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TANGENCY AND OPENNESS OF MULTIFUNCTIONS IN BANACH SPACES

BY

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Let X and Y be Banach spaces and let F be a multifunction from X into Y. For every point $(x, y) \in X \times Y$, we consider the multifunction $K_F(x, y)$ from X into Y which assigns to every point $u \in X$ the set of all points $v \in Y$ such that

$$\emptyset \neq ((x, y) + (0, r)((u, v) + W)) \cap \text{graph}\,(F),$$

for every neighborhood W of the origin in $X \times Y$ and for every $r > 0$. In other words graph $(K_F(x, y))$ equals $K_{\text{graph}(F)}(x, y)$, the tangent set in the sense of B o u l i g a n d [1, p. 32] and S e v e r i [8, p. 99] to the set graph(F) at the point (x, y). According to the elementary properties of these tangent sets, graph$(K_F(x, y))$ is a closed cone which is not empty if and only if $(x, y) \in$ closure(graph(F)).

Our main results relate openness of the multifunction F at some points $(x, y) \in X \times Y$ to openness of the corresponding multifunctions $K_F(x, y)$ at the origin in $X \times Y$. Recall, on this occasion, some definitions concerning openness of a multifunction F from a topological space X into a topological space Y at a point $(x, y) \in X \times Y$. The multifunction F is said to be open at the point (x, y) if, for every neighborhood U of x, there exists a neighborhood V of y such that $V \subseteq F(U)$. Other variants of this definition make use of the inclusions $V \subseteq$ closure$(F(U))$, $V \cap F(X) \subseteq F(U)$, or $V \cap F(X) \subseteq$ closure $(F(U))$.

Now we return to our initial setting. In the statement below, for every point c of a Banach space and for every $r > 0$, $B(c, r)$ stands for the open ball with centre c and radius r. And, for every subset W of $X \times Y$, we denote

$$Z(W) = \bigcap_{(x, y) \in \text{graph}(F) \cap W} (y + \text{closure}\ (K_F(x, y)\,(X))).$$

Theorem 1. *Let W be an open subset of $X \times Y$ such that the set* graph$(F) \cap$ closure(W) *is closed, and let $\lambda > 0$ such that*

$$B(0, \lambda) \cap K_F(x, y)(X) \subseteq \text{closure}(K_F(x, y)\,(B(0, 1))) \tag{1}$$

for every $(x, y) \in$ graph$(F) \cap W$. *Then*

$$B(y, \lambda\varepsilon) \cap Z(W) \subseteq F(B(x, \varepsilon)) \tag{2}$$

for every $(x, y) \in$ graph$(F) \cap W$ *and for every $\varepsilon > 0$ satisfying*

$$B(x, \varepsilon) \times B(y, \lambda\varepsilon) \subseteq W. \tag{3}$$

Proof. Let $(x, y)\in\mathrm{graph}(F)\cap W$ and $\varepsilon>0$ satisfying (3), and let $z\in B(y, \lambda\varepsilon)\cap Z(W)$. Consider $\mu>1$ such that $\mu||z-y||<\lambda\varepsilon$. Following an idea of Penot [7, p. 30], we endow the linear space $X\times Y$ with the norm

$$(p, q)\in X\times Y\to\max\{\lambda||p||, ||q||\},$$

and apply the variational principle of Ekeland [5, p. 324] to the function

$$(p, q)\in\mathrm{graph}(F)\cap \mathrm{closure}(W)\to\mu||z-q||\in R,$$

in order to obtain a point $(a, b)\in\mathrm{graph}(F)\cap\mathrm{closure}(W)$ such that

$$\mu||z-b||+||(a, b)-(x, y)||\leqslant\mu||z-y||,$$

and such that

$$\mu||z-b||\leqslant\mu||z-q||+||(p, q)-(a, b)||,$$

for every $(p, q)\in\mathrm{graph}(F)\cap\mathrm{closure}(W)$. Note that

$$||(a, b)-(x, y)||<\lambda\varepsilon,$$

hence $a\in B(x, \varepsilon)$, $b\in B(y, \lambda\varepsilon)$, $(a, b)\in W$,

$$z\in b+\mathrm{closure}(K_F(a, b)(X)),$$

and, according to (1),

$$B(0, \lambda)\cap K_F(a, b)(X)\subseteq\mathrm{closure}(K_F(a, b)(B(0, 1))).$$

Now we shall show that $z=b$. Denote $v=z-b$. Consider a sequence $(r_n)_{n\in N}$ in $(0, +\infty)$ such that it converges to 0. Then, for every $n\in N$,

$$v\in B(0, ||v||+r_n)\cap\mathrm{closure}(K_F(a, b)(X))\subseteq\mathrm{closure}(B(0, ||v||+r_n)\cap$$
$$\cap K_F(a, b)(X))\subseteq\mathrm{closure}(K_F(a, b)(B(0, (||v||+r_n)/\lambda))).$$

Consider a sequence $(u_n, v_n)_{n\in N}$ in $X\times Y$ such that $(v_n)_{n\in N}$ converges to v and such that, for every $n\in N$,

$$v_n\in K_F(a, b)(u_n),\ \text{and}\ u_n\in B(0, (||v||+r_n)/\lambda).$$

Consider, further, a sequence $(s_n, p_n, q_n)_{n\in N}$ in $(0, +\infty)\times X\times Y$ such that it converges to the origin in $R\times X\times Y$ and such that, for every $n\in N$,

$$(a, b)+s_n((u_n, v_n)+(p_n, q_n))\in\mathrm{graph}(F).$$

We may suppose, by passing to subsequences, that, for every $n\in N$, $s_n\in(0, 1)$ and

$$(a, b)+s_n((u_n, v_n)+(p_n, q_n))\in W.$$

Since, for every $n\in N$,

$$\mu||v||\leqslant\mu||s_n(v_n+q_n)-v||+s_n||(u_n, v_n)+(p_n, q_n)||\leqslant\mu((1-s_n)||v||+s_n||v_n+$$
$$+q_n-v||)+s_n||(u_n, v_n)+(p_n, q_n)||$$

and

$$\mu||v||\leqslant\mu||v_n+q_n-v||+\max\{||v||+r_n, ||v_n||\}+||(p_n, q_n)||,$$

we conclude $\mu||v||\leqslant||v||$, $v=0$, $z=b\in F(a)\subseteq F(B(x, \varepsilon))$, (2) holds, and the theorem is proved.

Next we discuss two particular cases, $Z(W)=Y$ and $F(X)\subseteq Z(W)$. The first case, $Z(W)=Y$, holds if and only if

$$\mathrm{closure}(K_F(x, y)(X))=Y \tag{4}$$

for every $(x, y)\in\mathrm{graph}(F)\cap W$. Note that (1) and (4) is equivalent to

$$B(0, \lambda)\subseteq\mathrm{closure}(K_F(x, y)(B(0, 1))). \tag{5}$$

In that case, $Z(W)=Y$, (2) is equivalent to

$$B(y, \lambda\varepsilon)\subseteq F(B(x, \varepsilon)), \tag{6}$$

and we obtain a first corollary to Theorem 1.

Corollary 1. *Let W be an open subset of $X\times Y$ such that the set* $\mathrm{graph}(F)\cap\mathrm{closure}(W)$ *is closed, and let $\lambda>0$ such that* (5) *holds for every* $(x, y)\in\mathrm{graph}F)\cap W$. *Then* (6) *holds for every* $(x, y)\in\mathrm{graph}(F)\cap W$ *and for every* $\varepsilon>0$ *satisfying* (3).

Corollary 1 improves a result of Penot [7, p. 30] Where $W=U\times Y$ for some open subset U of X. The second case, $F(X)\subseteq Z(W)$, holds if and only if

$$F(X)\subseteq\mathrm{closure}(K_F(x, y)(X)) \tag{7}$$

for every $(x, y)\in\mathrm{graph}(F)\cap W$. Note that (7) is equivalent to

$$\mathrm{closure}\ (K_F(x, y)(X))=K_{F(X)}(y), \tag{8}$$

and

$$F(X)\subseteq y+K_{F(X)}(y). \tag{9}$$

In that case, $F(X)\subseteq Z(W)$, (2) implies

$$B(y, \lambda\varepsilon)\cap F(X)\subseteq F(B(x, \varepsilon)), \tag{10}$$

and we obtain a second corollary to Theorem 1.

Corollary 2. *Let W be an open subset of $X\times Y$ such that the set* $\mathrm{graph}(F)\cap\mathrm{closure}(W)$ *is closed and such that* (8) *and* (9) *hold for every* $(x, y)\in\mathrm{graph}(F)\cap W$, *and let $\lambda>0$ such that* (1) *holds for every* $(x, y)\in\mathrm{graph}(F)\cap W$. *Then* (10) *holds for every* $(x, y)\in\mathrm{graph}(F)\cap W$ *and for every $\varepsilon>0$ satisfying* (3).

Before, we continue, we observe that (7), hence (8) and (9), holds for every $(x, y)\in\mathrm{graph}(F)$ if the set $\mathrm{graph}(F)$ is convex since, according to the properties of tangent sets to convex sets,

$$\mathrm{graph}(F)\subseteq(x, y)+K_{\mathrm{graph}\,(F)}(x, y),$$

for every $(x, y)\in\mathrm{closure}\,(\mathrm{graph}(F))$. Next we prove a certain converse of Theorem 1 in case the space X is finite dimensional.

Theorem 2. *Let the space X be finite dimensional, let $(x, y)\in X\times Y$, and let $Z\subseteq Y$ and $\lambda>0$ such that there exists $\delta>0$ such that*

(11) $$B(y,\lambda\varepsilon)\cap Z\subseteq \mathrm{closure}\,(F(B(x,\varepsilon))),$$

for every $\varepsilon\in(0,\delta)$. *Then*

(12) $$B(0,\lambda)\cap K_Z(y)\subseteq K_F(x,y)(B(0,1)).$$

Proof. Let $v\in B(0,\lambda)\cap K_Z(y)$, and consider a sequence $(s_n,q_n)_{n\in N}$ in $(0,+\infty)\times Y$ such that it converges to the origin in $R\times Y$ and such that $y+s_n(v+q_n)\in Z$ for every $n\in N$. We may suppose, by passing to subsequences, that $v+q_n\in B(0,\lambda)$ and

$$B(y,\lambda s_n)\cap Z\subseteq \mathrm{closure}(F(B(x,s_n))),$$

for every $n\in N$. Then

$$y+s_n(v+q_n)\in \mathrm{closure}(F(B(x,s_n))),$$

for every $n\in N$, and we get a sequence $(u_n,\hat{q}_n)_{n\in N}$ in $X\times Y$ such that $(\hat{q}_n)_{n\in N}$ converges to the origin in Y and such that

$$y+s_n(v+q_n+\hat{q}_n)\in F(x+s_nu_n)$$

and $u_n\in B(0,1)$ for every $n\in N$. We may suppose, by passing to subsequences again, that $(u_n)_{n\in N}$ converges. Denote by u its limit. Then $v\in K_F(x,y)(u)$,

$$B(0,\lambda)\cap K_Z(y)\subseteq K_F(x,y)(\mathrm{closure}(B(0,1))),$$

and the conclusion follows since, for every $\mu\in(0,1)$,

$$B(0,\lambda\mu)\cap K_Z(y)\subseteq K_F(x,y)(\mathrm{closure}(B(0,\mu)))\subseteq K_F(x,y)(B(0,1)).$$

Again we discuss two particular cases, $Z=Y$ and $Z=F(X)$.

In the first case, $Z=Y$, (11) becomes

(13) $$B(y,\lambda\varepsilon)\subseteq \mathrm{closure}(F(B(x,\varepsilon))),$$

and (12) becomes

(14) $$B(0,\lambda)\subseteq K_F(x,y)(B(0,1)),$$

so we obtain a first corollary to Theorem 2.

Corollary 3. *Let the space* X *be finite dimensional, and let* $\lambda>0$ *such that there exists* $\delta>0$ *such that* (13) *holds for every* $\varepsilon\in(0,\delta)$. *Then* (14) *holds.*

In the second case, $Z=F(X)$, (11) becomes

(15) $$B(y,\lambda\varepsilon)\cap F(X)\subseteq \mathrm{closure}(F(B(x,\varepsilon))),$$

and (12) implies

(16) $$B(0,\lambda)\cap K_F(x,y)(X)\subseteq K_F(x,y)(B(0,1)),$$

since $K_F(x,y)(X)\subseteq K_{F(X)}(y)$, so we obtain a second corollary to Theorem 2.

Corollary 4. *Let the space* X *be finite dimensional, and let* $\lambda>0$ *such that there exists* $\delta>0$ *such that* (15) *holds for every* $\varepsilon\in(0,\delta)$. *Then* (16) *holds.*

Summing up, if the space X is finite dimensional, W is an open subset of $X\times Y$ such that the set $\mathrm{graph}(F)\cap\mathrm{closure}(W)$ is closed, and $\lambda>0$, then the following four conditions are equivalent:

(5) holds for every $(x,y)\in\mathrm{graph}(F)\cap W$;

(6) holds for every $(x,y)\in\mathrm{graph}(F)\cap W$ and for every $\varepsilon>0$ satisfying (3);

(13) holds for every $(x,y)\in\mathrm{graph}(F)\cap W$ and for every $\varepsilon>0$ satisfying (3);

(14) holds for every $(x,y)\in\mathrm{graph}(F)\cap W$.

Moreover, if (8) and (9) hold for every $(x,y)\in\mathrm{graph}(F)\cap W$, then the following four conditions are equivalent too:

(1) holds for every $(x,y)\in\mathrm{graph}(F)\cap W$;

(10) holds for every $(x,y)\in\mathrm{graph}(F)\cap W$ and for every $\varepsilon>0$ satisfying (3);

(15) holds for every $(x,y)\in\mathrm{graph}(F)\cap W$ and for every $\varepsilon>0$ satisfying (3);

(16) holds for every $(x,y)\in\mathrm{graph}(F)\cap W$.

To close this paper we transpose Theorems 1 and 2 in the setting of functions and of sets.

Let, first, f be a function from a subset of X into Y. For every point $x\in\mathrm{domain}(f)$, we consider the multifunction $K_f(x)$ from X into Y which assigns to every point $u\in X$ the set of all points $v\in Y$ such that

$$\varnothing\neq((x,f(x))+(0,r)((u,v)+W))\cap\mathrm{graph}(f)$$

for every neighborhood W of the origin in $X\times Y$ and for every $r>0$. In other words $\mathrm{graph}(K_f(x))$ equals $K_{\mathrm{graph}(f)}(x,f(x))$, the tangent set in the sense of Bouligand and Severi to the set $\mathrm{graph}(f)$ at the point $(x,f(x))$. The multifunction $K_f(x)$ may be easily calculated in case the function f is Severi differentiable at the point x (see Severi [9, p. 10], and [10, p. 200]) since the graph of the Severi differential of f at x equals $K_{\mathrm{graph}(f)}(x,f(x))$ too (see [10, p. 201]). A similar result has been proved earlier by F l e t t [6, p. 525] in the particular case when x is an interior point of the set $\mathrm{domain}(f)$ and f is a Fréchet differentiable function at the point x.

In the following, for every subset U of X, we denote

$$Z(U)=\bigcap_{u\in\mathrm{domain}(f)\cap U}(f(x)+\mathrm{closure}(K_f(x)(X))).$$

Theorem 3. *Let* U *be an open subset of* X *such that the restriction of the function* f *to the set* domain $(f)\cap$ closure(U) *has a closed graph, and let* $\lambda>0$ *such that*

$$B(0,\lambda)\cap K_f(x)(X)\subseteq \mathrm{closure}(K_f(x)(B(0,1)))$$

for every $x\in\mathrm{domain}(f)\cap U$. *Then*

$$B(f(x),\lambda\varepsilon)\cap Z(U)\subseteq f(\mathrm{domain}(f)\cap B(x,\varepsilon))$$

for every $x\in\mathrm{domain}(f)\cap U$ *and for every* $\varepsilon>0$ *satisfying*

$$B(x,\varepsilon)\subseteq U.$$

Theorem 4. *Let the space* X *be finite dimensional, let* $x\in\mathrm{domain}(f)$, *and let* $Z\subseteq Y$ *and* $\lambda>0$ *such that there exists* $\delta>0$ *such that*

$$B(f(x),\lambda\varepsilon)\cap Z\subseteq \mathrm{closure}(f(\mathrm{domain}(f)\cap B(x,\varepsilon)))$$

for every $\varepsilon\in(0,\delta)$. *Then*

$$B(0,\lambda)\cap K_Z(y)\subseteq K_f(x)(B(0,1)).$$

Proof of Theorems 3 and 4. Take $F(x)=\{f(x)\}$ for every $x\in\text{domain}(f)$, $F(x)=\varnothing$ for every $x\notin\text{domain}(f)$, $W=U\times Y$, and apply Theorems 1 and 2.

Let, finally, S be a subset of Y. For every subset V of Y, we denote

$$Z(V)=\bigcap_{y\in S\cap V}(y+K_S(y)).$$

Theorem 5. *Let V be an open subset of Y such that the set $S\cap\text{closure}(V)$ is closed. Then*

$$B(y,\varepsilon)\cap Z(V)\subseteq S,$$

for every $y\in S\cap V$ and for every $\varepsilon>0$ satisfying

$$B(y,\varepsilon)\subseteq V.$$

Theorem 6. *Let $y\in Y$, and let $Z\subseteq Y$ and $\lambda>0$ such that*

$$B(y,\lambda)\cap Z\subseteq\text{closure}(S).$$

Then

$$K_Z(y)\subseteq K_S(y).$$

Proof of Theorems 5 and 6. Take $X=R$, $F(x)=S$ for every $x\in X$, $W=X\times V$, and apply Theorems 1 and 2.

Theorem 3 improves a result of Cramer and Ray [2, p. 41] where f is defined and Gâteaux differentiable on X, since the graph of the Gâteaux differential of f at a point x is a subset of $K_{\text{graph}(f)}(x,f(x))$. Theorem 5 contains a result in [11, p. 183] which deals with locally closed sets and which we have proved by using the drop theorem of Daneš [3, p. 371]. The fact is that the drop theorem of Danes and the variational principle of Ekeland are equivalent (see Danes [4, p. 453]). Theorem 6 is elementary in the theory of tangent sets.

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ON GENERALIZED p-ORTHOGONAL PROJECTIONS ON BANACH SPACES

BY

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1. Introduction. Various notions of projection operators on Banach spaces have been introduced and developed by Berkson [3], Kalton [21], Ando [1], Papini [30], Phadke [29] and others. Here the notion of generalized p-orthogonal ($1<p<\infty$) projections on Banach spaces is introduced and studied using generalized semiinner product techniques. It is observed that the class of idempotent self adjoint (adjoint abelian) operators is too narrow to furnish an interesting generalization of the class of orthogonal projections from Hilbert to Banach spaces. This defect is somewhat overcome by the introduction of the class of generalized p-orthogonal projections. Further, the different classes of projections on Banach spaces, introduced by various authors, mentioned above, ultimately reduce to the class of orthogonal projections on Hilbert spaces. But the same is not true (as shown in section 3) for the class of generalized p-orthogonal ($1<p<\infty$, $p\neq 2$) projections.

In section 3, the main results are the following:

1. The generalized p-orthogonal projection operators on an l_p-sum ($1<p<\infty$, $p\neq 2$) of a sequence of Banach spaces are precisely the direct sums of generalized p-orthogonal projection operators on the summands.

2. There exists no non-trivial generalized p_1-orthogonal projection on an l_p-sum ($1<p$, $p_1<\infty$, $p\neq 2$, $p\neq p_1$) of a sequence of smooth Banach spaces.

In section 4, the characterization of generalized p-orthogonal projections on $L^p(\Omega,\Sigma,\mu)$, $1<p<\infty$, $p\neq 2$, on a localizable measure space, is given. In this context, the characterization of Hermiteian operators on

$$L^p(\Omega,\Sigma,\mu),\ 1\leqslant p<\infty,\ p\neq 2,$$

on a localizable measure space, is obtained.

In section 5, it is shown that there exists no non-trivial generalized p-orthogonal projection on some concrete Banach spaces.

In section 6, some properties of generalized p-orthogonal projections are studied.

Some queries are placed to the readers at appropriate places.

Terminology and Notation. For literature on s.i.p. and g.s.i.p. we refer to Lumer [24], Giles [16], Nath [28], Miličić [26] and Sen ([33], [35], [37]). For literature on complex strict convexity, the reader is referred to Throp [41], Istrăţescu ([18], [19]), Miličić [27],