h-correspondence on the set of smooth manifolds with quasi-connections is an equivalence one.*

Definition 3. *A smooth diffeomorphism $h: V \to V$ is called a quasi-affine-transformation with respect to D^M if it is a quasi-affine-collineation and if the quasi-structure M on V is invariant with respect to h (i.e. $Mh_* = h_*M$).*

Then it is not hard to see that:

$$(2'')\qquad D^M_{h_*X}(h^{-1})^*f = h^{*-1}(D^M_X f), \quad D^M_{h_*X}h_*Y = h_*(D^M_X Y).$$

Proposition 3. *Let V, V' be smooth manifolds, $h: V \to V'$ a diffeomorphism, (M_1, M_1'), (M_2, M_2') two pairs of quasi-h-correspondent structures and D^{M_1}, $D^{M_1'}$, D^{M_2}, $D^{M_2'}$, the corresponding quasi-derivatives. If h is a quasi-affine-collineation for the pairs $(D^{M_1}, D^{M_1'})$, $(D^{M_2}, D^{M_2'})$ then it is also a quasi-affine-collineation for the pairs of linear derivations $(\lambda_1 D^{M_1} + \lambda_2 D^{M_2}, \lambda_1 D^{M_1'} + \lambda_2 D^{M_2'})$. In particular it is an affine-collineation for the deformation quasi-connections $(D^{M_2-M_1}, D^{M_2'-M_1'})$.*

It is easy to verify the following properties.

Proposition 4. *Let V and V' be a pair of smooth diffeomorphic manifolds and $h: V \to V'$ a diffeomorphism between them. Let ∇ and ∇' be two derivations such that h is an affine-collineation with respect to them. Let M and $M' = h_*Mh_*^{-1}$ be two quasi-h-correspondent structures. Then the quasi-derivations $D^{M,\nabla}$ and $D^{M'\nabla'}$, are quasi-affine collineations and conversely.*

Indeed, let $\nabla'_{h_*X}h_*Y = h_*(\nabla_X Y)$, $D^{M',\nabla'}_{X'}Y' = \nabla'_{M'X'}Y' = \nabla'_{h_*MX}h_*Y = h_*\nabla_{MX}Y = h_*S^{M,\nabla}_X Y$. In other words, D^M and $D^{M'}$ $(M' = h_*Mh_*^{-1})$ corresponding to ∇, ∇' respectively have the property that h is an affine collineation with respect to them.

Conversely, $\nabla'_{X'}Y' = \nabla'_{h_*X}h_*Y = \nabla'_{(h_*Mh_*^{-1})(h_*Mh_*^{-1})}h_*Y = D^{M',\nabla'}_{(M'^{-1}X)}h_*Y = h_*D^{M,\nabla}_{M^{-1}X}Y = h_*\nabla_X Y$.

In particular, for the affine transformations we have:

$$D^M_{h_*X}h_*Y = h_*D^M_X Y \Leftrightarrow \nabla_{h_*MX}h_*Y = h_*\nabla_{MX}Y.$$

Proposition 5. *Let M, M' be a pair of quasi-structures. Then M is an M-structure if and only if M' has the same property.*

Proposition 6. *The M-torsion and the M-curvature of the M-derivation law $D^{M'}$, which is induced by h on V' can be expressed by the formulae:*

$$(3)\qquad \begin{aligned} \mathcal{T}^{M'}(X', Y') &= h_*\mathcal{T}^M(h_*^{-1}X', h_*^{-1}Y'), \\ \mathcal{R}^{M'}(X', Y')Z' &= h_*\mathcal{R}^M(h_*^{-1}X', h_*^{-1}Y')\, h_*^{-1}Z'. \end{aligned}$$

The proof is a straightforward calculation.

Definition 4. *The M-Lie derivative $L^M_X D^M$ of the operator D^M is given by the formula:*

$$(4)\qquad (L^M_X D^M)_Y = [L^M_X, D^M_Y] - D^M_{M^{-1}[MX, MY]}.$$

Its value on a vector field Z is obtained by:

$$(4')\qquad (L^M_X D^M)_Y Z = M^{-1}[MX, MD^M_Y Z] - D^M_Y(M^{-1}[MX, MZ]) - D^M_{M^{-1}[MX, MY]}Z.$$

Now, let us consider a smooth manifold V with an M-derivation D^M on it.

Definition 5. *The operator*

$$(5)\qquad DL^M_X = L^M_X - L_{MX},$$

is called the deformation of the Lie derivative associated to the M-derivation D^M with respect to the vector field X.

Proposition 7. *The deformation of the Lie derivative is zero if and only if $L_X M = ML_X$.*

Indeed, $L^M_X = L_{MX} \Rightarrow M^{-1}L_{MX}M = L_{MX}$ and conversely.

Proposition 8. *The Lie derivative L_X corresponding to the derivation and the M-Lie derivative D^M are related by the relation:*

$$(6)\qquad (L^M_X D^M)_Y = (L_{MX}\nabla)_{MY} + [DL^M_X, \nabla_{MY}].$$

Proposition 9. *If $L_X M = ML_X$, X a vector field on V, then the M-Lie derivative of the M-connection D^M with respect to X is equal with the Lie derivative of the connection ∇ with respect to the vector field MX.*

Lemma 1. *Let M be a tensor on V. Then there exist in an unique way a vector field X on V such that $ML_X = L_X M$.*

The vector field X is given locally by the equations:

$$M^h_q \frac{\partial X^q}{\partial x^j} - M^q_j \frac{\partial X^h}{\partial x^q} + \frac{\partial M^h_j}{\partial x^p} X^p = 0.$$

Lemma 2. *A tensor field M on V satisfies the commutation condition $(ML_X = L_X M)$ for each X, if and only if $M = kI$ (k = constant).*

The commutation condition can be written $M[X, Y] = [X, MY]$, and it is obvious that $M = kI$ satisfies this relation. Conversely, suppose that the commutation relation is satisfied. Then for each X we have:

$$\frac{\partial M^h_j}{\partial x^p} = 0, \quad M^h_q \delta^q_j - M^q_j \delta^h_q = 0,$$

and then M^i_j is constant and therefore $M = kI$.

Proposition 10. *Let M be a vector field on V which commutes with the Lie derivative L_X and $X'=MX$. Let φ_t be the flow of the vector field X. Then φ_t is a quasi-affine-transformation with respect to an M-linear connection D^M defined on V if and only if we have:*

$$(L^M_{M^{-1}X}D^M)_Y=0. \tag{7}$$

Proof. If $(L^M_{M^{-1}X}D^M)_Y=0$, then by using (6) it follows that $(L_X\nabla)_Y=0$, and then φ_t is an affine transformation; we have: $\varphi_{t*}(\nabla_X Y)=\nabla_{\varphi_{t*}X}\,\varphi_{t*}Y$, and therefore $D^M_{\varphi_{t*}(M^{-1}X)}\varphi_{t*}Y=\varphi_{t*}D^M_{M^{-1}X}Y$. Conversely, let $D^M_{\varphi_{t*}(M^{-1}X)}\varphi_{t*}Y=$ $=\varphi_{t*}D^M_{M^{-1}X}Y=\varphi_{t*}(\nabla_X Y)=\nabla_{\varphi_{t*}X}\varphi_{t*}Y$. Then $(L_X\nabla)_Y=0$ so that $(L^M_{M^{-1}X}D^M)_Y=0$.

Definition 6. *The above vector field X is called an M-infinitesimal affine collineation.*

Proposition 11. *The set of M-infinitesimal affine collineations with respect to an M-connection D^M is an R-subalgebra of $\mathcal{X}(V)$.*

Proof. We have:

$$1^\circ \qquad (L^M_{M^{-1}(k^1X_1+k^2X_2)}D^M)_Y=k^1(L^M_{M^{-1}X_1}D^M)_Y+k^2(L^M_{M^{-1}X_2}D^M)_Y=0,$$

$$2^\circ \qquad (L^M_{M^{-1}[MX,MY]}D^M)_Z=(L_{[X,Y]}\nabla)_Z=0.$$

The equality 2° follows via the property that $[X, Y]$ is an M-infinitesimal affine collineation.

Let D^M be an M-derivation. Then we can construct the operator $A^M_X=$ $=L^M_X-D^M_X$, and one gets:

$$A^M(X, Y)=A^M_X Y=-\mathcal{T}^M(X, Y)-D^M_Y X. \tag{8}$$

It follows that $A^M(X, Y)$ is an $\mathcal{F}$-linear operator with respect to the second variable and A^M_X defines a tensor field of type (1, 1) on V.

Proposition 12. *Let X be a vector field on V. Then it is an M-affine collineation with respect to the M-derivation D^M if and only if we have:*

$$R^M(M^{-1}X, Y)=D^M_Y A^M_{M^{-1}X}. \tag{9}$$

Proof. Let $M^{-1}X=\overline{X}$. Then $R^M(\overline{X}, Y)=[D^M_{\overline{X}}, D^M_Y]-D^M_{M^{-1}[\overline{X},Y]}=$ $=[L^M_{\overline{X}}-A^M_{\overline{X}}, D^M_Y]-D^M_{M^{-1}[M\overline{X},MY]}=[L^M_{\overline{X}}, D^M_Y]-D^M_{M^{-1}[M\overline{X},MY]}-[A^M_{\overline{X}}, D^M_Y]=$ $D^M_Y A^M_{\overline{X}}-A^M_{\overline{X}}A^M_Y=D^M_Y A^M_{\overline{X}}: Z\to(D^M_Y A^M_{M^{-1}X})Z$, and conversely.

Proposition 13. *If the vector fields X and Y generate M-infinitesimal affine collineations then we have:*

$$R^M(M^{-1}X, M^{-1}Y)=[A^M_{M^{-1}X}, A^M_{M^{-1}Y}]-A^M_{M^{-1}[X,Y]}. \tag{10}$$

Proof. $[A^M_{M^{-1}X}, A^M_{M^{-1}Y}]-A^M_{M^{-1}[X,Y]}=[L^M_{M^{-1}X}-D^M_{M^{-1}X}, L^M_{M^{-1}Y}-D^M_{M^{-1}Y}]-$ $-(L^M_{M^{-1}[X,Y]}-D^M_{M^{-1}[X,Y]})=\ldots=R^M(M^{-1}X, M^{-1}Y)$ (with $[L^M_{\overline{X}}, L^M_{\overline{Y}}]=L^M_{M^{-1}[M\overline{X},M\overline{Y}]}$ see [4]).

Conversely, if X (resp. Y) is an M-infinitesimal affine collineation and the relation (10) is satisfied then it is not hard to see that Y (resp. X) has the same property.

2. **Induced quasi-connections on smooth submanifolds.** Let $\tilde V$ be a smooth ($=C^\infty$), n-dimensional manifold and V a p-dimensional submanifold of $\tilde V$. We consider the pair $(\tilde V, V)$ and the following configurations:

1° The tangent bundle $T\tilde V=\bigcup_{x\in V}T_x\tilde V$ of the manifold $\tilde V$,

2° The tangent bundle $TV=\bigcup_{x\in V}T_xV$ of the manifold V,

3° The hypertangent bundle $HV=\bigcup_{x\in V}T_x\tilde V=T\tilde V/V$ of the manifold V,

4° We suppose that the normal bundle $NV=\bigcup_{x\in V}N_xV$, has the property $T_x\tilde V=T_xV\oplus N_xV$. In this case we say that the submanifold V of $\tilde V$ has a normalisation.

The vector bundles TV and NV are sub-vector-bundles of the hypertangent vector bundle $HV=TV\oplus NV$.

We shall say that a field of frames (I_α) of HV is adapted to the pair (TV, NV) if $I_i=\partial/\partial x^i$ is the canonical frame of the manifold V associated to the chart (U,χ) and I_a is a field of frames in NV over U.

Using the embedding of V in $\tilde V$, each vector field $X\in\mathcal{X}(V)$ can be extended in a natural way to a vector field $\tilde X\in\mathcal{X}(\tilde V)$ [2]. Then we can say that HV is the restriction of $T\tilde V$ to V.

Let V be a smooth manifold with a quasi-structure $\tilde M:\mathcal{X}(\tilde V)\to\mathcal{X}(\tilde V)$. We shall say that this quasi-structure is compatible with the submanifold V if and only if for each $X\in\mathcal{X}(V)$ we have $\tilde M\tilde X/V\in\mathcal{X}(V)$. Using the diagram:

$$X\in\mathcal{X}(V)\xrightarrow{\text{prol.}}\tilde X\xrightarrow{\tilde M}\tilde M\tilde X\xrightarrow{\text{rest.}}\tilde M\tilde X/V\in\mathcal{X}(V),$$

it is easy to see that there exists a tensor field:

$$M: X\to MX\overset{\text{def}}{=}\tilde M\tilde X/V,$$

and moreover it is independent of the extension $\tilde X$ of X.

In an adapted frame the matrix of $\tilde M$ has the form

$$\begin{pmatrix} M^j_i & 0\\ M^a_i & M^a_b\end{pmatrix}, \qquad (i, j=\overline{1, p}\,;\ a, b=\overline{p+1, n}),$$

and then we can obtain the matrix (M^j_i) of M in the canonical frame $\partial/\partial x^i$).

The set of quasi-structures which are compatible with V is a vector subspace of $\mathcal{T}^1_1(V)$ for which the map $\tilde M\to M$ is a surjective one.

Proposition 1. *Let $\tilde V$ be a smooth manifold and V a smooth submanifold of $\tilde V$. Let $D^{\tilde M}$ be a quasi-derivation on $\tilde V$. Then its restriction to V is a quasi-derivation on the hypertangent bundle and it is denoted by D^M.*

Indeed, we have the restrictions:

$$(1)\qquad \begin{aligned} &D^M_X f=D^{\tilde M}_{\tilde X}\tilde f/V=\tilde M\tilde X(\tilde f)/V=\tilde MX(f),\ \tilde f \text{ is an extension of } f \text{ to } \tilde V,\\ &D^M_X y=D^{\tilde M}_{\tilde X}y/V.\end{aligned}$$

and then D_X^M can be extended to the tensor algebra of HV. Moreover we have:

$$D_X^M(u\otimes v)=D_X^M u\otimes v+u\otimes D_X^M v.$$

Proposition 2. *Let $D^{\tilde{M}}$ be a quasi-derivation on $\tilde{V}$ and NV the normal vector bundle on V. Then there exist two bilinear maps:*

$$\begin{aligned}\alpha^M &: (X, Y)\in\mathcal{X}(V)\times\mathcal{X}(V)\to\alpha^M(X, Y)\in\mathcal{X}(N),\\ \beta^M &: (X, y)\in\mathcal{X}(V)\times\mathcal{X}(N)\to\beta^M(X, y)\in\mathcal{X}(V),\end{aligned}\tag{2}$$

if and only if the quasi-structure $\tilde{M}$ is compatible with V.

Proof. Let $\tilde{M}$ be a quasi-structure compatible with V and M its restriction to V. The operators D^{hM}, D^{vM}, α^M, β^M, given by:

$$\begin{aligned}&D_X^{hM}f=MX(f)/V,\quad D_X^{vM}(f)=MX(f),\quad D_X^{vM}Y=\mathrm{pr}_{TV}D_X^M Y,\\ &D_X^{vN}y=\mathrm{pr}_{NV}D_X^M y,\quad \alpha^M(X, Y)=\mathrm{pr}_{NV}\, D_X^M Y,\quad \beta^M(X, y)=-\mathrm{pr}_{TV}\, D_X^M y,\end{aligned}\tag{3}$$

satisfy the conditions of the proposition.

Conversely, $D_X^{hN}(fY)=\mathrm{pr}_{TV}(D_X^M fY)=\mathrm{pr}_{TV}(fD_X^M Y+MX(f)\cdot Y)=fD_X^{hM}Y+$ $+MX(f).Y=fD_X^{hM}Y+hMX(f)Y$, and then $hMX=MX$, X, so that $hM=M$ $(MX\in\mathcal{X}(V))$.

Proposition 3. Let $\{\tilde{M}\}$ *be the set of quasi-structures on $\tilde{V}$ which are compatible with V. The maps $D^{\tilde{M}}\to D^M$, $D^M\to D^{hM}$, $D^M\to D^{vM}$, $D^M\to\alpha^M$, $D^M\to$ $\to\beta^M$ are $\mathcal{F}$-linear.*

The quasi-derivations and the bilinear forms on V given by Proposition 2 satisfy by definition the Gauss-Weingarten equations:

$$\begin{aligned}&D_X^M Y=D_X^{hM}Y+\alpha^M(X, Y), &&\text{Gauss equation},\\ &D_X^M y=D_X^{vM}y-\beta^M(X, y), &&\text{Weingarten equation}.\end{aligned}\tag{4}$$

Proposition 4. *Let D^{hM} and D^{vM} be quasi-derivations on TV and NV respectively, and α^M, β^M the corresponding bilinear forms. Then there exists in a canonical way a quasi-derivation D^M on HV.*

Indeed, let $\xi=Y+y$, and $D_X^M\xi=D_X^M Y+D_X^M y=D_X^{hM}Y+D_X^{vM}y+$ $+\alpha^M(X, Y)-\beta^M(X, y)$. Then it is easy to verify that it satisfies the conditions of the Proposition.

α^M is called the second fundamental bilinear form of the triple submanifold, normalization, quasi-structure.

Proposition 5. *Let $\tilde{M}$ be a quasi-structure on $\tilde{V}$, ∇ a derivation on V and D^M its corresponding quasi-derivation. Let ∇^h [resp. D^{hM}] the induced derivation [resp. the induced quasi-derivation] on the submanifold V of $\tilde{V}$. Then D^{hM} is the associated quasi-derivation of the connection ∇^h.*

Indeed, we have $\nabla_X^h Y=\mathrm{pr}_{TV}\nabla_X Y$, $D_X^{hM}Y=\mathrm{pr}_{TV}D_X^M Y$ and $D_X^M Y=\nabla_{MX}Y$, hence $D_X^{hM}Y=\mathrm{pr}_{TV}D_X^M Y=\mathrm{pr}_{TV}\nabla_{MX}Y=\nabla_{MX}^h Y=\nabla_{hMX}^h Y$. In other words the following diagram is commutative:

$$\begin{tikzcd}(M, \nabla) \arrow[r, dashed] \arrow[d, dashed] & D^M \arrow[d, dashed]\\ (hM, \nabla^h) \arrow[r, dashed] & D^{hM}\end{tikzcd}$$

For functions we have: $\nabla_X^h(f)=Xf=\nabla_X(f)$, and $D_X^{hM}(f)=\nabla_{hMX}(f)=$ $=hMX(f)=MX(f)=\nabla_{MX}(f)=D_X^M(f)$.

If M is an M-structure then in the above diagram the orizontal arrows can be reversed.

Proposition 6. *If the quasi-structure M is an M-structure, then hM and vM have the same property.*

Proposition 7. *The operators d^M and L_X^M are invariant under the projection on V: $d^{hM}=d^M$, $L_X^{hM}=L_X^M$, $(\forall NV)$.*

In other words, their projection are identically with their restriction to V.

Proposition 8. *The following relation is valid:*

$$\mathcal{T}^M(X, Y)=\mathcal{T}^{hM}(X, Y)+[\alpha(X, Y)-\alpha(Y, X)].\tag{5}$$

Using (5) it follows that $\mathrm{pr}_{TV}\mathcal{T}^M(X, Y)=\mathcal{T}^{hM}(X, Y)$, $\mathrm{pr}_{NV}\mathcal{T}^M(X, Y)=$ $=\alpha(X, Y)-\alpha(Y, X)=\mathcal{T}^{NV}$. This expression is called the vertical torsion.

Corrollary. *The tensor $\mathcal{T}^M$ is trivial if and only if $\mathcal{T}^{NV}$ and α are trivial.*

Definition 1. *The vector field Y is called M-conjugate with respect to X if $\alpha^M(X, Y)=0$.*

It is easy to see that Y is M-conjugate to X if and only if $D_X^M Y=D_X^{hM}Y$.

Remark. The relation of M-conjugation can not be in generally reversed.

Definition 2. *A vector field X is called M-autoparallel if $D_X^M=0$. If c is a curve of the manifold V it is M-autoparallel if $D_X^M X=0$, where $X=0$. It is called M-asimptotic if $\alpha^M(X, X)$.*

Proposition 9. *A curve c on V which is M-autoparallel with respect to D^M is M-autoparallel with respect to D^{hM} and asimptotic.*

The converse statement is also true.

Proposition 10. *The curvature tensor R^M [resp. R^{hM}] of the derivation law D^M on HV [resp. D^{hM} on V] satisfies the following relation:*

$$R^M(X, Y)Z=R^{hM}(X, Y)Z+D_X^{vM}(\alpha^M(Y, Z))-D_Y^{vM}(\alpha^M(X, Z))+\alpha^M(X, D^h{}_Y Z^M)-$$
$$-\alpha^M(Y, D_X^{hM}Z)-\beta^M(X, \alpha^M(X, Z))-\beta^M(Y, \alpha^M(X, Z))-\alpha^M(M^{-1}[MX, MY], Z).$$

Using the projection we can deduce the following relations:

$$\mathrm{pr}_{TV}R^M(X, Y)Z=R^{hM}(X, Y)Z-[\beta^M(X, \alpha^M(Y, Z))-\beta^M(Y, \alpha^M(X, Z))],$$
$$\mathrm{pr}_{NV}R^M(X, Y)Z=D_X^{vM}(\alpha^M(Y, Z))-D_X^{vM}(\alpha^M(X, Z))+\alpha^M(X, D_Y^{hM}Z)-$$
$$-\alpha^M(Y, D_X^{hM}Z)-\alpha^M(M^{-1}[MX, MY], Z).$$

Remarks: If $R^M(X, Y)Z=0$, then:

1° $R^{hM}(X, Y)Z=\beta^M(X, \alpha^M(Y, Z))-\beta^M(Y, \alpha^M(X, Z))$,

2° $D_X^{vM}(\alpha^M(Y, Z))-D_Y^{vM}(\alpha^M(X, Z))=-[\alpha^M(X, D_Y^{hM}Z)-\alpha^M(Y, D_X^{hM}Z)]+$ $+\alpha^M(M^{-1}[MX, MY], Z)$.

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Department of Mathematics
Faculty of Natural Sciences
University of Timişoara
Timişoara, România