ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
Tomul XXXIV, s. I a. Matematică, 1988, f. 3

EXISTENCE AND REGULARITY FOR A COUPLED SYSTEM OF LINEAR PARABOLIC EQUATIONS

BY

LIVIU I. NICOLAESCU

1. **Motivation.** The problem we shall consider in the sequel arises in the integrate circuit design (see M a r i n o v & N e i t t a a n m ä k i [4]). Neglecting inductive effects one arrive to the following system

$$\text{(S)} \qquad \frac{\partial u_k}{\partial t} = \frac{1}{r_k c_k}\frac{\partial^2 u_k}{\partial x^2} - \frac{g_k}{c_k} u_k \qquad 0 < x < d,\ t > 0,$$

(r_k, c_k, g_k are positive constants $\forall k = 1, 2, \ldots, n$) with boundary conditions

$$\text{(B)} \qquad \begin{bmatrix} -\frac{1}{r_1}\frac{\partial u_1}{\partial x}(0,t) \\ \frac{1}{r_1}\frac{\partial u_1}{\partial x}(d,t) \\ --- \\ -\frac{1}{r_n}\frac{\partial u_n}{\partial x}(0,t) \\ \frac{1}{r_n}\frac{\partial u_n}{\partial x}(d,t) \end{bmatrix} + G \begin{bmatrix} u_1(0,t) \\ u_1(d,t) \\ \vdots \\ u_n(0,t) \\ u_n(d,t) \end{bmatrix} = B(t),$$

(here G is a $2n \times 2n$ constant matrix and B is a $2n$-dimensional column vector) and initial data

$$\text{(I)} \qquad u_k(0, x) = v_k(x) \qquad 0 < x < d, \qquad k = 1, 2, \ldots, n.$$

(B) is a coupling condition between $u_1, u_2, \ldots, u_n$.

In the same field of integrate circuits design, M a r i n o v & M o r o ş a n u [5] arrived at the following problem:

(P) „Find $u_1, u_2, \ldots, u_n : [0, d] \times [0, T] \to \mathbf{R}$ and

$$u_{n+1}, \ldots, u_{n+m} : [0, T] \to \mathbf{R} \quad \text{such that}$$

$u_1, u_2, \ldots, u_n$ satisfy (S) and (I) and $u_1, \ldots, u_n, u_{n+1}, \ldots, u_{n+m}$ satisfy the following system of coupling boundary conditions

(B')
$$\begin{bmatrix} -\frac{1}{r_1}\frac{\partial u_1}{\partial x}(0,t) \\ \frac{1}{r_1}\frac{\partial u_1}{\partial x}(d,t) \\ \vdots \\ -\frac{1}{r_n}\frac{\partial u_n}{\partial x}(0,t) \\ \frac{1}{r_n}\frac{\partial u_n}{\partial x}(d,t) \\ c_1\frac{du_{n+1}}{dt}(t) \\ \vdots \\ c_m \frac{du_{n+m}}{dt}(t)\end{bmatrix} + G \begin{bmatrix} u_1(0,t)\\ u_1(d,t)\\ \vdots \\ u_n(0,t)\\ u_n(d,t)\\ u_{n+1}(t)\\ \vdots\\ u_{n+m}(t)\end{bmatrix} = B(t),$$

while $u_{n+1},\dots,u_{n+m}$ satisfy the initial conditions

(I') $$u_{n+i}(0)=w_i\in\mathbf{R} \quad i=1,2,\dots,m.$$

We shall transform problem (S), (B'), (I, I') using the following device (cf. [5]). Let us consider the block decomposition of G and B

$$G=\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22}\end{bmatrix} \qquad B=\begin{bmatrix} B_1 \\ B_2\end{bmatrix}.$$

Here G_{11} is $2n\times 2n$, $G_{22}=m\times m$ and B_1 is $2n\times 1$.

Set $a_k=u_k(0,t)$, $b_k=u_k(d,t)$, $k=1,2,\dots,n$. Then u_{n+i}, $i=1,m$ satisfy the Cauchy problem

(CP)
$$\left\{\begin{array}{l} \begin{bmatrix} c_1\frac{du_{n+1}}{dt} \\ \vdots \\ c_m\frac{du_{n+m}}{dt}\end{bmatrix} + G_{22}\begin{bmatrix} u_{n+1} \\ \vdots \\ u_{n+m}\end{bmatrix} = -G_{21}\begin{bmatrix} a_1 \\ b_1 \\ \vdots \\ a_n \\ b_n\end{bmatrix} + B_2(t), \\ u_{n+i}(0)=w_i \quad i=1,2,\dots,m \end{array}\right.$$

and $u_1,\dots,u_n$ satisfy boundary conditions similar to (B).

(B'')
$$\begin{bmatrix} -\frac{1}{r_1}\frac{\partial u_1}{\partial x}(0,t) \\ \frac{1}{r_1}\frac{\partial u_1}{\partial x}(d,t) \\ \vdots \\ -\frac{1}{r_n}\frac{\partial u_n}{\partial x}(0,t) \\ \frac{1}{r_n}\frac{\partial u_n}{\partial x}(d,t)\end{bmatrix} + G_{11}\begin{bmatrix} a_1 \\ b_1 \\ \vdots \\ a_n \\ b_n\end{bmatrix} + G_{12}\begin{bmatrix} u_{n+1} \\ \vdots \\ u_{n+m}\end{bmatrix} = B_1(t).$$

If in (CP) we use the variation of constants formula, we get the following system of boundary conditions

(BC)
$$\begin{bmatrix} -\frac{1}{r_1}\frac{\partial u_1}{\partial x}(0,t) \\ \frac{1}{r_1}\frac{\partial u_1}{\partial x}(d,t) \\ \vdots \\ -\frac{1}{r_n}\frac{\partial u_n}{\partial x}(0,t) \\ \frac{1}{r_n}\frac{\partial u_n}{\partial x}(d,t)\end{bmatrix} + G_{11}\begin{bmatrix} a_1 \\ b_1 \\ \vdots \\ a_n \\ b_n\end{bmatrix} + \int_0^t K(t-s)\begin{bmatrix} a_1(s) \\ b_1(s) \\ \vdots \\ a_n(s) \\ b_n(s)\end{bmatrix} ds = B_1(t).$$

Here $t\to K(t)$ is the matrix valued function:

$$t\to\exp(-tA), \quad A=\mathrm{diag}(c_1^{-1},\dots,c_m^{-1})G_{22}.$$

Clearly it is a C^∞ mapping. BC is a coupling condition for $u_1,u_2,\dots,u_n$.

Solving (S)+(BC)+(I), we can also solve the initial problem (P), since u_{n+i} can be found from $(u_k)_{1\le k\le n}$ via (CP). We have thus reduced problem (P) to the following one (P_1) Find $u_1,\dots,u_n:[0,d]\times[0,T]\to\mathbf{R}$ satisfying (S)+(BC)+(I).

The dificulty of problem (P_1) comes from the presence of the integral in (BC).

2. **Statement of the Main Result.** In this paper we shall deal with the following type of systems

(2.1) $$\frac{\partial u}{\partial t}(x,t)-\Delta u(x,t)+Du(x,t)=f(x,t) \text{ in } Q=\Omega\times]0,T[.$$

$$\vec{u}(x,0)=\vec{u}_0(x), \text{ in} \tag{2.2}$$

$$\frac{\partial \vec{u}}{\partial \nu}(x,t)+\int_\Gamma G(x,y)\,\vec{u}(y,t)\,d\sigma_y=\int_0^t ds\int_\Gamma B(x,y,t-s)\vec{u}(y,s)d\sigma_y+ \tag{2.3}$$

$$+\vec{g}(x,t) \text{ on } \Sigma=\partial\Omega\times]0,T[.$$

where $\Omega\subset\mathbf{R}^N$ is a bounded domain with C^∞-boundary $\Gamma=\partial\Omega$.

$\vec{u}:Q\to\mathbf{R}^n$ is a vector valued function. $D=\mathrm{diag}(d_1,\ldots,d_n)$. $d_i\geq 0$, $i=1,2,\ldots,n$. $\vec{f}:\to\mathbf{R}^n$, $\vec{u}_0:\Omega\to\mathbf{R}^n$, $\vec{g}:\Sigma\to\mathbf{R}^n$.
$G:\Gamma\times\Gamma\to M_n(\mathbf{R})$ $B:\Gamma\times\Sigma\to M_n(\mathbf{R})$ are smooth mappings.

Let us observe that when $N=1$, $\int_\Gamma$ reduced to a discrete sum since in this case $\Gamma=\{0,d\}$. Problem (2.1)-(2.3) reduces then to the problem (P_1) for G and B suitable chosen.

Before stating the main result let us recall a definition. (M will denote either a compact C^∞-hypersurface in $\mathbf{R}^N$ or an open subset of $\mathbf{R}^N$).

Definition 2.1. *Let $M\subset\mathbf{R}^N$ as above. Denote $Q=M\times]0,T[$. Then $H^{r,s}(Q)$ denotes the nonisotropic Sobolev space*

$$H^{r,s}(Q)=L^2(0,T;H^r(M))\cap H^s(0,T;L^2(M)).$$

(where $r,s>0$ (see Lions [2]).

The main result of this paper is the following theorem similar to Fredholm's alternative.

Theorem 2.2. *Let $\Omega\subset\mathbf{R}^N$ be a bounded domain with C^∞-boundary $\Gamma=\partial\Omega$. Let $G=\Gamma\times\Gamma\to M_n(\mathbf{R})$, $B=\Gamma\times\Sigma\to M_n(\mathbf{R})$. be C^1-mappings. Suppose that the homogeneous problem (2.1)-(2.3) (i.e. the problem (2.1)-(2.3) in which $\vec{f}=0$, $\vec{u}_0=0$, $\vec{g}=0$) has only the trivial solution $\vec{u}=0$.* Then for every

$$\vec{f}\in[L^2(Q)]^n,\quad \vec{u}_0\in[H^1(\Omega)]^n,\quad \vec{g}\in[H^{1/2,1/4}(\Sigma)]^n.$$

the problem (2.1)-(2.3) has a unique solution $\vec{u}\in[H^{2,1}(Q)]^n$.

3. **The Proof of Theorem 2.2.** Before proving theorem 2.2 let us recall some basic facts concerning Sobolev spaces and parabolic equations.

Trace theorem. *If $u\in H^{2,1}(Q)$ then $u|_\Sigma\in H^{3/2,3/4}(\Sigma)$, $\frac{\partial u}{\partial\nu}\in H^{1/2,1/4}(\Sigma)$, $u|_{\Omega\times\{0\}}\in H^1(\Omega)$.*

Existence theorem for parabolic P.D.E. *Let $f\in L^2(Q)$, $g\in H^{1/2,1/4}(\Sigma)$, $u_0\in H^1(\Omega)$. Then the following problem has a unique solution $u\in H^{2,1}(Q)$*

$$\left\{\frac{\partial u}{\partial t}-\Delta u+\alpha u=f \text{ in } Q,\quad u(x,0)u_0 \text{ in } \Omega,\quad \frac{\partial u}{\partial\nu}=g \quad\text{on } \Sigma.\right.$$

Moreover there is $C>0$ independent of f, u_0, g such that

$$\|u\|_{H^{2,1}(Q)}\leq C(\|f\|_{L^2(Q)}+\|u_0\|_{H^1(\Omega)}+\|g\|_{H^{1/2,1/4}(\Sigma)}). \tag{E}$$

Our proof relies on Fredholm's alternative. In order to apply this theorem we need a compactness result. This is supplied by the following

Lemma 3.1. *The imbedding*

$$H^{3/2,3/4}(\Sigma)\to H^{1/2,1/4}(\Sigma) \tag{3.1}$$

is compact and continuous.

Proof of Lemma 3.1. By a well known compactness result of Aubin (see e.g. Lions [3]) we get the compact imbedding

$$H^{r,\gamma}(\Omega)\xrightarrow[\text{compact}]{}H^{q,\alpha}(\Sigma),\ \text{for } 0\leq q\leq r,\ 0<\gamma<1. \tag{3.2}$$

Hence for $l\in]3/4,1[$, $H^{2l,l}(\Sigma)\xrightarrow[\text{compact}]{}L^2(\Sigma)=H^{0,0}(\Sigma)$. Denote $X=H^{2l,l}(\Sigma)$ and $Y=L^2(\Sigma)$. Then there are $0<\theta_1<\theta_2<1$ such that

$$3/2=(1-\theta_1)2l,\ 3/4=(1-\theta_1)l,\ 1/2(1-\theta_2)2l,\ 1/4=(1-\theta_2)l,$$

and by a known interpolation result (cf. Lions [2]):

$$[X,Y]_{\theta_1}=H^{3/2,3/4}(\Sigma),\ [X,Y]_{\theta_2}=H^{1/2,1/4}(\Sigma).$$

Since $X\xrightarrow[\text{compact}]{}Y$ and $\theta_1<\theta_2$ we infer

$$[X,Y]_{\theta_1}\xrightarrow[\text{compact}]{}[X,Y]_{\theta_2}$$

and this is our lemma. q.e.d.

From the previously quoted existence result about parabolic P.D.E we get that for every $\vec{f}\in[L^2(Q)]^n$, $[\vec{g}\in H^{1/2,1/4}(\Sigma)]^n$, $\vec{u}_0\in[H^1(\Omega)]^n$ the following problem has a unique solution $\vec{u}\in[H^{2,1}(Q)]^n$

$$\frac{\partial\vec{u}}{\partial t}-\Delta\vec{u}+D\vec{u}=\vec{f} \text{ in } Q,\quad \vec{u}(x,0)=\vec{u}_0 \text{ in } \Omega,\quad \frac{\partial\vec{u}}{\partial\nu}=\vec{g} \quad\text{on } \Sigma. \tag{3.3}$$

We shall denote this solution by $\vec{u}=U(\vec{f},\vec{u}_0,\vec{g})$; by (E), U depends continuously on $\vec{f}$, $\vec{u}_0$, $\vec{g}$. For simplicity we make the following notations

$$\vec{u}(x,t)\to(F\vec{u})(x,t)=-\int_\Gamma G(x,y)\vec{u}(y,t)d\sigma_y,\ (x,t)\in\Sigma.$$

$$\vec{u}(x,t)\to(K\vec{u})(x,t)=\int_0^t ds\int_\Gamma B(x,y,t-s)\vec{u}(y,s)d\sigma_y,\quad (x,t)\in\Sigma.$$

Problem (2.1)–(2.3) can be rewritten as

$$\frac{\partial\vec{u}}{\partial t}-\Delta\vec{u}+D\vec{u}=\vec{f} \quad\text{in } Q,\quad \vec{u}(x,0)=\vec{u}_0 \quad\text{on } \Omega,\quad \frac{\partial\vec{u}}{\partial\nu}=F\vec{u}+K\vec{u}+\vec{g} \text{ on } \Sigma. \tag{3.4}$$

We may assume $\vec{f}\equiv 0$, $\vec{u}_0\equiv 0$. Indeed on the contrary one can find $\vec{v}\in[H^{2,1}(Q)]^n$ such that $\vec{v}=\vec{u}_0$ on $\Omega\times\{0\}$ and

$$\frac{\partial \vec{v}}{\partial t} - \Delta \vec{v} + D\vec{v} = \vec{f} \text{ (Lions [2] vol. II).}$$

Then $\vec{u}_1 = \vec{u} - \vec{v}$ satisfies (3.4) with $\vec{f} \equiv 0$, $\vec{u}_0 \equiv 0$. Hence one can presume that (3.4) is actually

$$(3.4') \qquad \frac{\partial \vec{u}}{\partial t} - \Delta \vec{u} + D\vec{u} = 0, \quad \vec{u}(x, 0) = 0, \quad \frac{\partial \vec{u}}{\partial \nu} = F\vec{u} + K\vec{u} + \vec{g}.$$

Let U be the operator

$$U : [H^{1/2,\,1/4}(\Sigma)]^n \to [H^{2,1}(Q)]^n$$

$g \to U(0, 0, g)$ — the solution of (3.3).

U is a continuous operator. The trace operator

$$T : [H^{2,1}(Q)]^n \to [H^{3/2,\,3/4}(\Sigma)]^n, \ \vec{u} \to \vec{u}|_\Sigma$$

is also continuous. Therefore the mapping

$$g \in [H^{1/2,\,1/4}(\Sigma)]^n \xrightarrow{T} U(0, 0, \vec{g})|_\Sigma \in [H^{3/2,\,3/4}(\Sigma)]^n$$

is continuous.

Let us make some apriori remarks. Let $\vec{u}$ be a solution of (3.4). Then $u = U(0, 0, \vec{\varphi})$, where $\vec{\varphi} = Fu + Ku + g$. Here $[H^{1/2,\,1/4}(\Sigma)]^n$, $u|_\Sigma = T\vec{\varphi} \in [H^{3/2,\,3/4}(\Sigma)]^n$. Hence

$$(3.5) \qquad \vec{\varphi} = F(T\vec{\varphi}) + K(T\vec{\varphi}) + \vec{g}.$$

From the regularity assumptions on G and B, we get that F and K are bounded linear operators

$$F, K : [H^{3/2,\,3/4}(\Sigma)]^n \to [H^{5/2,\,5/4}(\Sigma)]^n.$$

Since the imbedding

$$i : H^{5/2,\,5/4}(\Sigma) \to H^{1/2,\,1/4}(\Sigma)$$

is compact and since (3.5) is equivalent to

$$\vec{\varphi} = i \circ F(T\vec{\varphi}) + i \circ K(T\vec{\varphi}) + \vec{g}.$$

we get that $\vec{\varphi}$ satisfies a Fredholm type equation

$$(3.6) \qquad \vec{\varphi} = \mathcal{A}\vec{\varphi} + \vec{g}, \ \mathcal{A} : [H^{1/2,\,1/4}(\Sigma)]^n \to [H^{1/2,\,1/4}(\Sigma)]^n.$$

$\mathcal{A} = i \circ F \circ T + i \circ K \circ T$ is a compact operator. Hence if $\vec{u}$ is a solution to (3.4') then $\vec{u} = U(0, 0, \vec{\varphi})$ where $\vec{\varphi}$ is a solution to (3.6) and conversely if $\vec{\varphi}$ is a solution to (3.6), then $u = U(0, 0, \vec{\varphi})$ is a solution to (3.4').

Now we apply Riesz-Fredholm's theorem (see Brézis [1]): (3.6) has a solution for every g if and only if the homogeneous equation has only the trivial solution. In our case this is equivalent to the fact that the homogeneous problem (2.1)–(2.3) has only the trivial solution. Q.E.D.

Theorem 2.2 reduces the problem of existence and regularity to a question of uniqueness. Hence we shall get an effective result of existence if we set some extra assumptions on B and G implying uniqueness. It is natural to impose some monotonicity conditions

$$(M1) \qquad \int_\Gamma < \int_\Gamma G(x, y)u(y)d\Gamma_y, \ u(x) > d\Gamma_x \geq 0, \quad \forall u \in L^2(\Gamma, \mathbb{R}^n)$$

$$(M2) \qquad \int_\Sigma < \int_0^t ds \int_\Gamma B(x, y, t-s)u(y, s)d\Gamma_y, \ u(x, t) > d\Sigma_x \geq 0 \ \forall u \in L^2(\Sigma, \mathbb{R}^n).$$

It is a routine exercise, using Green's formula to prove the following:

Corollary 3.2. *Suppose that G and B satisfy together with the previous assumptions the monotonicity conditions M1, M2. Then $\forall \vec{f} \in [L^2(Q)]^n$, $\vec{u}_0 \in [H^1(\Omega)]^n$, $\vec{g} \in [H^{1/2,\,1/4}(\Sigma)]^n$ the problem (2.1)–(2.3) has a unique solution $\vec{u} \in [H^{2,1}(Q)]^n$.*

4. Acknowledgements. I wish to express my gratitude to prof. G. Moroşanu who proposed this problem and has given me useful advices.

REFERENCES

1. Brézis, H. — *Analyse Fonctionnelle*, Masson, Paris, 1983.
2. Lions, J. L., Magenes, E. — *Problèmes aux Limite Non-Homogène et Applications*, Dunod, Paris, vol. I, II, 1967.
3. Lions, J. L. — *Quelques Méthodes de Résolutions des Problèmes aux Limites Non-Linéaires*, Dunod, Paris, 1969.
4. Marinov, C., Neittaanmäki, P. — *The problem of response time in electrical circuits with resistively coupled distributed structures*, Report 8/1985, Department of Physics and Mathematics, Lappeenrata University of Technology, Finland.
5. Marinov, C., Moroşanu, G. — *Faultless models for electrical networks with distributed parameters*, to appear

Received 19.I.1987

Faculty of Mathematics
University of Iaşi
6600 — Iaşi, România