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FILIPPOV SOLUTIONS FOR ELLIPTIC VARIATIONAL INEQUALITIES

BY

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§0. **Introduction.** This paper is concerned with the existence of non-trivial Filippov solutions of the elliptic equation

$$
\begin{cases} -\Delta u + \lambda u + \beta(u) \ni f(u) \text{ on } \Omega, \\ u \mid \partial\Omega = 0, \end{cases} \tag{0.1}
$$

where β is a maximal monotone graph in $R \times R$ and $f: R \to R$ is a measurable function. Here, Ω is a bounded and open subset of the Euclidean space R^n with a sufficiently smooth boundary.

The main ingredient of the proof is the mountain pass theorem (see [1]). For other related results see [4] and [5].

§ 1. **Preliminaries.** We shall use the following hypotheses: $f: R \to R$ is a measurable function, β is a maximal monotone graph with $0 \in \beta(0)$ (therefore there exists a function $j: R \to [0, +\infty]$ proper, convex and lower semicontinuous with $j(0) = 0$ and $\partial j = \beta$, where ∂j is the subdifferential of j) and they satisfy the following conditions:

$$
f(x) \leq \begin{cases} a_1 + a_2 |x|^p \text{ for } n > 2, \text{ where } 1 < p < (n+2)/(n-2), \\ a_3 e^{a(x)} \text{ for } n = 2, \text{ where } \lim\limits_{|x| \to \infty} \dfrac{a(x)}{x^2} = 0. \end{cases} \tag{C$_1$}
$$

$$
\lim_{x \to 0} \frac{f(x)}{x} = 0. \tag{C$_2$}
$$

(C$_3$) If $F(x) = \int_0^x f(t)dt$ then there exist $a_4 > 0$ and $\theta \in]0, 1/2[$ such that

$$
F(x) \leq \theta f(x) x \text{ for all } |x| \geq a_4. \tag{1.1}
$$

$$
\lim_{x \to +\infty} \frac{F(x) - j(x)}{x^2} = +\infty \text{ or } \lim_{x \to -\infty} \frac{F(x) - j(x)}{x^2} = +\infty. \tag{C$_4$}
$$

(C$_5$) There exists $a_5 > 0$ such that

$$
j(x) \geq (1/2) x \beta(x) \text{ for all } x \in D(j), \text{ with } |x| \geq a_5. \tag{1.2}
$$

Remark. For $n=1$ condition (C_1) becomes $f\in L^1_{loc}(\Omega)$. We have denoted by $\|\,.\,\|$ the norm in $H_0^1(\Omega)$.

For any $\varepsilon>0$ and $s\in R$ let

$$\overline{f}_\varepsilon(s)=\operatorname*{ess\,sup}_{|t-s|<\varepsilon} f(t),\qquad \underline{f}_\varepsilon(s)=\operatorname*{ess\,inf}_{|t-s|<\varepsilon} f(t).$$

For s fixed, $\overline{f}_\varepsilon(s)$ is a decreasing function of ε and $\underline{f}_\varepsilon(s)$ is increasing in ε. Let

$$\overline{f}(s)=\lim_{\varepsilon\to 0}\overline{f}_\varepsilon(s),\quad \underline{f}(s)=\lim_{\varepsilon\to 0}\underline{f}_\varepsilon(s).$$

The function $\overline{f}$ is upper semicontinuous and $\underline{f}$ is lower semicontinuous. We define the multivalued function

$$\hat{f}(s)=[\underline{f}(s),\overline{f}(s)]. \tag{1.3}$$

Definition. *If there is $u\in H_0^1(\Omega)$ and $\gamma\in L^{(p+1)/p}(\Omega)$ such that $\gamma(x)\in\hat{f}(u(x))$ a.e. in Ω and*

$$\int_\Omega \nabla u\nabla(v-u)dx+\lambda\int_\Omega u(v-u)dx+\int_\Omega g(v-u)dx+$$

$$+\int_\Omega (j(v)-j(u))dx\geq\int_\Omega \gamma(v-u)dx,\ \text{for all}\ v\in H_0^1(\Omega),$$

then u is called Filippov solution of problem (0.1).

§ 2. **The main result.** Consider the following boundary value problem

$$\begin{cases}-\Delta u+\lambda u+\beta(u)\ni f(u)\ \text{on}\ \Omega,\\ u|_{\partial\Omega}=0.\end{cases} \tag{2.1}$$

Theorem 2.1. *If $\lambda\geq 0$ and conditions $(C_1)-(C_3)$ are satisfied then problem (2.1) has at least one nontrivial Filippov solution.*

Consider the following functions for $0<\varepsilon\leq 1$:

$$f^\varepsilon(x)=\begin{cases}0, & \text{for}\ |x|<2\varepsilon,\\ f(x), & \text{for}\ |x|\geq 2\varepsilon.\end{cases} \tag{2.2}$$

$$f_\varepsilon(x)=\int_R f^\varepsilon(x-\varepsilon\tau)\rho(\tau)d\tau, \tag{2.3}$$

where $\rho\in C_0^\infty(R)$ is a C^∞-mollifier on R, i.e.,

$$\rho(\tau)=0\ \text{for}\ |\tau|>1,\ 0\leq\rho\leq 1,\ \int_R\rho\,d\tau=1.$$

The function f_ε may be written (see [2]) as

$$f_\varepsilon(x)=\varepsilon^{-1}\int_R f^\varepsilon(\tau)\rho((x-\tau)\varepsilon^{-1})d\tau$$

Consider the regularization of the convex function j:

$$j_\varepsilon(x)=\inf\{|x-y|^2/2\varepsilon+j(y):\ y\in R\},\ x\in R. \tag{2.4}$$

The functions j_ε are convex and differentiable (see e.g. [3]). We begin by establishing the following lemmas:

Lemma 2.1. *There exist $\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,\tilde{a}_4\in R^+$ independent on ε and $\tilde{a}(x)$, $\tilde{\theta}\in[0,1/2)$ such that*

$$(\tilde{C}_1)\qquad |f_\varepsilon(x)|\leq\begin{cases}\tilde{a}_1+\tilde{a}_2|x|^p & \text{for}\ n>2,\ \text{where}\ 1<p<(n+2)/(n-2),\\ \tilde{a}_3e^{\tilde{a}(x)} & \text{for}\ n=2,\ \text{where}\ \lim_{|x|\to\infty}\tilde{a}(x)/x^2=0.\end{cases}$$

$(\tilde{C}_3)$ *There is $c\in R^+$ such that*

$$F_\varepsilon(x):=\int_0^x f_\varepsilon(\tau)d\tau\leq c\varepsilon+\tilde{\theta}f_\varepsilon(x)x\ \text{for}\ |x|\geq\tilde{a}_4.$$

Proof. It is clear that f^ε verifies the condition (C_1) for any $\varepsilon>0$. ($|f^\varepsilon(x)|\leq|f(x)|$, for any $\varepsilon>0$). Let $A_\varepsilon=\{\tau\in[-1,1]:\ |x-\varepsilon\tau|\geq 2\varepsilon\}$. Then

$$|f_\varepsilon(x)|\leq\int_{-1}^{1}|f^\varepsilon(x-\varepsilon\tau)|\rho(\tau)d\tau\leq\int_{A_\varepsilon}(a_1+a_2|x-\varepsilon\tau|^p)|\rho(\tau)d\tau\leq a_1+$$

$$+a_2\int_{-1}^{1}2^p|x|^p\rho(\tau)d\tau=a_1+2^pa_2|x|^p,\quad\forall x\in R.$$

For $n=2$ and $\tilde{a}(x)=\sup_{\tau\in[-1,1]}a(x+\tau)$, we have

$$|f_\varepsilon(x)|\leq a_3\int_{-1}^{1}e^{a(x-\varepsilon\tau)}\rho(\tau)d\tau\leq a_3e^{\tilde{a}(x)},\quad\forall x\in R,$$

$$\lim_{|x|\to\infty}\frac{\tilde{a}(x)}{x^2}=0. \tag{2.5}$$

By Fubini's theorem we have

$$F_\varepsilon(x)=\int_0^x\int_{-1}^{1}f^\varepsilon(y-\varepsilon\tau)\rho(\tau)d\tau\,dy=\int_{-1}^{1}\int_0^{\varepsilon\tau}f^\varepsilon(y-\varepsilon\tau)dy\rho(\tau)d\tau\leq c\varepsilon+$$

$$+\int_{-1}^{1}\int_{\varepsilon\tau}^{x}f(y-\varepsilon\tau)dy\rho(\tau)d\tau\leq c\varepsilon+\int_{-1}^{1}\theta f(x-\varepsilon\tau)(x-\varepsilon\tau)\rho(\tau)d\tau\leq c\varepsilon+\tilde{\theta}f_\varepsilon(x)\cdot x$$

(where $\tilde{\theta}$ is a constant in $[0,1/2)$), for $|x|$ sufficiently large, $\forall\varepsilon>0$, and $c=\operatorname*{ess\,sup}_{x\in[-1,1]}f(x)$.

Lemma 2.2. *The functions f_ε satisfy (C_2) and (C_4) uniformly relative to ε, for all ε sufficiently small.*

Proof. For $|x|$ sufficiently large we have

$$f_\varepsilon(x)=\int_{-1}^{1} f(x-\varepsilon\tau)\rho(\tau)d\tau.$$

It follows that f_ε satisfy (C_4) uniformly relative to ε. For $|x|<\varepsilon$ we have that $f_\varepsilon(x)=0$. For $|x|\geqslant\varepsilon$ and $A_\varepsilon=\{\tau\in[-1,1]\,;\ |x-\tau\varepsilon|\geqslant 2\varepsilon\}$, we have

$$\left|\frac{f_\varepsilon(x)}{x}\right|\leqslant\int_{A_\varepsilon}\left|\frac{f(x)-\varepsilon\tau}{x}\right|\rho(\tau)d\tau\leqslant 2\int_{A_\varepsilon}\left|\frac{f(x-\varepsilon\tau)}{x-\varepsilon\tau}\right|\rho(\tau)d\tau\leqslant 2\sup_{|\tau|\leqslant\varepsilon}\left|\frac{f(x-\tau)}{x-\tau}\right|.$$

So $\left|\dfrac{f_\varepsilon(x)}{x}\right|\leqslant 2\sup\limits_{|\tau|\leqslant 2|x|}\left|\dfrac{f(\tau)}{\tau}\right|$. It follows that f_ε satisfy (C_2) uniformly relative to ε (because f satisfies (C_2)).

Lemma 2.3. *There exists $\tilde a_0>0$ independent on ε such that $j_\varepsilon(x)\geqslant\tilde{0}j'_\varepsilon(x).x$, for all $|x|\geqslant a_{\tilde 0}$.*

Proof. For any $x>0$ we have

$$\frac{|x-y|^2}{2\varepsilon}+j(y)\geqslant j(x),\ \forall y\geqslant x \text{ and } \frac{|x-y|^2}{2\varepsilon}+j(y)\geqslant\frac{|x|^2}{2\varepsilon}\ \forall y\leqslant 0.$$

So $j\,(x)=\inf\{|x-y|^2/2\varepsilon+j(y)\,;\ y\in[0,x]\}$.
It follows (because j is lower semicontinuous) that there exists $y_x\in[0,x]$ such that $j_\varepsilon(x)=|x-y_x|^2/(2\varepsilon)+j(y_x)$. Consider the following two situations:

1° $\quad y_x\geqslant a_5$ and $y_x\in\operatorname{int} D(j)$.

In this case we have $(1/2)\,x\,j'_\varepsilon(x)=(1/2)\,x\lim\limits_{\tau\searrow 0}(j_\varepsilon(x+\tau)-j_\varepsilon(x))/\tau$. But

$$x(j_\varepsilon(x+\tau)-j_\varepsilon(x))/\tau\leqslant x\left(\frac{1}{2\varepsilon}\left(x-y_x+\frac{x-y_x}{x}\tau\right)^2+j\left(y_x+\frac{y_x}{x}\tau\right)-\right.$$

$$\left.-\frac{(x-y_x)^2}{2\varepsilon}-j(y_x)\right)\Big/\tau=\frac{1}{2\varepsilon}(x-y_x)\left(\left(x-y_x+\frac{x-y_x}{x}\tau\right)^2-(x-y_x)^2\right)\Big/\left(\frac{x-y_x}{x}\tau\right)+$$

$$+y_x\left(j\left(y_x+\frac{y_x}{x}\tau\right)-j(y_x)\right)\Big/\left(\frac{y_x}{x}\tau\right).$$

(If $y_x=x$ or $y_x=0$ the proof is trivial). Since

$$\lim_{\tau\downarrow 0}\frac{1}{4\varepsilon}(x-y_x)\left(\left(x-y_x+\frac{x-y_x}{x}\tau\right)^2-(x-y_x)^2\right)\Big/\left(\frac{x-y_x}{x}\tau\right)=\frac{1}{2\varepsilon}(x-y_x)^2$$

and

$$y_x\left(j\left(y_x+\frac{y_x}{x}\tau\right)-j(y_x)\right)\Big/\left(\frac{y_x}{x}\tau\right)=y_x(\,j(y_x))\,|\,y_x$$

(j is convex), it follows that

$$xj'_\varepsilon(x)\leqslant 2j_\varepsilon(x).$$

2° $\quad y_x<a_5$ or $y_x\,\overline{\in}\operatorname{int} D(j)$.

$$\tilde{0}x(j_\varepsilon(x+\tau)-j_\varepsilon(x))/\tau\leqslant\tilde{0}x((x+\tau-y_x)^2-(x-y_x)^2)/(2\varepsilon\tau).$$

For $\tau\searrow 0$, $\tilde{0}xj'_\varepsilon(x)\leqslant 2x\dfrac{x-y_x}{2\varepsilon}\leqslant\dfrac{(x-y_x)^2}{2\varepsilon}\leqslant\dfrac{(x-y_x)^2}{2\varepsilon}+j(y_x)$ for all $x>a_{\tilde 0}$, for any $\varepsilon>0$. It follows that there is an $\tilde a_0>0$ such that

$$j_\varepsilon(x)\geqslant\tilde{0}j'_\varepsilon(x)x,\quad\forall x\geqslant a_{\tilde 0},\quad\forall\varepsilon>0.$$

By analogy it follows that

$$j_\varepsilon(x)\geqslant\tilde{0}j'_\varepsilon(x)x,\quad\forall x<-\tilde a_{\tilde 0},\quad\forall\varepsilon>0.$$

Let $I_\varepsilon: H_0^1(\Omega)\to R$ be defined by

$$I_\varepsilon(u)=\frac{1}{2}\|u\|^2+\frac{\lambda}{2}\int_\Omega u^2dx+\int_\Omega j_\varepsilon(u)dx-\int_\Omega F_\varepsilon(u)dx. \tag{2.6}$$

It is easy to see that $I_\varepsilon\in C^1(H_0^1(\Omega),R)$ and $I_\varepsilon(0)=0$.

Lemma 2.4. *There exist $\alpha>0$ and $\rho>0$ such that $I_\varepsilon(u)\geqslant\alpha$ for all $u\in\partial B_\rho$ and all $\varepsilon>0$.*

Proof. Consider only the case $n>2$ (for $n=2$ is analogous). It suffices to show $\lim\limits_{u\to 0}h_\varepsilon(u)/\|u\|^2\leqslant 0$ uniformly relative to $\varepsilon>0$, where

$$h_\varepsilon(u)=\int_\Omega[F_\varepsilon(u)-j_\varepsilon(u)]dx. \tag{2.7}$$

By (C_2) there exists $\delta>0$ independent on ε such that

$$|F_\varepsilon(x)|\leqslant\tilde\varepsilon|x|^2,\ \text{if}\ |x|\leqslant\delta.$$

By $(\tilde C_1)$ it follows that $|F_\varepsilon(x)|\leqslant a_6|x|^{p+1}$ for $|x|\geqslant\delta$. Using the Poincaré and Sobolev inequalities, it follows:

$$\left|\int_\Omega F_\varepsilon(u)dx\right|\leqslant\int_\Omega(\tilde\varepsilon u^2+a_6|u|^{p+1})dx\leqslant a_7(\tilde\varepsilon\|u\|^2+\|u\|^{p+1}_{L^{p+1}}),\ \text{where}\ p>1.$$

Hence $\int_\Omega F_\varepsilon(u)dx/\|u\|^2\to 0$ for $u\to 0$ in $H_0^1(\Omega)$, uniformly with respect to $\varepsilon>0$.

Since $j_\varepsilon\geqslant 0$ the conclusion follows.
The proofs of the following two lemmas are similar to those of lemma 3.4, respectively lemma 3.6 in [1].

Lemma 2.5. *There exists $e \in C_0^\infty(\Omega)$, $e(x) \in D(j)$ for all $x \in \Omega$, e independent on ε, such that $I_\varepsilon(e) < 0$.*

Lemma 2.6. *The functions I_ε satisfy the following condition of Palais-Smale type:*
(PS) *If $\{u_n\}$ is a sequence of $H_0^1(\Omega)$, $I_\varepsilon(u_n) \to c \in R$ and $I'_\varepsilon(u_n) \to 0$, then $\{u_n\}$ has at least one convergent subsequence.*

Proof of Theorem 1. Using the Mountain Pass Theorem, it follows the existence of a critical point of I_ε, $u_\varepsilon \in H_0^1(\Omega)$ such that

$$I_\varepsilon(u_\varepsilon) = \inf_{g \in C([0,1];\, H_0^1(\Omega))} \sup_{t \in [0,1]} I_\varepsilon(g(t)),$$

where

$$I_\varepsilon(u_\varepsilon) = \frac{1}{2}\|u_\varepsilon\|^2 + \frac{\lambda}{2}\int_\Omega u_\varepsilon^2 dx + \int_\Omega j_\varepsilon(u_\varepsilon)dx - \int_\Omega F_\varepsilon(u_\varepsilon)dx,$$

$$I'_\varepsilon(u_\varepsilon) = -\Delta u_\varepsilon + \lambda u_\varepsilon + j'_\varepsilon(u_\varepsilon) - f_\varepsilon(u_\varepsilon) = 0.$$

Consider the function $\chi(t) = te$, $t \in [0, 1]$, $\chi \in C([0, 1]; H_0^1(\Omega))$

$$I_\varepsilon(u_\varepsilon) = \inf_{g \in \Gamma_1} \sup_{t \in [0,1]} I_\varepsilon(g(t)) \leqslant \sup_{t \in [0,1]} I_\varepsilon(\chi(te)).$$

Using $(\tilde{C}_1)$ it follows that $\alpha \leqslant I_\varepsilon(u_\varepsilon) < \zeta$ where $\zeta > 0$ is independent on ε. Taking $\Omega_\varepsilon = \{x \in \Omega;\ |u_\varepsilon(x)| \geqslant \tilde{a}_4\}$ we have

$$\zeta > I_\varepsilon(u_\varepsilon) \geqslant \frac{1}{2}\|u_\varepsilon\|^2 + \frac{\lambda}{2}\int_{\Omega_\varepsilon} u_\varepsilon^2 dx + \int_{\Omega_\varepsilon} j_\varepsilon(u_\varepsilon)dx - \tilde{\theta}\int_{\Omega_\varepsilon} f_\varepsilon(u_\varepsilon)u_\varepsilon dx +$$

$$+ \tilde{c}\varepsilon - \int_{\Omega_\varepsilon} F_\varepsilon(u_\varepsilon)dx \geqslant \frac{1}{2}\|u_\varepsilon\|^2 + \frac{\lambda}{2}\int_\Omega u_\varepsilon^2 dx - \tilde{\theta}\int_\Omega f_\varepsilon(u_\varepsilon)u_\varepsilon dx + \tilde{\theta}\int_\Omega j'_\varepsilon(u_\varepsilon)u_\varepsilon dx + \tilde{c}\varepsilon +$$

$$+ A \geqslant \left(\frac{1}{2} - \tilde{\theta}\right)(\|u_\varepsilon\|^2 + \lambda\int_\Omega u_\varepsilon^2 dx) + \tilde{A} + (I'_\varepsilon(u_\varepsilon), u_\varepsilon)$$

with A, $\tilde{A} \in R$. It follows that $\{u_\varepsilon\}$ is bounded in $H_0^1(\Omega)$. Thus there exists a subsequence (again denoted $\{u_\varepsilon\}$) such that

$$u_\varepsilon \to u^* \text{ weakly in } H_0^1(\Omega),$$
$$u_\varepsilon \to u^* \text{ strongly in } L^{p+1}(\Omega),$$
$$u_\varepsilon \to u^* \text{ a.e. in } \Omega$$

For $n > 2$ we have $|f_\varepsilon(x)| \leqslant \tilde{a}_1 + \tilde{a}_2|x|^p$ and it follows that $\{f_\varepsilon(u_\varepsilon)\}$ is bounded in $L^{(p+1)/p}(\Omega)$.

We may conclude that there exists $\gamma \in L^{(p+1)/p}(\Omega)$ such that $f_\varepsilon(u_\varepsilon) \to \gamma$ in $L^{(p+1)/p}(\Omega)$.

For $n = 2$ we have $|f_\varepsilon(x)| \leqslant \tilde{a}_3 e^{\tilde{a}(x)} = \psi(x)$ and it follows that $|f_\varepsilon(u_\varepsilon)| \leqslant \tilde{a}_3 e^{\tilde{a}|u_\varepsilon|} \Rightarrow$ $\Rightarrow |f_\varepsilon(u_\varepsilon)|^2 < \tilde{a}_3^2 e^{2\tilde{a}(u_\varepsilon)}$.

Since $\{u_\varepsilon\}$ is bounded in $H_0^1(\Omega)$ and $H_0^1(\Omega)$ is compact in $L^{p+1}(\Omega)$, (see e.g. [6]), then using $(\tilde{C}_1)$ it follows that $\{f_\varepsilon(u_\varepsilon)\}$ is bounded in $L^2(\Omega)$ and there is $\gamma \in L^2(\Omega)$ such that on a subsequence $f_\varepsilon(u_\varepsilon) \to \gamma$ in $L^2(\Omega)$. Thus $\gamma \in L^{(p+1)/p}(\Omega)$. Let $\nu > 0$ be a positive number. Since $u_\varepsilon \to u^*$ a.e. in Ω, there is $\omega \subseteq \Omega$ measurable such that

$$\text{meas}(\omega) < \nu \text{ and } u_\varepsilon \to u^* \text{ uniformly on } \Omega \setminus \omega.$$

Then for any $\eta > 0$ there is an $\varepsilon_0 > 0$ sufficiently small such that $|u_\varepsilon - u^*| < \eta/2$ for all $x \in \Omega \setminus \omega$ and $0 < \varepsilon < \varepsilon_0$. We have

$$\underline{f}_\eta(u^*(x)) \leqslant \overline{f}_\varepsilon(u_\varepsilon(x)) \leqslant \overline{f}_\eta(u^*(x)).$$

If $h \in L^\infty(\Omega \setminus \omega)$, $h \geqslant 0$ a.e. we have

$$\int_{\Omega\setminus\omega} \underline{f}_\eta(u^*(x))h\,dx \leqslant \int_{\Omega\setminus\omega} f_\varepsilon(u_\varepsilon(x))h\,dx \leqslant \int_{\Omega\setminus\omega} \overline{f}_\eta(u^*(x))h\,dx.$$

Let $\varepsilon \searrow 0$; using the weak convergence of $\{f_\varepsilon(u_\varepsilon)\}$ we obtain

$$\int_{\Omega\setminus\omega} \underline{f}_\eta(u^*)h\,dx \leqslant \int_{\Omega\setminus\omega} \gamma h\,dx \leqslant \int_{\Omega\setminus\omega} \overline{f}_\eta(u^*)h\,dx.$$

Using the Lebesgue theorem we obtain

$$\int_{\Omega\setminus\omega} \underline{f}(u^*)h\,dx \leqslant \int_{\Omega\setminus\omega} \gamma h\,dx \leqslant \int_{\Omega\setminus\omega} \overline{f}(u^*)h\,dx.$$

Since $h \geqslant 0$ was arbitrary it follows that

$$\underline{f}(u^*(x)) \leqslant \gamma \leqslant \overline{f}(u^*(x)) \text{ a.e. } x \in \Omega \setminus \omega.$$

Since ν may be chosen arbitrary small it follows that

$$\underline{f}(u^*) \leqslant \gamma \leqslant \overline{f}(u^*) \text{ a.e. in } \Omega \Rightarrow \gamma(x) \in \hat{f}(u^*(x)) \text{ a.e. in } \Omega.$$

We will prove now that $u^* \neq 0_{H_0^1(\Omega)}$. Indeed we have

$$0 < \alpha \leqslant \|u_\varepsilon\|^2 + \lambda\int_\Omega u_\varepsilon^2 dx + \int_\Omega j_\varepsilon(u_\varepsilon)dx - \int_\Omega F_\varepsilon(u_\varepsilon)dx \leqslant \|u_\varepsilon\|^2 +$$

$$+ \lambda\int_\Omega u_\varepsilon^2 dx + \int_\Omega j'_\varepsilon(u_\varepsilon)u_\varepsilon dx - \int_\Omega f_\varepsilon(u_\varepsilon)u_\varepsilon dx + \int_\Omega f_\varepsilon(u_\varepsilon)u_\varepsilon dx - \int_\Omega F_\varepsilon(u_\varepsilon)dx = (I'_\varepsilon(u_\varepsilon), u_\varepsilon) +$$

$$+ \int_\Omega f_\varepsilon(u_\varepsilon)u_\varepsilon dx - \int_\Omega F_\varepsilon(u_\varepsilon)dx \leqslant \overline{a}_1\int_\Omega u_\varepsilon^2 dx + \overline{a}_2\int_\Omega |u_\varepsilon|^{p+1} dx.$$

(We prove only for $n > 2$). Since $u_\varepsilon \to u^*$ strongly in $L^{p+1}(\Omega)$, we infer that $u^* \neq 0$. Now we have for every $v \in H_0^1(\Omega)$:

$$(u_\varepsilon, v-u_\varepsilon)+\lambda\int_\Omega u_\varepsilon(v-u_\varepsilon)dx+\int_\Omega j'_\varepsilon(u_\varepsilon)(v-u_\varepsilon)dx=\int_\Omega f_\varepsilon(u_\varepsilon)(v-u_\varepsilon)dx.$$

So,

$$\int_\Omega f_\varepsilon(u_\varepsilon)(v-u_\varepsilon)dx\leqslant(u_\varepsilon, v-u_\varepsilon)+\lambda\int_\Omega u_\varepsilon(v-u_\varepsilon)dx+\int_\Omega(j_\varepsilon(v)-j_\varepsilon(u_\varepsilon))dx.$$

We see that

$$\lim_{\varepsilon\searrow 0}\left[\int_\Omega f_\varepsilon(u_\varepsilon)(v-u_\varepsilon)dx-(u_\varepsilon, v-u_\varepsilon)+\lambda\int_\Omega u_\varepsilon(v-u_\varepsilon)dx\right]\geqslant$$

$$\geqslant\int_\Omega\gamma(v-u^*)dx-(u^*, v-u^*)+\lambda\int_\Omega u^*(v-u^*)dx.$$

By the Fatou lemma we have

$$\lim_{\varepsilon\searrow 0}\int_\Omega j_\varepsilon(u_\varepsilon)\,dx\geqslant\int_\Omega j(u^*)dx.$$

As $j_\varepsilon(x)\leqslant j(x)$ for all $x\in R$, it follows that

$$\overline{\lim_{\varepsilon\searrow 0}}\int_\Omega j_\varepsilon(v)dx\leqslant\int_\Omega j(v)dx.$$

We may conclude that for any $v\in H_0^1(\Omega)$

$$\int_\Omega(j(v)-j(u^*))dx+(u^*, v-u^*)+\lambda\int_\Omega u^*(v-u^*)dx\geqslant\int_\Omega\gamma(v-u^*).$$

So, problem (2.1) has at least one nontrivial Filippov solution.

§3. Some particular cases. For $j(x)=0$ for $x\geqslant 0$, $+\infty$ for $x<0$ we obtain the following obstacle problem :

$$\text{(3.1)}\qquad\begin{cases}-\Delta u+\lambda u-f(u)\leqslant 0 \text{ on } \Omega,\\ u\geqslant 0,\\ (-\Delta u+\lambda u-f(u))u=0,\\ u|\partial\Omega=0,\end{cases}$$

and by Theorem 1 we conclude

Theorem 2. *If conditions $(C_1)-(C_5)$ (with j defined as above) are satisfied, then there exist $u^*\in H_0^1(\Omega)$, $u^*\geqslant 0$, $u^*\neq 0$ and $\gamma\in L^{(p+1)/p}(\Omega)$ such that $\gamma(x)\in\hat{f}(u^*(x))$ a.e. in Ω and*

$$\text{(3.1)}'\qquad\int_\Omega\nabla u^*\nabla(v-u^*)dx+\lambda\int_\Omega u^*(v-u^*)dx\geqslant\int_\Omega\gamma(v-u^*)dx,\ \forall v\in H_0^1(\Omega),\ v\geqslant 0 \text{ a.e. in } \Omega$$

(3.1)′ is the weak form of the obstacle problem (3.1).

For $j\equiv 0$ we have the following problem for $\lambda\geqslant 0$:

$$\text{(3.2)}\qquad\begin{cases}-\Delta u+\lambda u=f(u) \text{ on } \Omega,\\ u|\partial\Omega=0.\end{cases}$$

Corollary. *If conditions $(C_1)-(C_4)$ are satisfied (with $j\equiv 0$), then problem (3.2) has at least one nontrivial Filippov solution.*

§ 4. A problem of Signorini type. Let consider the boundary value problem

$$\text{(4.1)}\qquad\begin{cases}-\Delta u+\lambda u=f(u) \text{ on } \Omega,\\ \dfrac{\partial u}{\partial\nu}+\beta(u)\ni 0 \text{ on } \partial\Omega\end{cases}$$

where $\lambda>0$.

Theorem 3. *If conditions $(C_1)-(C_5)$ are satisfied, then there exist $u^*\in H^1(\Omega)$, $u^*\neq 0$, $u^*(x)\in D(j)$ a.e. in $\partial\Omega$ and $\gamma\in L^{(p+1)/p}(\Omega)$ such that $\gamma(x)\in\hat{f}(u^*(x))$ a.e. in Ω and $\int_\Omega\nabla u^*\nabla(v-u^*)dx+\lambda\int_\Omega u^*(v-u^*)dx+\int_{\partial\Omega}(j(v)-j(u^*))dx\geqslant\int_\Omega\gamma(v-u^*)dx$, for any $v\in H^1(\Omega)$.*

Proof. We consider the functions $I_\varepsilon: H^1(\Omega)\to R$,

$$I_\varepsilon(u)=\frac{1}{2}\int_\Omega|\nabla u|^2dx+\frac{\lambda}{2}\int_\Omega u^2dx+\int_{\partial\Omega}j_\varepsilon(u)dx-\int_\Omega F_\varepsilon(u)dx.$$

It is possible to prove as in Theorem 1 that I_ε satisfy condition (PS) and there exist $\alpha, \rho>0$ independent of ε such that $I_{\varepsilon|\partial B_\rho}\geqslant\alpha$. It is easy to see that $I_\varepsilon\in C^1(H^1(\Omega); R)$ and $I_\varepsilon(0)=0$. By analogy we may prove the existence of a critical point of I_ε, u_ε (Mountain Pass Theorem) with $\{u_\varepsilon\}$ bounded in $H^1(\Omega)$. Then we have $u_\varepsilon\to u^*$ in $L^{p+1}(\Omega)$, $u^*\in H^1(\Omega)$, $u^*\in D(j)$ a.e. in $\partial\Omega$, $u^*\neq 0$ and the existence of $\gamma\in L^{(p+1)/p}(\Omega)$ such that $\gamma(x)\in\hat{f}(u^*(x))$ a.e. in Ω and

$$\int_\Omega\nabla u^*\nabla(v-u^*)dx+\lambda\int_\Omega u^*(v-u^*)dx+\int_{\partial\Omega}(j(v)-j(u^*))dx\geqslant\int_\Omega\gamma(v-u^*)dx,\ \forall v\in H^1(\Omega).$$

We may conclude that problem (4.1) has at least one nontrivial solution.

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ON A TWO SPECIES ECOLOGICAL SYSTEM WITH DIFFUSION DEPENDING ON AGES

BY

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1. **Introduction. Statement of the Problem.** In many papers in last years, the study of ecological systems of populations structured on ages and involving diffusion is performed (see e.g. [3, 4, 7, 10, 12]).

The diffusion coefficients are considered in most of these papers constant; in some others they depend on the time and on the point of the space. In a recent paper [9], we have studied the evolution of a population structured on ages with diffusion coefficient depending on the density u of the same population.

In the present paper, we will investigate an ecological system, consisting on two species, structured on ages — one of them, a predator, the other a prey — but with diffusion coefficients which depend on the density of the age a population, and on the densities of both entire populations in the environment.

Such a problem has not a classical solution, since, when the density of the age a population is zero, the diffusion coefficient must be considered zero, and so the fundamental equations of the system are degenerated. Such equations have been studied, also in relation with porous media, in [14, 6, 13, 1].

The space in which the diffusion takes place is R^2; a more dimensional space leads only to computational complications but not to other theoretical difficulties.

The mathematical model contains the following equations:

1. The evolution equations:

$$\frac{\partial u_i}{\partial a}+\frac{\partial u_i}{\partial t}-\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left(d_i\frac{\partial u_i}{\partial x_j}\right)-\Lambda_i u_i=0,\quad (i=1,2); \tag{1.1}$$

2. The offspring equations:

$$b_i(x,t)=\int_0^{1}\beta_i(x,t,\alpha)u_i(x,t,\alpha)d\alpha,\quad (i=1,2); \tag{1.2}$$

3. The initial repartition of the populations:

$$u_i(x,0,a)=u_{i0}(x,a),\qquad (i=1,2); \tag{1.3}$$