Proof of Theorems 3 and 4. Take $F(x)=\{f(x)\}$ for every $x\in \mathrm{domain}(f)$, $F(x)=\varnothing$ for every $x\notin \mathrm{domain}(f)$, $W=U\times Y$, and apply Theorems 1 and 2.

Let, finally, S be a subset of Y. For every subset V of Y, we denote

$$Z(V)=\bigcap_{y\in S\cap V}(y+K_S(y)).$$

Theorem 5. *Let V be an open subset of Y such that the set $S\cap \mathrm{closure}(V)$ is closed. Then*

$$B(y,\varepsilon)\cap Z(V)\subseteq S,$$

for every $y\in S\cap V$ and for every $\varepsilon>0$ satisfying

$$B(y,\varepsilon)\subseteq V.$$

Theorem 6. *Let $y\in Y$, and let $Z\subseteq Y$ and $\lambda>0$ such that*

$$B(y,\lambda)\cap Z\subseteq \mathrm{closure}(S).$$

Then

$$K_Z(y)\subseteq K_S(y).$$

Proof of Theorems 5 and 6. Take $X=R$, $F(x)=S$ for every $x\in X$, $W=X\times V$, and apply Theorems 1 and 2.

Theorem 3 improves a result of Cramer and Ray [2, p. 41] where f is defined and Gâteaux differentiable on X, since the graph of the Gâteaux differential of f at a point x is a subset of $K_{\mathrm{graph}(f)}(x,f(x))$. Theorem 5 contains a result in [11, p. 183] which deals with locally closed sets and which we have proved by using the drop theorem of Daneš [3, p. 371]. The fact is that the drop theorem of Daneš and the variational principle of Ekeland are equivalent (see Daneš [4, p. 453]). Theorem 6 is elementary in the theory of tangent sets.

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ON GENERALIZED p-ORTHOGONAL PROJECTIONS ON BANACH SPACES

BY

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1. **Introduction.** Various notions of projection operators on Banach spaces have been introduced and developed by Berkson [3], Kalton [21], Ando [1], Papini [30], Phadke [29] and others. Here the notion of generalized p-orthogonal ($1<p<\infty$) projections on Banach spaces is introduced and studied using generalized semiinner product techniques. It is observed that the class of idempotent self adjoint (adjoint abelian) operators is too narrow to furnish an interesting generalization of the class of orthogonal projections from Hilbert to Banach spaces. This defect is somewhat overcome by the introduction of the class of generalized p-orthogonal projections. Further, the different classes of projections on Banach spaces, introduced by various authors, mentioned above, ultimately reduce to the class of orthogonal projections on Hilbert spaces. But the same is not true (as shown in section 3) for the class of generalized p-orthogonal ($1<p<\infty$, $p\neq 2$) projections.

In section 3, the main results are the following:

1. The generalized p-orthogonal projection operators on an l_p-sum ($1<p<\infty$, $p\neq 2$) of a sequence of Banach spaces are precisely the direct sums of generalized p-orthogonal projection operators on the summands.

2. There exists no non-trivial generalized p_1-orthogonal projection on an l_p-sum ($1<p$, $p_1<\infty$, $p\neq 2$, $p\neq p_1$) of a sequence of smooth Banach spaces.

In section 4, the characterization of generalized p-orthogonal projections on $L^p(\Omega,\Sigma,\mu)$, $1<p<\infty$, $p\neq 2$, on a localizable measure space, is given. In this context, the characterization of Hermiteian operators on

$$L^p(\Omega,\Sigma,\mu),\quad 1\leqslant p<\infty,\ p\neq 2,$$

on a localizable measure space, is obtained.

In section 5, it is shown that there exists no non-trivial generalized p-orthogonal projection on some concrete Banach spaces.

In section 6, some properties of generalized p-orthogonal projections are studied.

Some queries are placed to the readers at appropriate places.

Terminology and Notation. For literature on s.i.p. and g.s.i.p. we refer to Lumer [24], Giles [16], Nath [28], Miličič [26] and Sen ([33], [35], [37]). For literature on complex strict convexity, the reader is referred to Thorp [41], Istrăţescu ([18], [19]), Miličič [27], Jamison [20] and Sen [38]. By X we mean a complex Banach space unless otherwise stated. By an operator on X we mean a bounded linear transformation from X into X. The notations $R(T)$, $N(T)$, $\sigma(T)$, $\sigma_r(T)$ and $\sigma_p(T)$ will denote, respectively, the range, kernel, spectrum, residual spectrum and point spectrum of T. By a positive operator on Banach spaces we mean an operator with non-negative numerical range (Sen [34]). An element x is said to be orthogonal to y with respect to a g.s.i.p. $[.]$ iff $[y, x]=0$.

2. Generalized p-orthogonal projections

Definition 2.1. *Let X be a complex (real) vector space. We recall that a complex (real) p-generalized semiinner product (in short, p-g.s.i.p), $1<p<\infty$, is defined on X, if to any $x, y\in X$, there corresponds a complex (real) number $[x, y]_p$ satisfying the following properties: for any $x, y, z\in X$ and λ complex (real):*

(i)
$$[x+y, z]_p=[x, z]_p+[y, z]_p.$$

(ii)
$$[x, x]_p>0 \text{ for } x\neq 0.$$

(iii)
$$[\lambda x, y]_p=\lambda\cdot[x, y]_p,\quad [x, \lambda y]_p=|\lambda|^{p-2}\cdot\bar{\lambda}\cdot[x, y]_p.$$

(iv)
$$|[x, y]_p|\leq[x, x]_p^{1/p}\cdot[y, y]_p^{(p-1)/p}.$$

Then X is called a p-generalized semiinner product space (in short, p-g.s.i.p.s.). For $p=2$, p-generalized semiinner product reduces to semiinner product.

Note 2.2. A p-g.s.i.p. space is a normed space with the norm $[x, x]_p^{1/p}$.

For each normed space $(X, \|\cdot\|)$ and each $p\in(1, \infty)$ there exists a p-g.s.i.p. $[.]_p$ such that it is compatible with the norm, i.e. $[x, x]_p^{1/p}=\|x\|$, $\forall x\in X$. For a smooth Banach space, p-g.s.i.p for each p, is unique. Throughout the paper we assume that all p-g.s.i.p are compatible with the norm.

If $[.]_p$ is any p-g.s.i.p. on X then the relation, $[x, y]_{p'}=\|y\|^{p'-p}\cdot[x, y]_p$, $\forall x, y\in X$, $y\neq 0$, defines a p'-g.s.i.p on X.

Definition 2.3. *An operator T on X is said to be generalized p-selfadjoint (or just p-selfadjoint), in short p-s.a., iff there exists a p-g.s.i.p. $[.]_p$ on X such that $[Tx, y]_p=[x, Ty]_p$, $\forall x, y\in X$.*

Note 2.4. A notion of generalized p-self adjoint operator on a smooth, strictly convex and reflexive Banach space, is defined and studied in Sen [37].

Definition 2.5. *An operator P on X is said to be generalized p-orthogonal (or just p-orthogonal) projection iff $P^2=P$ and P is p-self adjoint.*

Note 2.6. These operators will be termed „orthogonal projections" when $p=2$. The operators O and I will be called trivial p-orthogonal projections.

Note 2.7. Orthogonal projections on X are idempotent adjoint abelian operators in the sense of Stampfli [39] and Fleming [10].

Note 2.8. In non-smooth Banach spaces, it would be interesting to know whether an operator T may be p-selfadjoint or p-orthogonal projection with respect to two distinct p-g.s.i.p.

The following result is straight forward.

Proposition 2.9. *Every p-orthogonal projection P on X is positive and $R(P)$ and $N(P)$ are mutually orthogonal (with respect to any p-g.s.i.p. associated with P) closed subspaces of X such that $R(P)\oplus N(P)=X$.*

Note 2.10. If $P^2=P$ and $R(P)$ and $N(P)$ are mutually orthogonal with respect to some g.s.i.p. $[.]$, then, it would be interesting to know whether P is p-orthogonal.

3. Characterization of p-orthogonal projections on an l_p-sum of Hilbert spaces. We now give a characterization of p-orthogonal projections on an l_p-sum $(1<p<\infty,\ p\neq 2)$ of Hilbert spaces. Before proving this result, we first show that there exists no non-trivial p-orthogonal $(p\neq 2)$ projection on Hilbert spaces.

Proposition 3.1. *Let T be a p-selfadjoint operator $(p\neq 2)$ on X. Then $\sigma_p(T)\subset\{-1, 0, 1\}$ and the eigenvectors corresponding to distinct eigenvalues are orthogonal with respect to any p-g.s.i.p. associated with T.*

Proof. The proof is straight forward.

Theorem 3.2. *Let X be a smooth, strictly convex and reflexive Banach space. If T is a p-selfadjoint operator on X then $\sigma_r(T)$ is empty.*

Proof: Case I. $p=2$. The result is in Theorem 2.20 of Sen [37].

Case II. $p\neq 2$. Let $\lambda\in\sigma_r(T)$ then the range of $(T-\lambda I)$ is not dense in X and by proposition 2 of Sen [33], there exists an element $y\neq 0$ such that

$$[(T-\lambda I)x, y]_p=0,\ \forall x\in X \text{ or } [x, ky]_p=[x, Ty]_p,$$

where $|k|^{p-2}\cdot\bar{k}=\lambda$. Then by Theorem 4 of Miličić [26], $Ty=ky$. As $y\neq 0$, $k\in\sigma_p(T)$. By our proposition 3.1, $k=\lambda$ and so a contradiction.

Note 3.3. The above Theorem 3.2 answers the query raised in Note 2.21 of Sen [37].

Lemma 3.4. (a) *Let X be a smooth Banach space. Then $T\in L(X)$ is p-self adjoint for $p=p_1, p_2\in(1, \infty)$, $p_1\neq p_2$ iff $T=0$ or T is 2-s.a. and $T^2=I$.*

(b) *Let X be a Hilbert space. Then $T\in L(X)$ is p-s.a. with $p\in(1,\infty)\setminus\{2\}$ iff $T=0$ or $T=T^*=T^{-1}$.*

Proof. The sufficiency of (a) and (b) is obvious. (a) „$\Rightarrow$". Let $T\neq 0$. We take $Ty\neq 0$; for $\bar{x}=Ty$ we have $0\neq[\bar{x}, Ty]_{p_i}=[T\bar{x}, y]_{p_i}$ and so $\|y\|^{p_2-p_1}[T\bar{x}, y]_{p_1}=\|Ty\|^{p_2-p_1}[\bar{x}, Ty]_{p_1}$. Hence $\|Ty\|=\|y\|$. We take now $x\in N(T)$. Then $T(y+tx)\neq 0$ and so, as above, $\|y+tx\|=\|T(y+tx)\|=\|Ty\|=\|y\|$ for all $t\in C$. Therefore $x=0$. It follows that $\|Tx\|=\|x\|$, $\forall x\in X$ if $T\neq 0$.

Hence T is 2-s.a. and $[(T^2-I)x, x]_2=0, \forall x\in X$. Application of Theorem 5 of Lumer [24] shows that $T^2=I$ and so $T=T^{-1}$.

(b) „$\Rightarrow$" As $\|y\|^{p-2}\cdot\langle Tx, y\rangle=\|Ty\|^{p-2}\cdot\langle x, Ty\rangle$ we obtain, $\|x\|^{p-2}\cdot\langle Tx, x\rangle=\|Tx\|^{p-2}\overline{\langle Tx, x\rangle}$, and so $\langle Tx, x\rangle\in R$ $\forall x\in X$. If $\langle Tx, x\rangle=0$ for all $x\in X$ then $T=0$. Suppose that $T\neq 0$ and we take $\langle T\bar{x}, \bar{x}\rangle\neq 0$. From the above relations we have that $\|T\bar{x}\|=\|\bar{x}\|$ and $\langle Tx, \bar{x}\rangle=\langle x, T\bar{x}\rangle\Leftrightarrow\langle\bar{x}, Tx\rangle=\langle T\bar{x}, x\rangle$, $\forall x\in X$. As $\|x\|^{p-2}\langle T\bar{x}, x\rangle=\|Tx\|^{p-2}\langle\bar{x}, Tx\rangle$, it follows that $\|Tx\|=\|x\|$ for $x\in\{u\,|\,\langle u, Tx\rangle\neq 0\}$. As this set is dense in X, $\|Tx\|=\|x\|$ for all $x\in X$. Therefore $T=T^*=T^{-1}$.

Note 3.5. The Lemma 3.4 above answers the query raised in Note 3.5 of Sen [37].

Note 3.6. It would be interesting to know whether "smoothness condition" in Lemma 3.4 above can be dropped.

Corollary 3.7. *If S_p denotes the set of all non-trivial p-orthogonal projections on a smooth Banach space X, then $S_{p_1}\cap S_{p_2}=\varnothing$ (empty set) for $p_1, p_2\in(1, \infty)$, $p_1\neq p_2$.*

Corollary 3.8. *Let X be smooth. If T is p-s.a. for $p=p_1, p_2 \in (1, \infty)$, $p_1 \neq p_2$ and positive then $T=I$.*

Proof. From Lemma 3.4 above, $T^2=I$. As T is positive, by Theorem 3 of Foguel [15, section 5], $T=I$.

Corollary 3.9. (a) *Let X be a smooth, strictly convex and reflexive Banach space. Suppose $T(\neq 0, I)$ is p-s.a. for $p=p_1, p_2$, $p_1 \neq p_2$.*

(i) *If $0 \in \sigma(T)$ then it must be an eigenvalue.*

(ii) *T can not be positive.*

(b) *p-s.a. ($p \neq 2$) operators ($\neq 0, I$) on Hilbert spaces can not be positive.*

Proof. The results follow from Theorem 3.2, Lemma 3.4 and Corollary 3.8 above.

Theorem 3.10. *In Hilbert spaces, there does not exist any non trivial p-orthogonal ($p \neq 2$) projection.*

Proof. The result follows from Lemma 3.4 above.

Theorem 3.11. *If X is an l_p-sum $(1<p<\infty)$, $p \neq 2)$ of a sequence of Banach spaces $\{B_i\}$, then T is a p-orthogonal projection on X iff $T(B_i) \subseteq B_i$ and the restriction of T on B_i is a p-orthogonal projection on B_i.*

Proof. First let T be a p-orthogonal projection on X. Then an application of Theorem 2.1 of Berkson [4], yields the desired result. Conversely, suppose the condition is satisfied. As T_i is a p-orthogonal projection on B_i then there exists a p-g.s.i.p. $[,]_i$ on B_i such that $[T_i x_i, y_i]_i = [x_i, T_i y_i]_i$, $\forall x_i, y_i \in B_i$. Now for $x=\{x_i\}$, $y=\{y_i\}$, $x_i, y_i \in B_i$, the series $\sum_{i=1}^{\infty} [x_i, y_i]_i$ is absolutely convergent since, $\sum_{i=1}^{\infty} |[x_i, y_i]_i| \leqslant \sum_{i=1}^{\infty} \|x_i\| \cdot \|y_i\|^{p-1} \leqslant \left(\sum_{i=1}^{\infty} \|x_i\|^p\right)^{1/p} \cdot \left(\sum_{i=1}^{\infty} \|y_i\|^{(p-1)q}\right)^{1/q} = \|x\| \cdot \|y\|^{p/q} = \|x\| \cdot \|y\|^{p-1}$ [by Holder's inequality].

Therefore it can be easily verified that $[x, y] = \sum_{i=1}^{\infty} [x_i, y_i]_i$ defines a p-g.s.i.p. on X, compatible with the norm. The rest of the proof is straightforward.

Theorem 3.12. *Let X be an l_p-sum $(1<p<\infty, p \neq 2)$ of a sequence of Hilbert spaces $\{H_i\}$. Then an operator T on X is p-orthogonal iff $T=0$ or I on each H_i. In particular, an operator P on l_p $(1<p<\infty, p\neq 2)$ is p-orthogonal iff there exists a sequence $\{\alpha_i\}$ of 0 and 1 such that $Px=\{\alpha_i x_i\}$ for $x=\{x_i\}$.*

Proof. The result follows from Theorems 3.10, 3.11.

Theorem 3.13. *There exists no non-trivial p_1-orthogonal projection on an l_p-sum $(1<p, p_1<\infty, p\neq 2, p \neq p_1)$ of a sequence of smooth Banach spaces.*

Proof. Let $\{B_i\}$ be a sequence of smooth Banach spaces. Suppose $X = l_p$-sum of $\{B_i\}$. If possible let T be a non-trivial p_1-orthogonal projection on X. Then by Theorem 2.1 of Berkson [4], $T(B_i) \subseteq B_i$. Let the restriction of T on B_i be T_i. Then each T_i is p_1-orthogonal on B_i. Now applying the same proof as in Theorem 3.11, we get that

$$[x, y] = \|y\|^{p_1-p} \cdot \sum_{i=1}^{\infty} [x_i, y_i]_i \cdot \|y_i\|^{p-p_1}$$

is the unique p_1-g.s.i.p. on X, where $x=\{x_i\}$, $y=\{y_i\}$ and $[,]_i$ is the p_1-g.s.i.p. on B_i. To prove our result it suffices to consider the following cases.

Case I. $T_i \neq 0, I$ for all i. Let $x_1 \in R(T_1)$, $x_2 \in R(T_2)$, $y_2 \in N(T_2)$, $x_1, x_2, y_2 \neq 0$. Then $[x_2, y_2]_2 = 0$. Suppose $x=\{x_i\}$, $x_i=0$ for $i \neq 1, 2$, $y=\{y_i\}$, $y_i=0$ for $i \neq 1, 2$, $y_1 = x_1$.

Using $[Tx, y]_{p_1} = [x, Ty]_{p_1}$, we get $y_2=0$, a contradiction.

Case II. For some $i=k, l$, $T_k \neq 0, I$, $T_l = 0$ or I.

Subcase (a): $T_l = 0$. Let $x_k \in R(T_k)$, $x_k \neq 0$.

Suppose, $x=\{x_i\}$, $x_i=0$ for $i \neq k, l$. Then $Tx=\{x_i'\}$, $x_i'=0$ for $i \neq k$. As $[Tx, x] = \|Tx\|^{p_1}$, we get $x_l=0$, a contradiction.

(b) $T_l = I$. Let $x_k \in R(T_k)$, $y_k \in N(T_k)$, $x_k, y_k \neq 0$, then $[x_k, y_k]_k = 0$. Suppose, $x=\{x_i\}$, $x_i=0$ for $i \neq l, k$, $y=\{y_i\}$, $y_i=0$ for $i\neq l, k$, $y_l = x_l \neq 0$.

Then $Tx=\{x_i'\}$, $x_i=0$ for $i \neq l, k$ and $Ty=\{y_i'\}$, $y_i'=0$ for $i \neq l$.

Using $[Tx, y]_{p_1} = [x, Ty]_{p_1}$ and $y_l' = y_l$, we get, $y_k=0$, a contradiction.

Case III. $T_i = 0$ for I for all i. The method of proof is similar as in subcase (a) of Case II.

Note 3.14. It would be interesting to know whether "smoothness condition" in Theorem 3.13 above can be dropped.

Corollary 3.15. *There exists no non-trivial p_1-orthogonal projection on an l_p-sum $(1<p, p_1<\infty, p_1 \neq p)$ of a sequence of Hilbert spaces. In particular, there exists no non-trivial p_1-orthogonal projection on $l_p (1<p, p_1<\infty, p_1\neq p)$.*

4. The characterization of p-orthogonal projection on L^p.

We now give the characterization of p-orthogonal projections on L^p. We first generalize Theorem 3 of Fleming [12]. We recall the following definition from Kelley ([22], p. 129).

Definition 4.1. *Let (Ω, Σ, μ) be a measure space and Σ_0 be the collection of measurable subsets of Ω of finite measure. A scalar valued function f on Ω is called locally measurable if $f \cdot \chi_A$ is measurable for $A \in \Sigma_0$.*

A property is said to hold locally almost everywhere iff the set on which it fails intersects each of Σ_0 in a set of measure zero. The functions on Ω which are locally measurable and bounded locally almost everywhere, form a linear space, denoted by $L^\infty(\Omega, \Sigma, \mu)$. For such a function f, if $\|f\|_\infty = \inf\{K : |f(x)| \leqslant K$ locally almost everywhere$\}$ then $|f(x)| \leqslant \|f\|_\infty$ locally almost everywhere.

The measure space (Ω, Σ, μ) will be called localizable iff to each family $\{g_A : A \in \Sigma_0\}$ of scalar valued functions, such that g_A is defined and measurable on A and $g_A(x) = g_B(x)$ for almost all x in $A \cap B$, there corresponds a locally measurable function g on Ω coinciding with g_A on A except for a set of measure zero for each A in Σ_0.

From now on we consider L^p on a localizable measure space (Ω, Σ, μ). Every measure space can be made into a complete measure space. So, throughout the paper, without loss of generality, we assume that all the measure spaces are complete.

Theorem 4.2. *Let $X = L^p$, $1 \leqslant p < \infty$, $p \neq 2$. An operator T on X is Hermitian iff there exists a real valued function $h \in L^\infty$ such that $Tf = hf$ a.e., $\forall f \in X$.*

Proof. Let T be Hermitian and $A \in \Sigma_0$. Any function f in $L^p(A, \Sigma \cap A, \mu)$, where $\Sigma \cap A = \{K \cap A, K \in \Sigma\}$ can be considered to be an element of $L^p(\Omega, \Sigma, \mu)$ by difining $f(x)=0$ outside A. Therefore $L^p(A, \Sigma \cap A, \mu)$ can be considered as a subspace of $L^p(\Omega, \Sigma, \mu)$. Let T_A be the restriction of T on $L^p(A, \Sigma \cap A, \mu)$ in the sense that $T_A f = \chi_A Tf$, $\forall f \in L^p(A, \Sigma \cap A, \mu)$. Then by Lemma 8 of

Lumer [25]. T_A is Hermitian. Hence by Theorem 3 of Fleming [12], $T_A f = h_A f$ a.e., $\forall f \in L^p(A, \Sigma \cap A, \mu)$ where $h_A \in L^\infty(A, \Sigma \cap A, \mu)$, using the restriction of T to $L^p(A \cap B, \Sigma \cap A \cap B, \mu)$, $A, B \in \Sigma_0$, it follows $h_A = h_B$ a.e. on $A \cap B$. So by the definition of localizability, there exists a locally measurable function h such that $h = h_A$ a.e., $\forall A \in \Sigma_0$. Now let $f \in X$. Suppose the supports of f and Tf are B_1 and B_2 respectively. Let $B = B_1 \cup B_2$. Then the set B is of σ-finite measure and $B = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \Sigma_0$. Hence it can be easily shown that $Tf = hf$ and $h \in L^\infty$.

Theorem 4.3. *Let $X = L^p$, $1 < p < \infty$, $p \neq 2$. Then an operator P on X is p-orthogonal iff $Pf = h_M f$ a.e., where h_M is the characteristic function of some locally measurable set M.*

Proof: Suppose P is p-orthogonal on X. Then P is Hermitian and by Theorem 4.2 above, $Pf = hf$ a.e., $\forall f \in X$, with $h \in L^\infty$. As $P^2 = P$ and h is locally measurable it follows that h is the characteristic function of some locally measurable set M. For the converse let us note that the p-g.s.i.p. on X is given by $[f, g]_p = \int_\Omega f \cdot |g|^{p-1} \cdot \operatorname{sgn} g \, du$, $\forall f, g \in X$. Using this p-g.s.i.p., it is easy to see that the condition is sufficient.

Corollary 4.4. *An operator T on L^p is a p-orthogonal projection iff T is a Hermitian projection in the sense of Berkson [3].*

Proposition 4.5. *An operator T on $L^p (1 < p < \infty)$ is a p-orthogonal projection iff T^* (Banach space adjoint) on L^q is a q-orthogonal projection, where* $1/p + 1/q = 1$.

Proof. The result follows from Theorem 4.3 above.

Note 4.6. From Theorem 4.3 above, it follows that any p-orthogonal projection T on L^p is positive in the sense of Ando [1].

Note 4.7. It is clear from Theorem 4.3 above that the set of all p-orthogonal projections on L^p forms a Boolean algebra with respect to the usual operations, $P \vee Q = P + Q - PQ$, $P \wedge Q = PQ$.

Note 4.8. Let $X = L^p$, $1 < p < \infty$. Let S be the set of all p-orthogonal projections on X. Then from Note 4.7 above, it can be easily verified that every functional $x^* \in X^*$ corresponding to $x \in X$ is a Bade functional with respect to S in the sense of Berkson [2].

Theorem 4.9. *Let T be p_1-self adjoint on $X = L^p$, $1 < p$, $p_1 < \infty$, $p \neq p_1$. Then there exists a non-negative constant λ such that $T^2 = \lambda I$. Further $\lambda = 0$ or 1 for $p_1 \neq 2$.*

Proof. Case I : $p = 2$.

The result follows from Lemma 3.4.

Case II : $p \neq 2$. Suppose T be p_1-self adjoint on X. Then T^2 is Hermitian and p_1-selfadjoint. Then by our Theorem 4.2, $T^2 f = hf$ a.e., $\forall f \in X$, $h(x) \geqslant 0$ a.e. on Ω. The p_1-g.s.i.p. on X, is given by

$$[f, g]_{p_1} = ||g||^{p_1 - 1} \cdot \int_\Omega f \cdot \left(\frac{g}{||g||}\right)^{p-1} \cdot \operatorname{sgn} g \, du.$$

Then $[T^2 f, g]_{p_1} = [f, T^2 g]_{p_1}$, $\forall f, g \in X$. Let $f \in X$. Suppose the support of f is B. Then the set B is of σ-finite measure. If $M = L^p(B, B \cap \Sigma, \mu)$ then $T^2 M \subseteq \subseteq M$. Therefore the restriction T_B^2 of T^2 on M is Hermitian and p_1-self adjoint.

Then $T_B^2 g = h_B g$ a.e., $\forall g \in M$, $h_B \in L^\infty(B)$. Now applying the same proof as in Theorem 4 of Fleming [12], we get $h_B = \lambda$ a.e., where λ is a non-negative constant. Since h_B's are essentially restrictions of h to B, it follows that the constant λ is independent of B. $T_B^2 g = \lambda g$ a.e., $\forall g \in M$. As $f \in M$, $T^2 f = T_B^2 f = = \lambda f$, $\forall f \in X$, and so $T^2 = \lambda I$.

Theorem 4.10. *There exists no non-trivial p_1-orthogonal projection on L^p, $1 < p$, $p_1 < \infty$, $p \neq p_1$.*

Proof: The result follows from Theorem 4.9 above.

Note 4.11. It would be interesting to know whether there exists any non-trivial p-orthogonal projection on L_1. Partly we answer this query as follows: A p-g.s.i.p. on L_1 is given by, $[f, g]_p = ||g||^{p-1} \cdot \int_\Omega f \cdot \operatorname{sgn} g \, du$. Applying the same proof as in Theorems 4.9, 4.10 above, it can be shown that there exists no non-trivial p-orthogonal projection on L_1 with respect to this p-g.s.i.p.

5. Non-existence of p-orthogonal projections. In this section we now show that there exists no non-trivial p-orthogonal projection on some concrete Banach spaces.

Theorem 5.1. *There exists no non-trivial p-orthogonal projection on $C(K)$ where K is a compact Hausdorff space.*

Proof: Let $X = C(K)$. If possible let P be a non-trivial p-orthogonal projection on X, $1 < p < \infty$. As P is Hermitian then by Theorem 3 of Bonsall ([6], p. 91), $Pf = hf$, $\forall f \in X$, where h is a real valued continuous function on K. As $P^2 = P$, h is the characteristic function of some clopen set $A \subset K$.

Since P is p-orthogonal, there exists a p-g.s.i.p. $[\ ,\ ]_p$ on X such that $[Pg, f]_p = [g, Pf]_p$, $\forall f, g \in X$. Now this p-g.s.i.p. can be written as $[g, f]_p = ||f||^{p-2} \cdot \int_k g d\nu_f$, where ν_f is a regular complex Borel measure for each $f \in X$, Rudin ([31], Theorem 6.19).

Now as $P \neq 0$, I then the clopen set $A \neq \Phi$, K.

Let us define a function g on K by $g(t) = 1/2$, $t \in A$, $= 1$, $t \in K - A$. Since A is clopen, g is continuous. Now, $[Pg, g]_p = [hg, g]_p = ||g||^{p-2} \cdot \int_k hg d\nu_g = = ||g||^{p-2} \cdot \int_A g d\nu_g = 0$, as we know from Fleming ([12], Lemma 2) that if G be any open set containing the peak set of g then $|\nu_g|(K - G) = 0$. Moreover, $[g, Pg]_p = [g, hg]_p = ||hg||^{p-2} \cdot \int_k g d\nu_{hg} = ||hg||^{p-2} \cdot \int_A 1/2 \, d\nu_{hg} = ||hg||^{p-2} \cdot 1/2 \cdot \cdot \nu_{hg}(A) \neq 0$.

Therefore $[Pg, g]_p \neq [g, Pg]_p$, a contradiction. Hence the result.

Note 5.2. It might be of interest to note that a non-trivial Hermitian projection exists on $C(K)$ iff K is disconnected.

Proposition 5.3. *The following spaces have only trivial p-orthogonal projections:*

(i) $C_0(X)$, *where X is a locally compact Hausdorff space.*

(ii) $L^\infty(\Omega, \Sigma, \mu)$, *where (Ω, Σ, μ) is a positive measure space.*

(iii) *Any abstract M-space with a strong unit.*

(iv) $C'[0, 1]$, $\mathrm{Lip}[0, 1]$, $AC[0, 1]$, $\mathrm{lip}\,\alpha (0 < \alpha < 1)$, $H^p(D)$ *(where D is the unit disc, $1 \leqslant p < \infty$) and* $S_p(\alpha)(1 < p < \infty$, $p \neq 2$, $\alpha > 0)$.

Proof. (i) Let $Y = C_0(X)$ then any p-g.s.i.p. on Y is given by $[g, f]_p = = ||f||^{p-2} \cdot \int_X g d\nu_f$, $f, g \in Y$, where ν_f is a regular complex Borel measure on X, Rudin ([31], Theorem 6.19), and by a result of Bonsall ([6], p. 92), an operator T on Y is Hermitian iff $Tf=hf$, $\forall f\in Y$, where h is a real valued continuous function on X. Now applying the same proof as in Theorem 5.1 above, we get the result.

(ii) By Theorem 11 of Dunford ([8], p. 445), $L^{\infty}(\Omega, \Sigma, \mu)$ is isometrically isomorphic onto $C(K)$ for some compact Hausdorff space K. Suppose α is that isometric isomorphism. If $[\,,\,]_p$ is any p-g.s.i.p. on L^{∞} then $[f, g]'_p = [\alpha^{-1}f, \alpha^{-1}g]_p$, $\forall f, g\in C(K)$, defines a p-g.s.i.p. on $C(K)$. Hence the result follows from Theorem above.

(iii) As this space is order isometric to a $C(K)$ space, Lindenstrauss ([23], Theorem 1.b.6, p. 16).

(iv) As these spaces admit only trivial bounded Hermitian operators i.e. real multiples of the identity operator I, Fleming ([11], p. 80) and Corollary 4.6 of Fleming [14].

Proposition 5.4. *Let $1<\alpha\leqslant p_i\leqslant\beta<\infty$, and furthermore suppose $p_i=2$ for at most one i. Then there exists no non trivial p-orthogonal projection ($1<p<\infty$) on $l(p_i)$.*

Proof. The result follows from Corollary 2 and the proof of Theorem 4 of Fleming [13].

Note 5.5. Let E be a complex Banach space, K a compact Hausdorff space and $C(K, E)$ the Banach space of continuous functions on K with values in E provided with the supremum norm. The characterization of Hermitian operators on $C(K, E)$ is given in Fleming [11]. It would be interesting to know whether there exists any non-trivial p-orthogonal projection on $C(K, E)$.

Note 5.6. The characterization of Hermitian operators on C_p, $1\leqslant p\leqslant\infty$, is given in Berkson [5]. It would be interesting to know whether there exists any non-trivial p-orthogonal projection on C_p.

6. Properties of p-orthogonal projections. The following result is straight forward.

Proposition 6.1. *If $P(\neq 0, I)$ is a p-orthogonal projection on X, then $||P|| = 1$, and $||I-P||=1$.*

Note 6.2. If $P^2=P$ and $||P||=1$ then P is not necessarily generalized p-self adjoint, $1<p<\infty$, as follows from an example given by Berkson [3].

Theorem 6.3. *Let X be a complex strictly convex space and P be a p-orthogonal projection on X. If $||x||=||Px||$ for some $x\in X$ then $x\in R(P)$.*

Proof. We can suppose that $||x||=||Px||=1$.

As P is positive then by Theorem 1 of Vidav [42], $||e^{itP}x||=||x||$ for all real t, or $||x-Px+e^{it}Px||=1$, or $||Px+e^{it}(x-Px)||=1$ for all real t. Then by the definition of complex strict convexity, $x-Px=0$, which implies that $x\in R(P)$.

Corollary 6.4. *Let X be a complex strictly convex space and P be a Hermitian projection on X. If $||x||=||Px||$ for some $x\in X$ then $x\in R(P)$.*

Note 6.5. Corollary 6.4 above has been proved in Theorem 2.25 of Berkson [3] for strictly convex spaces.

Note 6.6. It can be shown by considering the l_p-sum ($1<p<\infty$, $p\neq 2$) of two copies of $C(K)$, that the converse of Theorem 6.3 above is false.

Proposition 6.7. *If X is a weakly sequentially complete Banach space then every p-orthogonal projection on X is a scalar type spectral operator.*

Proof. The proof is the same as that of Theorem 2.25 of Sen [37].

Note 6.8. It is not known whether p-orthogonal projections on arbitrary Banach space are spectral operators in the sense of Dunford [9], though they are generalized spectral operators by Theorem 4.5 of Colojoară and Foiaş ([7], p. 160) and Theorem 2.1 of Sen [36].

Proposition 6.9. *Let P and Q be two p-orthogonal projections on X.*

i) If they commute and their associated p-g.s.i.p. are the same then PQ is p-orthogonal and $R(PQ)=R(P)\cap R(Q)$.

ii) $R(P)$ is invariant under an operator T iff $TP=PTP$.

iii) If H be Hermitian then PHP is also Hermitian.

iv) If $||P-Q||<1$ then the Hamel dimensions of $R(P)$ and $R(Q)$ are equal.

Proof. (i), (ii), (iii) are straight forward and (iv) follows from Halmos ([17], Problem 43).

Proposition 6.10. *If every one dimensional subspace of a Banach space X is the range of some orthogonal projection on X then X is a Hilbert space.*

Proof. Though the result follows directly from the Corollary 4.4 of Kalton [21], we give here a shorter and independent proof. Let $x, y\neq 0$ and P be the orthogonal projection onto $C\cdot y$ and $[\,,\,]$ be a s.i.p. associated with P. Then $Px=ty$ with $t\in C$ and $[x, y]=[x, Py]=[Px, y]=[t\,y, y]=\overline{[y, ty]}=\overline{[y, px]}=[py, x]=\overline{[y, x]}$. Hence the result.

Proposition 6.11. *If $P_i (i=1, 2, \ldots, n)$ are distinct p-orthogonal ($1<p<\infty$) projections on a smooth Banach space X then there exist no set of positive numbers t_i with $\sum_{i=1}^{n} t_i=1$ such that $P=\sum_{i=1}^{n} t_iP_i$ is a p-orthogonal projection.*

Proof. Applying Holder's inequality and Theorem 5 of Lumer [24] to the proof of Steffen [40], we get the result.

Definition 6.12. *Let P and Q be two p-orthogonal projections on a smooth Banach space X, $1<p<\infty$. We define $P\leqslant Q$ iff $[Px, x]_p\leqslant[Qx, x]_p$, $\forall x\in X$, iff $Q-P$ is positive.*

Note 6.13. By our Proposition 6.1, $0\leqslant P\leqslant I$ for any p-orthogonal projection on X.

Note 6.14. Let S_p, $1<p<\infty$, be the set of all p-orthogonal projections on X. Then S_p forms a partially ordered set with respect to this order relation '$\leqslant$'. The following results are straight forward.

Proposition 6.15. *Let X be smooth. If P and Q are p-orthogonal projections on X and commute then the following statements are all equivalent: (1) $P\leqslant Q$, (2) $||Px||\leqslant||Qx||$, $\forall x\in X$, (3) $R(P)\subseteq R(Q)$, (4) $QP=P$.*

Proposition 6.16. *Let X be smooth, strictly convex and reflexive. If P and Q are p-orthogonal projections on X then the following statements are all equivalent. (1) $P\leqslant Q$, (2) $||Px||\leqslant||Qx||$ $\forall x\in X$, (3) $R(P)\subseteq R(Q)$, (4) $QP=P$, (5) $PQ=P$.*

Proposition 6.17. *Let X be smooth, strictly convex and reflexive. If P and Q are p-orthogonal projections on X such that $P\leqslant Q$ then $Q-P$ is idempotent and $R(Q)\cap N(P)=R(Q-P)$.*

Definition 6.18. *Two p-orthogonal projections P and Q are said to be mutually orthogonal, symbolically $P\perp Q$, iff $PQ=0=QP$.*

The following result is straight forward.

Proposition 6.19. *If P and Q are p-orthogonal projections on a smooth Banach space X then $R(P)\perp R(Q)\Leftrightarrow P\perp Q$.*

Theorem 6.20. *Suppose $\{P_i\}$ is a sequence of p-orthogonal projections on a smooth Banach space X. Then the following hold.*

(1) ΣP_i converges in norm iff $P_i=0$, $\forall i\geq k_0$.

(2) If the series $\Sigma P_i x$ converges, $\forall x\in X$, then the operator P is defined by $Px=\Sigma P_i x$, $x\in X$, is idempotent, iff P_i's are mutually orthogonal. Further,

$$M=R(P)=\sum_{i=1}^{\infty} R(P_i),\quad N=N(P)=\bigcap_{i=1}^{\infty} N(P_i) \text{ and } N \text{ is orthogonal to } M.$$

Proof of (1). Suppose ΣP_i converges in norm. Let $\Sigma P_i=\lim\limits_n S_n$ where $S_n=P_1+P_2+\ldots+P_n$. Then for every $\varepsilon>0$, there exists an integer $N>0$ such that, $||S_{n+p}-S_n||<\varepsilon$, $\forall n\geq N$ where p is any positive integer. Or, $||P_{n+1}+\ldots+P_{n+p}||<\varepsilon$ ... (i)

Let $L=P_{n+1}+\ldots+P_{n+p}$. As L is positive then by proposition 2 of Sinclair [32].

$||L||=|\sigma(L)|=|W(L)|=\text{Sup}\{|[Lx,x]_p|:||x||=1\}=\text{Sup}\{||P_{n+1}x||^2+\ldots+||P_{n+p}x||^2:||x||=1\}\geq 1$, by our proposition 6.1. Therefore from (i), $1\leq||L||<\varepsilon$, a contradiction.

Proof of (2). Let P be idempotent. As P_i's are positive then by Proposition 2 of Sinclair [32], $||P||=|\sigma(P)|$ and hence $||P||=1$. The rest of the proof follows from Theorem 5 of Zaanen [43, p. 249] and Proposition 2.3 of Kalton [21].

Note 6.21. We raise here the following question: Does there exist a Banach space X with two non-trivial projection operators of which one is p_1-orthogonal and other is p_2-orthogonal (where $1<p_1,\ p_2<\infty,\ p_1\neq p_2$)?

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