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ON A TWO SPECIES ECOLOGICAL SYSTEM WITH DIFFUSION DEPENDING ON AGES

BY

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1. Introduction. Statement of the Problem. In many papers in last years, the study of ecological systems of populations structured on ages and involving diffusion is performed (see e.g. [3, 4, 7, 10, 12]).

The diffusion coefficients are considered in most of these papers constant; in some others they depend on the time and on the point of the space. In a recent paper [9], we have studied the evolution of a population structured on ages with diffusion coefficient depending on the density u of the same population.

In the present paper, we will investigate an ecological system, consisting on two species, structured on ages — one of them, a predator, the other a prey — but with diffusion coefficients which depend on the density of the age a population, and on the densities of both entire populations in the environment.

Such a problem has not a classical solution, since, when the density of the age a population is zero, the diffusion coefficient must be considered zero, and so the fundamental equations of the system are degenerated. Such equations have been studied, also in relation with porous media, in [14, 6, 13, 1].

The space in which the diffusion takes place is R^2; a more dimensional space leads only to computational complications but not to other theoretical difficulties.

The mathematical model contains the following equations:

1. The evolution equations:

$$\frac{\partial u_i}{\partial a}+\frac{\partial u_i}{\partial t}-\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left(d_i\frac{\partial u_i}{\partial x_j}\right)-\Lambda_i u_i=0,\quad (i=1,2); \tag{1.1}$$

2. The offspring equations:

$$b_i(x,t)=\int_0^A \mu_i(x,t,\alpha)u_i(x,t,\alpha)d\alpha,\quad (i=1,2); \tag{1.2}$$

3. The initial repartition of the populations:

$$u_i(x,0,a)=u_{i0}(x,a),\qquad (i=1,2); \tag{1.3}$$

1. The compatibility equations:

$$u_{i0}(x, 0) = h_i(x, 0) = \int_0^A \beta_i(x, 0, \alpha) u_{i0}(x, \alpha) d\alpha, \quad (i = 1, 2); \tag{1.1}$$

where

a) $x = (x_1, x_2) \in R^2$ is a point in the space,

b) t is the time, $t \in [0, T]$, T — a positive constant,

c) a is the age $a \in [0, A]$, A — a positive constant,

d) $u_i(x, t, a)$ is the density of the age a population i, in x, at the moment t,

e) $\beta_i(x, t, a)$ is the fertility of the age a population i, in x, at the moment t,

f) d_i is the diffusion coefficient of the population i,

g) $u_{i0}: R^2 \times [0, A] \to R_+$, is the initial repartition of the age a population i, in x,

h) $h_i: R^2 \times [0, T]$ is the density of the offspring of the population i, in x, at the moment t,

i) Λ_i are proportionality factors.

Our aim is to investigate the behaviour of the system, when the diffusion coefficients d_i depend on the density u_i of the age a population, and on two functionals, each depending on the densities of the populations; to be more precise, and to avoid complicated calculations, we will suppose:

$$d_i = d_i(u_i, U_{ij}), \tag{1.5}$$

where

$$U_{ij} = \int_0^A k_{ij}(a, \alpha) u_j(x, t, \alpha) d\alpha, \quad (i, j = 1, 2;\ i \neq j), \tag{1.6}$$

k_{ij} being functions to be precised in the following.

Moreover, suggested by the Volterra theory of a prey-predator system, (see [17], ch. 1), we will suppose that the proportionality factors Λ_i are given by:

$$\Lambda_i = p_i - \int_0^t h_i(a, \alpha) u_j(x, t, \alpha) d\alpha, \quad (i, j = 1, 2,\ i \neq j), \tag{1.7}$$

$$p_1 > 0, \quad p_2 < 0, \quad h_1 > 0, \quad h_2 < 0, \quad p_i \text{ - constants},$$

which means that the first population ($i = 1$) is the prey, the second ($i = 2$) the predator. Indeed, the first population, being herbivorous, its proportionality factor Λ_i consists on a positive term p_1 — the natural increasing rate — diminished by another term, due to the existence of the predator living in the same environment, and hunting the first species; the effect of the existence of this second population, also depends on the age of its individuals, with a weight function h_1.

The second population — a carnivorous one — has a coefficient Λ_2, consisting on a negative constant term p_2, — since if it lives alone in the environment, it decreases for lack of food — majorized by a term due to the herbivorous species, hunted by the carnivorous individuals; the effect of the first species on the second is expressed in the form (1.7) for $i = 2$.

As we will see, the study of our system (1.1)—(1.4) is strongly connected with the non-linear system of parabolic equations

$$\frac{\partial u_i}{\partial t} - d_i \Delta u_i - \Lambda_i u_i = 0,$$

where, d_i, depending on u_i, vanish for $u_i = 0$. Such a system is a singular one, and as already said, it has not a classical, but only a weak solution.

2. **Preliminaries.** We will need in our paper, some properties of Hölder classes of functions; for these, we use the following notations, analogous to those of A. F r i e d m a n's monograph [5] on parabolic partial differential equations:

Let Ω be a domain in R^2, $x = (x_1, x_2) \in R^2$, A, $T > 0$ two constants and

$$Q = \Omega \times [0, T] \times [0, A],$$

$$Q_T = \Omega \times [0, T],$$

$$Q_A = \Omega \times [0, A]$$

cylinders in $R^2 \times R_+ \times R_+$, and $R^2 \times R_+$ respectively.

If $P(x, t, a)$, $P'(x', t', a') \in Q$, denote:

$$d(P, P') = (|x - x'|^2 + |t - t'| + |a - a'|)^{1/2}, \quad |x| = \sqrt{x_1^2 + x_2^2}.$$

Finally, if $f: Q \to R$, then

$$|f|_Q^{(0)} = \sup_Q |f|,$$

$$H_Q^{(\lambda)}(f) = \sup_{P, P' \in Q} \frac{|f(P) - f(P')|}{(d(P, P'))^{\lambda}},$$

$$|f|_Q^{(\lambda)} = |f|_Q^{(0)} + H_Q^{(\lambda)}(f).$$

The set of functions for which $|f|_Q^{(\lambda)}$ is finite is structured as a Banach space with norm $|f|_Q^{(\lambda)}$, and will be denoted by $C_{(\lambda)}$.

In an analogous manner, if $D_x^k f$, $D_t f$, $D_a f$ are arbitrary of order k derivatives with respect to x, of first order with respect to t, and of first order with respect to a, respectively, $|f|_Q^{(2+\lambda)}$ will be

$$|f|_Q^{(2+\lambda)} = |f|_Q^{(\lambda)} + |D_x f|_Q^{(\lambda)} + |D_x^2 f|_Q^{(\lambda)} + |D_t f|_Q^{(\lambda)} + |D_a f|_Q^{(\lambda)},$$

and the set of functions f, for which $|f|_Q^{(2+\lambda)}$ is finite, also is a Banach space, structured even by the norm $|f|_Q^{(2+\lambda)}$ and which will be denoted $C_{(2+\lambda)}(Q)$.

The notations $C_{(2+\lambda)}(Q_A)$, $C_{(2+\lambda)}(Q_T)$ will be used for particular Banach spaces $C_{(2+\lambda)}(Q)$ when t, respectively a will be constant.

With the metric

$$\delta(f_1, f_2) = |f_1 - f_2|_Q^{(2+\lambda)},$$

$C_{(2+\lambda)}(Q)$ is a complete metric space.

The following properties will be useful in our paper:

1. $C_{(\lambda)}(Q) \supset C_{(2+\lambda)}(Q)$ and $|f|_Q^{(2+\lambda)} \geqslant |f|_Q^{(\lambda)}$.

2. If $f \in C_{(2+\lambda)}(Q)$, then $f(\cdot, \ldots, a) \in C_{(2+\lambda)}(Q_T)$ for each $a \in [0, A]$ and $f(\cdot, \ldots, t) \in C_{(2+\lambda)}(Q_A)$ for each $t \in [0, T]$.

3. If $f \in C_{(2+\lambda)}(Q)$ and l is a linear transformation on $[0, T] \times [0, A]$, then $f \circ l \in C_{(2+\lambda)}(Q)$.

4. If $k(\cdot, \alpha) \in C_{(1+\lambda)}([0, A])$, $f(\cdot, \ldots, a) \in C_{(2+\lambda)}(Q_T)$ for each $a \in [0, A]$ and $k(a, \cdot)$, $f(x, t, \cdot)$ are integrable on $[0, A]$, then

$$g = \int_0^A k(a, \alpha) f(x, t, \alpha)\, d\alpha \in C_{(2+\lambda)}(Q).$$

5. If $g : [m, M] \to R$ is Lipschitz continuous and $f \in C_{(2+\lambda)}(Q)$, $m \leqslant f \leqslant M$, then $g \circ f \in C_{(2+\lambda)}(Q)$.

The hypotheses. In the following Ω will be R^2, so that $Q = R^2 \times [0, T] \times [0, A]$, $Q_T = R^2 \times [0, T]$, $Q_A = R^2 \times [0, A]$.

a) The two functions d_i, $(i = 1, 2)$ — the diffusion coefficients — satisfy the following conditions:

$$d_i(0, U_{ij}) = 0, \quad d_i(u_i, U_{ij}) > 0 \text{ for } u_i > 0,$$

i.e., if in a certain point of the space, at a given moment, the density of the age a population is zero, then there does not exist diffusion. In all other cases, the diffusion coefficients are positive.

We suppose also d_i differentiable with respect to all variables, and moreover,

$$\frac{\partial d_i}{\partial u_i}, \quad \frac{\partial d_i}{\partial U_{ij}} > 0,$$

i.e. they are increasing in all variables. The above derivatives are Lipschitz continuous with respect to u_i, U_{ij},

$$\left| \frac{\partial d_i}{\partial u_i}(u_i, U_{ij}) - \frac{\partial d_i}{\partial u_i}(u_i', U_{ij}') \right| \leqslant L\left(|u_i - u_i'| + \sum_{j=1}^{2} |U_{ij} - U_{ij}'|\right).$$

b) The functions $h_i(\cdot, \alpha) k_i(\cdot, \alpha) \in C_{(\lambda)}([0, A])$ for each $\alpha \in [0, A]$, and $h_i(a, \cdot)$, $k_i(a, \cdot)$ are integrable on $[0, A]$ for each $a \in [0, A]$.

Moreover

$$k_{ij} \geqslant 0, \quad h_1 > 0, \quad h_2 < 0,$$

$$\int_0^A k_{ij}(a, \alpha)\, d\alpha \leqslant \overline{k_{ij}}, \quad \int_0^A |h_i(a, \alpha)|\, d\alpha \leqslant \overline{h}_i.$$

c) p_1, p_2 are constants.

d) $\mu_i(\cdot, \ldots, \alpha) \in C_{(2+\lambda)}(Q_T)$ for each $\alpha \in [0, A]$ and $\mu_i(x, t, \cdot)$ are integrable on $[0, A]$ for each $(x, t) \in Q_T$, and

$$\mu_i \geqslant 0, \quad \overline{\overline{\mu}}_i \leqslant \int_0^A \mu_i(x, t, \alpha)\, d\alpha \leqslant \overline{\mu}_i, \quad \overline{\mu}_i, \overline{\overline{\mu}}_i \text{ constants.}$$

e) $u_{i0} : Q \to R_+$ belong to $C_{(2+\lambda)}(Q_A)$.

As our aim is to prove that the system (1.1) — (1.4) has a weak solution, we state now the following

Definition. *A weak solution of the system (1.1) — (1.4) (in the sense of the theory of distributions) is a system of four functions (u_1, u_2, b_1, b_2), which satisfy the following conditions:*

a) *$u_i : Q \to R_+$ are Lipschitz continuous, bounded functions on their domain of definition, with bounded first order generalized derivatives, with respect to x_i;*

b) *$b_i : Q_T \to R_+$ are bounded continuous functions on their domain of definition;*

c) *These four functions satisfy the relations:*

$$\iiiint_Q \left\{ u_i(\varphi_t + \varphi_a) - \sum_{j=1}^{2} \varphi_{x_j} d_i \frac{\partial u_i}{\partial x_j} + u_i \Lambda_i \varphi \right\} dx_1 dx_2 dt da =$$

(2.1)

$$= \iiint_{Q_A} \varphi(x, 0, a) u_{i0}(x, a)\, dx_1 dx_2 da + \iiint_{Q_T} \varphi(x, t, 0) b_i(x, t)\, dx_1 dx_2 dt,$$

(2.2)
$$b_i(x, t) = \int_0^A \mu_i(x, t, \alpha) u_i(x, t, \alpha)\, d\alpha, \quad (i = 1, 2),$$

where φ are C_0^∞ functions on Q.

In the following, obviously without restraining the generality, we will suppose $T = A$.

3. **Transformation of the system (2.1), (2.2).** We decompose the domain Q in two disjoint subdomains Q_1 and Q_2, $Q_1 \cup Q_2 = Q$, so that $Q_1 = Q \cap \{a \geqslant t\}$ and $Q_2 = Q \cap \{a < t\}$.

The integral in the left hand side of (2.1) will be the sum of two integrals, the first on Q_1, the second on Q_2. In the first, we perform the transformation:

$$a = \alpha + \tau, \quad t = \tau,$$

and denote:

$$u_i(x, t, a) = u_i(x, \tau, \alpha + \tau) = v_i(x, \tau, \alpha) = v_i(x, t, a - t);$$

in the second integral we take

$$t = \alpha + \tau, \quad a = \alpha,$$

and denote

$$u_i(x, t, a) = u_i(x, \alpha + \tau, \alpha) = w_i(x, \tau, \alpha) = w_i(x, t - a, a).$$

(2.1) becomes:

$$\iiiint\limits_{Q_1}\left(v_i\rho_\tau-\sum_{j=1}^{2}\bar{d}_i\rho_{x_j}v_{ix_j}+\rho\bar{\Lambda}_i v_i\right)dx_1dx_2d\tau d\alpha+$$

(3.1)
$$+\iiiint\limits_{Q_2}\left(w_i\sigma_\alpha-\sum_{j=1}^{2}\bar{\bar{d}}_i\sigma_{x_j}w_{ix_j}+\sigma\bar{\bar{\Lambda}}_i w_i\right)dx_1dx_2d\tau d\alpha=$$

$$=\iiint\limits_{Q_A}\rho(x,0,\alpha)u_{i0}(x,\alpha)dx_1dx_2d\alpha+\iiint\limits_{Q_T}\sigma(x,\tau,0)b_i(x,\tau)dx_1dx_2d\tau.$$

(3.2)
$$b_i(x,\tau)=\int_0^\tau\mu_i(x,\tau,\beta)w_i(x,\tau-\beta,\beta)d\beta+$$

$$+\int_\tau^A\mu_i(x,\tau,\beta)v_i(x,\tau,\beta-\tau)d\beta.$$

where

$$\bar{d}_i=d_i(v_i(x,\tau,\alpha),\ \bar{U}_{i1}(v_1,w_1),\ \bar{U}_{i2}(v_2,w_2)),$$

$$\bar{\bar{d}}_i=d_i(w_i(x,\tau,\alpha),\ \bar{\bar{U}}_{i1}(v_1,w_1),\ \bar{\bar{U}}_{i2}(v_2,w_2)),$$

(3.3)
$$\bar{U}_{ij}=\int_0^\tau K_{ij}(\alpha+\tau,\beta)w_j(x,\tau-\beta,\beta)d\beta+\int_\tau^A K_{ij}(\alpha+\tau,\beta)v_j(x,\tau,\beta-\tau)d\beta,$$

$$\bar{\bar{U}}_{ij}=\int_0^{\alpha+\tau}K_{ij}(\alpha,\beta)w_j(x,\alpha+\tau-\beta,\beta)d\beta+\int_{\alpha+\tau}^A K_{ij}(\alpha,\beta)v_j(x,\alpha+\tau,\beta-\alpha-\tau)d\beta,$$

$$\bar{\Lambda}_i(v_j,w_j)=p_i-\int_0^\tau h_i(\alpha+\tau,\beta)w_j(x,\tau-\beta,\beta)d\beta+$$

$$+\int_\tau^A h_i(\alpha+\tau,\beta)v_j(x,\tau,\beta-\tau)d\beta,$$

(3.4)
$$\bar{\bar{\Lambda}}_i(v_j,w_j)=p_i-\int_0^{\alpha+\tau}h_i(\alpha,\beta)w_j(x,\alpha+\tau-\beta,\beta)d\beta+$$

$$+\int_{\alpha+\tau}^A h_i(\alpha,\beta)v_j(x,\alpha+\tau,\beta-\alpha-\tau)d\beta.$$

(3.5)
$$\rho(x,\tau,\alpha)=\varphi(x,\tau,\alpha+\tau),$$
$$\sigma(x,\tau,\alpha)=\varphi(x,\alpha+\tau,\alpha).$$

4. The study of an auxiliary system. Suggested by the form of the relations (3.1), (3.2), we consider

a) the two sequences $\{\bar{u}_{i0}^{(n)}\}$ $(i=1,2)$ of functions on Q_A, such that

$$\bar{u}_{i0}^{(n)}(\cdot,\alpha)\in C^\infty([0,A]),\quad \bar{u}_{i0}^{(n)}(x,\cdot)\in C^\infty(\Omega),$$

and

$$\lim_{n\to\infty}\bar{u}_{i0}^{(n)}=u_{i0},$$

b) the set of cylinders $Q^{(n)}=\{(x,\tau,\alpha):\ |x|\leqslant n,\ \tau\in[0,T],\ \alpha\in[0,A]\}$,

c) the two sequences $\{u_{i0}^{(n)}\}$ belonging to the same classes as $\bar{u}_{i0}^{(n)}$ and moreover, satisfying:

$$\bar{u}_{i0}^{(n)}=u_{i0}^{(n)}\ \text{on}\ |x|\leqslant n-2,\ \tau=0,\ \alpha\in[0,A],$$

$$u_{i0}^{(n)}=M\ \text{on}\ |x|\geqslant n\ \ \tau=0,\ \alpha\in[0,A],$$

$$\lim_{n\to\infty}u_{i0}^{(n)}=u_{i0}\ \text{(uniformly)},$$

$$\bar{u}_{i0}^{(n+1)}\leqslant u_{i0}^{(n+1)}\leqslant u_{i0}^{(n)}\leqslant M,$$

$$\left|\frac{\partial u_{i0}^{(n)}}{\partial x_i}\right|\leqslant k';$$

here M is

$$M=\sup_{Q_A}\max u_{i0}(x,\alpha).$$

d) The system of equations:

$$\mathcal{L}_i v_i=\frac{\partial v_i}{\partial\tau}-\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left(\bar{d}_i\frac{\partial v_i}{\partial x_j}\right)-\bar{\Lambda}_i v_i=0,$$

$$\mathcal{M}_i w_i=\frac{\partial w_i}{\partial\alpha}-\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left(\bar{\bar{d}}_i\frac{\partial w_i}{\partial x_j}\right)-\bar{\bar{\Lambda}}_i w_i=0,\ \text{on}\ Q^{(n)},$$

$$v_i(x,0,\alpha)=u_{i0}^{(n)}(x,\alpha),\ |x|\leqslant n,\ \alpha\in[0,A],$$

(4.1)
$$b_i(x,\tau)=\int_0^t\mu_i(x,\tau,\beta)w_i(x,\tau-\beta,\beta)d\beta+\int_\tau^A\mu_i(x,\tau,\beta)v_i(x,\tau,\beta-\tau)d\beta.$$

$$v_i(x,\tau,\alpha)|_{x=n}=M,\qquad \alpha\in[0,A],$$

$$w_i(x,\tau,0)=b_i(x,\tau),\qquad \tau\in[0,T];$$

$$\underset{(i=1,2)}{w_i(x,\tau,\alpha)}|_{x=n}=\max_{|x|=n}b_i(x,\tau).$$

We can now prove our first theorem.

Theorem 1. *If d_i, Λ_i, K_{ij}, h_i, $u_{i0}^{(n)}$ satisfy the hypotheses stated in 2, the system (4.1) has a solution, belonging to $C_{(2+\lambda)}(Q^{(n)})$ for each n.*

Proof. The proof will be performed in three steps.

First step. Consider on the cylinder $Q^{(n)}$, the linear in $V_i^{(n)}$ system

$$
\begin{aligned}
&\mathcal{L}_i V_i^{(n)} = \frac{\partial V_i^{(n)}}{\partial \tau} - \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left[\bar{d}_i(v_i, U_{ij}(v_j, w_j))\frac{\partial V_i^{(n)}}{\partial x_j}\right] - \bar{\Lambda}_i(v_j, w_j)V_i^{(n)} = 0,\\
(4.2)\qquad & V_i^{(n)}(x, 0, \alpha) = u_{i0}^{(n)}(x, \alpha), \qquad (i, j = 1, 2,\ i \neq j)\\
& V_i^{(n)}(x, \tau, \alpha)\Big|_{|x| = n} = M,
\end{aligned}
$$

where $\bar{d}_i$, $\bar{\Lambda}_i$ are the functions defined in (3.3), (3.4), in which we suppose that v_i, w_i satisfy the following conditions:

$$
(4.3)\qquad v_i,\ w_i \in C_{(2+\lambda)}(Q_T^{(n)}) \ \text{for each}\ \alpha \in [0, A],
$$

and

$$
(4.4)\qquad \begin{cases} 0 < m_i \leq v_i \leq M_i, & 0 < r_i \leq w_i \leq R_i,\\ |v_i|_{Q_T}^{(2+\lambda)} \leq M_i, & |w_i|_{Q_A}^{(2+\lambda)} \leq R_i, \quad m_i, M_i, r_i, R_i \text{ - constants.} \end{cases}
$$

We remark now that, if we denote

$$
\begin{aligned}
&\bar{d}_i(x, \tau, \alpha) = d_i\Big(v_i(x, \tau, \alpha), \int_0^{\tau} K_{ij}(\alpha + \tau, \beta)w_j(x, \tau - \beta, \beta)d\beta + \\
&\qquad + \int_{\tau}^{A} K_{ij}(\alpha + \tau, \beta)v_j(x, \tau, \beta - \tau)d\beta\Big),\\
&\bar{\bar{d}}_i(x, \tau, \alpha) = d_i\Big(w_i(x, \tau, \alpha), \int_0^{\pi + \alpha} K_{ij}(\alpha, \beta)w_j(x, \alpha + \tau - \beta, \beta)d\beta + \\
&\qquad + \int_{\alpha + \tau}^{A} K_{ij}(\alpha, \beta)v_j(x, \alpha + \tau, \beta - \alpha - \tau)d\beta\Big),\\
(4.5)\qquad &\bar{l}_i(x, \tau, \alpha) = p_i - \int_0^{\tau} h_i(\alpha + \tau, \beta)w_j(x, \tau - \beta, \beta)d\beta + \\
&\qquad + \int_{\tau}^{A} h_i(\alpha + \tau, \beta)v_j(x, \tau, \beta - \tau)d\beta,\\
&\bar{\bar{l}}_i(x, \tau, \alpha) = p_i - \int_0^{\alpha + \tau} h_i(\alpha, \beta)w_j(x, \alpha + \tau - \beta, \beta)d\beta + \\
&\qquad + \int_{\alpha + \tau}^{A} h_i(\alpha, \beta)v_j(x, \alpha + \tau, \beta - \alpha - \tau)d\beta, \quad (i, j = 1, 2,\ i \neq j),
\end{aligned}
$$

then, as a consequence of the properties staded in 2. and the hypotheses on d_i, Λ_i, it follows:

$$
\bar{d}_i,\ \bar{\bar{d}}_i,\ \bar{l}_0,\ \bar{\bar{l}}_i \in C_{(2+\lambda)}(Q)
$$

We also remark that

$$
(4.6)\qquad \frac{\partial \bar{d}_i}{\partial x_j},\ \frac{\partial \bar{\bar{d}}_i}{\partial x_j},\ \frac{\partial \bar{l}_i}{\partial x_j},\ \frac{\partial \bar{\bar{l}}_i}{\partial x_j} \in C_{(1+\lambda)}(Q)
$$

as a consequence of the same hypotheses. Since $C_{(1+\lambda)}(Q)$, $C_{(2+\lambda)}(Q) \subset C_{(\lambda)}(Q)$, it follows that all the coefficients of the equations (4.2) belong to $C_{(\lambda)}(Q)$.

We remark finally, that by eventually changing v_i in $v_i e^{\omega_i \tau}$, by a suitable choice of ω_i, we can suppose $\bar{\Lambda}_i \geq 0$.

From the conditions imposed to v_i, w_i, and the hypotheses, it follows that the coefficients of each equation $\mathcal{L}_i v_i^{(n)} = 0$ satisfy the conditions:

A. $\bar{d}_i \leq K_1$, $\left|\dfrac{\partial \bar{d}_i}{\partial x_j}\right| < K_1$, $|\Lambda_i| \leq K_1$,

B. $\bar{d}_i > K_2 > 0$, K_1, K_2 constants on $Q_T^{(n)}$.

Moreover, the conditions on the frontier of $Q_T^{(n)}$ are continuous and $Q^{(n)}$ satisfy the conditions of Theorem 7 (p. 88) and Consequence 2 p. 95 of Friedmann's monograph [5] already quoted.

It results that the system (4.2) has a unique solution $V_i^{(n)} \subset C_{(2+\lambda)}(Q_T)$ for each $\alpha \in [0, A]$, and

$$
(4.7)\qquad |V_i^{(n)}|_{Q_T}^{(2+\lambda)} \leq K|V_i^{(n)}|^{(0)}.
$$

It remains to estimate $|V_i^{(n)}|^{(0)}$. Or, replacing in (4.2) $V_i^{(n)}$ by $V_i^{(n)} = \bar{V}_i^{(n)} e^{\Gamma_i \tau}$, we obtain the system

$$
\begin{aligned}
&\frac{\partial \bar{V}_i^{(n)}}{\partial \tau} - \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(\bar{d}_i \frac{\partial \bar{V}_i^{(n)}}{\partial x_j}\right) + (\Gamma_i - \bar{\Lambda}_i)\bar{V}_i^{(n)} = 0,\\
(4.8)\qquad &\bar{V}_i^{(n)}(x, 0, \alpha) = u_{i0}^{(n)}(x, \alpha),\\
&\bar{V}_i^{(n)}(x, \tau, \alpha)|_{|x| = n} = Me^{-\Gamma_i \tau}.
\end{aligned}
$$

By applying the maximum principle, it follows that, if we choose

$$
\Gamma_i \geq \max \bar{\Lambda}_i,
$$

it results that the maximum of $\bar{V}_i^{(n)}$ is on the frontier of $Q_T^{(n)}$, and then that

$$
(4.9)\qquad V_i^{(n)} \leq Me^{\Gamma_i \tau}.
$$

In an analogous manner, by substituting $V_i^{(n)}$ by

$$
(4.9')\qquad \bar{V}_i^{(n)} = V_i^{(n)} e^{\gamma_i \tau},\ \gamma_i > 0,
$$

and choosing $-\gamma_i \leq \min \bar{\Lambda}_i$, we deduce

$$V_i^{(n)} \geq m^{(n)} e^{-\gamma_i T},$$

where $m^{(n)} = \min_{|x| \leq n} |u_{i0}^{(n)}|$.

It follows now from (4.7) that

$$|V_i^{(n)}|_{Q_T^{(n)}}^{(2+\lambda)} \leq K M e^{\Gamma_i \tau} \tag{4.10}$$

and also

$$0 \leq m^{(n)} e^{-\gamma_i T} \leq V_i^{(n)} \leq M_i e^{\Gamma_i \tau}.$$

Remark 1. As a consequence of the above considerations, it results that the solutions $V_i^{(n)}$ of the equations (4.2) are positive.

Remark 2. Also as a consequence, it follows that

$$d_i(V_i^{(n)},\ U_{ij}(V_i^{(n)},\ w_i)) > 0$$

and so our system is non-degenerate.

Remark 3. Analogous bounds can be found in [11].

Second step. Considering the pair (w_1, w_2) fixed, we conceive now the system (4.2) as a mapping $\mathcal{T}$ of the space $\mathcal{V}$ of pairs of functions $\{v_i\}$ on the space $\mathcal{V}'$ of pairs functions $\{V_i^{(n)}\}$ — solutions of the system (4.2) — and will establish conditions under which this mapping satisfies Banach's fixed point theorem.

To this end we assume the set $\mathcal{V}$ as a space $C_{(2+\lambda)}(Q_T^{(n)})$ and the norm of the belonging functions satisfying (4.4). Then, as it results from (4.10) and (4.4), if we take

$$M_i = K M e^{\Gamma_i \tau}, \qquad m_i = m^{(n)} e^{-\gamma_i T} > 0,$$

$\mathcal{T}$ maps $\mathcal{V}$ into itself. Moreover, the last inequality proves that $\mathcal{T}$ maps a nondegenerate system (4.2) in a nondegenerate one. It remains to prove that under certain conditions $\mathcal{T}$ is a contraction with respect to the metric δ, induced by the norm $|\cdot|_{Q_T^{(n)}}^{(2+\lambda)}$.

To this end, we consider two pairs (v_{i1}), (v_{2i}), $(i=1, 2)$ satisfying (4.4), and the corresponding pairs $(V_{i1}^{(n)})$, $(V_{i2}^{(n)})$ of solutions of (4.2); then denoting by

$$Z_i = V_{i1}^{(n)} - V_{i2}^{(n)},$$

it is seen that Z_i satisfies the system

$$\frac{\partial Z_i}{\partial \tau} - \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(\bar{d}_{i1} \frac{\partial Z_i}{\partial x_j}\right) - \bar{\Lambda}_{i1} Z_i = F_i, \tag{4.11}$$

$$Z_1(x, 0, \alpha) = 0, \qquad Z_i(x, \tau, \alpha)|_{|x|=n} = 0,$$

where, with a notation which can be understood by itself:

$$F_i = (\bar{d}_{i1} - \bar{d}_{i2}) \Delta \bar{v}_{i2}^{(n)} + \left(\frac{\partial \bar{d}_{i1}}{\partial x_j} - \frac{\partial \bar{d}_{i2}}{\partial x_j}\right) \frac{\partial \bar{V}_{i2}}{\partial x_j} + (\bar{\Lambda}_{i1} - \bar{\Lambda}_{i2}) V_{i2}^{(n)}. \tag{4.12}$$

Or, from the conclusion of the quoted theorems in [5], we have:

$$|\Delta V_{i2}^{(n)}|,\ \left|\frac{\partial V_{i2}^{n}}{\partial x_j}\right|,\ |V_{i2}^{(n)}| \leq K |V_i^{(n)}|_{Q_T^{(n)}}^{(0)}. \tag{4.13}$$

On the other hand, from the hypotheses (§ 2) and from the expressions (4.7), it follows

$$\bar{d}_{i1} - \bar{d}_{i2} = \left(\frac{\partial \bar{d}_i}{\partial v_i}\right)^{*} (v_{i1} - v_{i2}) + \sum_{j=1}^{2} \left(\frac{\partial \bar{d}_i}{\partial U_{ij}}\right)^{*} \int_{\tau}^{A} k_{ij}(\alpha + \tau, \beta)[v_{j1}(x, \tau, \beta - \tau) - v_{j2}(x, \tau, \beta - \tau)]\, d\beta,$$

so that

$$|\bar{d}_{i1} - \bar{d}_{i2}| \leq L_1(|v_{11} - v_{12}|_{Q_T^{(n)}}^{(0)} + |v_{21} - v_{22}|_{Q_T^{(n)}}^{(0)}$$

where L_1 is a constant depending on $(\partial \bar{d}_i / \partial v_i)$, $(\partial \bar{d}_i / \partial U_{ij})$, k_{ij}. Analogously

$$\left|\frac{\partial \bar{d}_{i1}}{\partial x_j} - \frac{\partial \bar{d}_{i2}}{\partial x_j}\right| \leq L_2\left(\left|\frac{\partial v_{11}}{\partial x_j} - \frac{\partial v_{12}}{\partial x_j}\right|_{Q_T^{(n)}}^{(0)} + \left|\frac{\partial v_{21}}{\partial x_j} - \frac{\partial v_{22}}{\partial x_j}\right|_{Q_T^{(n)}}^{(0)},\right.$$

$$|\bar{\Lambda}_{i1} - \bar{\Lambda}_{i2}| \leq L_3 |v_{j1} - v_{j2}|_{Q_T^{(n)}}^{(0)},$$

L_2 depends on the Lipschitz constant from the hypotheses and L_3 depends on h_i;

Finally we have:

$$|F_i| \leq \tilde{L}\left(\sum_{j=1}^{2} |v_{j1} - v_{j2}|_{Q_T^{(n)}}^{(0)} + \sum_{j,k=1}^{2} \left|\frac{\partial v_{j1}}{\partial x_k} - \frac{\partial v_{j2}}{\partial x_k}\right|_{Q_T^{(n)}}^{(0)}\right),\ \tilde{L} \text{ — a constant.}$$

On the other hand, according to the hypotheses on $\bar{d}_i$ and their derivatives (the coefficients of $\mathcal{L}_i$), we have $F_i \in C_{(\lambda)}$. It follows that

$$|Z_i|_{Q_T}^{(2+\lambda)} \leq K |F_i|_{Q_T}^{(\lambda)}.$$

But obviously

$$|v_{j1} - v_{j2}|_{Q_T^{(n)}}^{(0)},\ \left|\frac{\partial v_{j1}}{\partial x_j} - \frac{\partial v_{j2}}{\partial x_j}\right|_{Q_T^{(n)}}^{(0)} \leq |v_{j1} - v_{j2}|_{Q_T^{(n)}}^{(2+\lambda)},$$

so that:

$$|Z_1|_{Q_T^{(n)}}^{(2+\lambda)} + |Z_2|_{Q_T^{(n)}}^{(2+\lambda)} = \sum_{j=1}^{2} |V_{j1} - V_{j2}|_{Q_T^{(n)}}^{(2+\lambda)} \leq 2KL \sum_{j=1}^{2} |v_{j1} - v_{j2}|_{Q_T^{(n)}}^{(2+\lambda)},$$

and if we suppose $2KL < 1$, it follows that the mapping $\mathcal{T}$ is a contraction, and, as a consequence, that it has a unique fixed point. This means that the system

$$\begin{gathered} \frac{\partial V_i^{(n)}}{\partial \tau} - \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(\bar{d}_i(V_i^{(n)},\ \bar{U}_{i1}(V_1^{(n)}, w_1),\ \bar{U}_{i2}(V_2^{(n)} w_2)\right) \frac{\partial V_1^{(n)}}{\partial x_j} - \\ - \Lambda_i(V_j^{(n)},\ w_j) V_1^{(n)} = 0, \\ V_i^{(n)}(x, 0, \alpha) = u_{i0}^{(n)}(x, \alpha), \\ V_i^{(n)}(x, \tau, \alpha)|_{|x|=n} = M \end{gathered} \tag{4.14}$$

has a unique solution, belonging to $C_{(2+\lambda)}(Q_T)$ for each $\alpha[0, A]$, and for each pair $w_1, w_2 \in C_{(2+\lambda)}(Q)$ which satisfy (4.4).

Remark 1. The boundedness of the derivatives (4.13) also results from a theorem of S.N. B e r n s t e i n [2].

Remark 2. The solution of the system (4.14) is obviously positive, as one can easily see from the fact that it is a limit of positive solutions of (4.2).

Remark 3. The solution of (4.15) is Hölder continuous of class $C_{(2+\lambda)}(Q^{(n)})$. To prove this assertion, we write two systems (4.2), one for $\alpha = \alpha_1$, the other for $\alpha = \alpha_2$, denote $v_{ij} = v_i(x, \tau, \alpha_j)$, $w_{ij} = w_i(x, \tau, \alpha_j)$, $V_{ij}^{(n)}$ the solutions of the corresponding systems (4.2), and finally $Z_i = V_{i1}^{(n)} - V_{i2}^{(n)}$. Then using a technique analogous to that already used to prove the contraction of $\mathcal{T}$, we obtain by the maximum principle

$$|Z_1| + |Z_2| = \sum_{i=1}^{2} |V_{i1} - V_{i2}| \leqslant L' \left\{ \sum_{i=1}^{2} |v_{i1} - v_{i2}|_{Q_T^{(n)}}^{(0)} + |w_{i1} - w_{i2}|_{Q_T^{(n)}}^{(0)} + \right.$$

$$\left. + \sum_{j=1}^{2} \left(\left| \frac{\partial v_{i1}}{\partial x_j} - \frac{\partial v_{i2}}{\partial x_j} \right|_{Q_T^{(n)}}^{(0)} + \left| \frac{\partial w_{i1}}{\partial x_j} - \frac{\partial w_{i2}}{\partial x_j} \right|_{Q_T^{(n)}}^{(0)} \right) \right\},$$

L' a new constant. Using the properties of V_{ij}, w_{ij}, the statement follows for the solution of (4.2). Since the solution of (4.15) is a limit of solutions of systems (4.2), and L' is a constant, the conclusion results.

For the following, we must estimate the difference between two solutions $V_{i1}^{(n)}$, $V_{i2}^{(n)}$ of (4.15), corresponding to two pairs of functions w_{i1}, w_{i2}. We denote

$$Y_i = V_{i1}^{(n)} - V_{i2}^{(n)};$$

this difference will satisfy the system:

$$\text{(4.15)} \quad \frac{\partial Y_i}{\partial \tau} - \sum_{j=1}^{2} \frac{\partial}{\partial x_j} \left(d_i(V_{i1}^{(n)}, U_{il}(V_{il}^{(n)}, w_{il})) \frac{\partial Y_i}{\partial x_j} \right) - \bar{\Lambda}_i(V_{k1}^{(n)} - V_{k2}^{(n)}) Y_i = G_i,$$

$$(i, k = 1, 2, \ i \neq k).$$

$$Y_i(x, 0, \alpha) = 0, \quad Y_i(x, \tau, \alpha)|_{|x| = n} = 0,$$

where, in this case, the function G_i is majorized by:

$$\sum_{i=1}^{2} |G_i| \leqslant K_5 \sum_{i=1}^{2} \left(|V_{i1}^{(n)} - V_{i2}^{(n)}|_{Q^{(n)}}^{(0)} + \sum_{j=1}^{2} \left| \frac{\partial V_{i1}^{(n)}}{\partial x_j} - \frac{\partial V_{i2}^{(n)}}{\partial x_j} \right|_{Q^{(n)}}^{(0)} + \right.$$

$$\left. + |w_{i1} - w_{i2}|_{Q^{(n)}}^{(0)} + \sum_{j=1}^{2} \left| \frac{\partial w_{i1}}{\partial x_j} - \frac{\partial w_{i2}}{\partial x_j} \right|_{Q^{(n)}}^{(0)} \right),$$

K_5 a constant dependent linearly on K_1, K, $\bar{h}$, and polynomialy on $\bar{k}_{ij}$.

As a consequence of the same theorem already used, it results

$$\sum_{i=1}^{2} |Y_i|_{Q^{(n)}}^{(2+\lambda)} = \sum_{i=1}^{2} |V_{i1}^{(n)} - V_{i2}^{(n)}|_{Q^{(n)}}^{(2+\lambda)} \leqslant K \sum_{i=1}^{2} |G_i|_{Q^{(n)}}^{(\lambda)} \leqslant$$

$$\leqslant KK_5 \sum_{i=1}^{2} (|V_{i1}^{(n)} - V_{i2}^{(n)}|_{Q^{(n)}}^{(2+\lambda)} + (w_{i1} - w_{i2}|_{Q^{(n)}}^{(2+\lambda)}).$$

Supposing now $KK_5 < 1$, we obtain

$$\text{(4.16)} \quad |V_{i1}^{(n)} - V_{i2}^{(n)}|_{Q^{(n)}}^{(2+\lambda)} \leqslant K_6 \sum_{i=1}^{2} |w_{i1} - w_{i2}|_{Q^{(n)}}^{(2+\lambda)}, \quad K_6 = \frac{KK_5}{1 - KK_5}.$$

Third step. We come now to the second part of the system (4.1), namely:

$$\frac{\partial w_i}{\partial \alpha} - \sum_{j=1}^{2} \frac{\partial}{\partial x_j} \left(\bar{\bar{d}}_i(w_i, \bar{\bar{U}}_{ij}(V_j^{(n)}, w_j)) \frac{\partial w_i}{\partial x_j} \right) - \bar{\bar{\Lambda}}_i(V_j^{(n)}, w_j) w_i = 0, \text{ on } Q_A, \ \tau \in [0, T],$$

$$w_i(x, \tau, 0) = h_i(x, \tau),$$

$$h_i(x, \tau) = \int_0^{\tau} \mu_i(x, \tau, \beta) w_i(x, \tau - \beta, \beta) d\beta + \int_{\tau}^{A} \mu_i(x, \tau, \beta) V_i^{(n)}(x, \tau, \beta - \tau) d\beta, \ (i = 1, 2),$$

where $V_i^{(n)}$ is the solution of (4.15) corresponding to the pair of functions w_i, and consider the linear in $W_i^{(n)}$ system:

$$\frac{\partial W_i^{(n)}}{\partial \alpha} - \sum_{j=1}^{2} \frac{\partial}{\partial x_j} \left(\bar{\bar{d}}_i(w_i, \bar{\bar{U}}_{ij}(V_j^{(n)}, w_j)) \frac{\partial W_i^{(n)}}{\partial x_j} \right) - \bar{\bar{\Lambda}}_i(V_j^{(n)}, w_j) W_i^{(n)} = 0,$$

$$(i = 1, 2, \ i \neq j)$$

$$\text{(4.17)} \quad W_i^{(n)}(x, \tau, 0) = h_i^{(n)}(x, \tau), \quad W_i^{(n)}(x, \tau, \alpha)|_{|x|=n} = \max h_i^{(n)}(x, \tau)|_{|x|=n}.$$

$$h_i^{(n)}(x, \tau) = \int_0^{\tau} \mu_i(x, \tau, \beta) w_i(x, \tau - \beta, \beta) d\beta + \int_{\tau}^{A} \mu_i(x, \tau, \beta) V_i^{(n)}(x, \tau, \beta - \tau) d\beta.$$

Suppose now $h_i^{(n)}$ a given Hölder continuous function on $Q_T^{(n)}$. Then by the same argument as in the first step, the system of the two first equations in (4.17) have a unique solution which satisfies:

$$|W_i|_{Q_A^{(n)}}^{(2+\lambda)} \leqslant K |h_i^{(n)}|_{Q^{(n)}}^{(0)}.$$

But from the last relation (4.17), we have:

$$h_i^{(n)} \leqslant \bar{\mu}_i(|w_i|_{Q_A^{(n)}}^{(0)}, |V_i^{(n)}|_{Q_A^{(n)}}^{(0)}) \leqslant \bar{\mu}_i(|w_i|_{Q_A}^{(2+\lambda)}, |V_i^{(n)}|_{Q_A}^{(2+\lambda)}),$$

and therefore

$$\text{(4.18)} \quad |W_i|_{Q_A}^{(2+\lambda)} \leqslant K\bar{\mu}_i \max |w_i|_{Q_A}^{(2+\lambda)}, |V_i^{(n)}|_{Q_1}^{(2+\lambda)}).$$

We conceive now, as in the second step, (4.17) as a mapping $\mathcal{T}_1$ of the space $\mathcal{W}$ of pairs (w_i), which satisfy (4.3), (4.4), on the space $\mathcal{W}'$ of solutions of (4.17), and will prove that this mapping transforms $\mathcal{W}$ into itself. By choosing suitably the constants K, μ_i, it is easily seen that the second relation (4.4) is satisfied by the pair $W_i^{(n)}$ and so the assertion follows, i.e. $\mathcal{W} \subset \mathcal{W}$. It remains to prove that $\mathcal{T}_1$ is a contraction. To this end, consider two pairs of functions (w_{i1}) and (w_{i2}), the corresponding pairs $(V_{i1}^{(n)})$, $(V_{i2}^{(n)})$ of solutions of (4.2) and finally the pairs $(W_{i1}^{(n)})$, $(W_{i2}^{(n)})$ of respectively solutions of (4.17). Denote by X_i the differences

$$X_i = W_{i1}^{(n)} - W_{i2}^{(n)}.$$

This differences satisfy :

$$\frac{\partial X_i}{\partial \alpha} - \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(\overline{\overline{d}}_i(V_{i1}^{(n)}, \overline{\overline{U}}_{ij}(V_{ij}^{(n)}, w_{ij}))\frac{\partial X_i}{\partial x_j}\right) - \overline{\overline{\Lambda}}(V_{j1}^{(n)}, w_{ji})X_i = H_i.$$

$$X_i(x, 0, \alpha) = b_{i1}^{(n)} - b_{i2}^{(n)}, \quad X_i(x, \tau, \alpha)|_{|x|=n} = (b_{i1}^{(n)} - b_{i2}^{(n)})|_{|x|=n},$$

$$b_{i1}^{(n)} - b_{i2}^{(n)} = \int_0^{\tau} \mu_i(x, \tau, \beta)(w_{i1}(x, \tau-\beta, \beta) - w_{i2}(x, \tau-\beta, \beta))d\beta +$$

$$+ \int_{\tau}^{A} \mu_i(x, \tau, \beta)(V_{i1}^{(n)}(x, \tau, \beta-\tau) - V_{i2}^{(n)}(x, \tau, \beta-\tau))d\beta,$$

where

$$H_i = (\overline{\overline{d}}_{i1} - \overline{\overline{d}}_{i2})\Delta W_{i2}^{(n)} + \sum_{j=1}^{2}\left(\frac{\partial \overline{\overline{d}}_{i1}}{\partial x_j} - \frac{\partial \overline{\overline{d}}_{i2}}{\partial x_j}\right)\frac{\partial W_{i2}^{(n)}}{\partial x_j} + (\overline{\overline{\Lambda}}_{i1} - \overline{\overline{\Lambda}}_{i2})W_{i2}^{(n)}.$$

According to the theorem several times quoted from [5], it follows that

$$|X_i|_{Q_A^{(n)}}^{(2+\lambda)} \leqslant K[|X_i|_{Q_A^{(n)}}^{(0)} + |H_i|_{Q_A^{(n)}}^{(\lambda)}.$$

But

$$|\overline{\overline{d}}_{i1} - \overline{\overline{d}}_{i2}| \leqslant L_2\left(|w_{i1} - w_{i2}|_{Q_A^{(n)}}^{(0)} + \sum_{j=1}^{2}\overline{k}_{ij}(|V_{j1}^{(n)} - V_{j2}^{(n)}|_{Q_A^{(n)}}^{(0)} + |w_{j1} - w_{j2}|^{(0)}\right)_{Q_A^{(n)}}$$

$$\left|\frac{\partial \overline{\overline{d}}_{i1}}{\partial x_j} - \frac{\partial \overline{\overline{d}}_{i2}}{\partial x_j}\right| \leqslant L_3\left\{\left|\frac{\partial w_{i1}}{\partial x_j} - \frac{\partial w_{i2}}{\partial x_j}\right|_{Q_A^{(n)}}^{(0)} + \sum_{j=1}^{2}\overline{k}_{ij}\left(\left|\frac{\partial V_{j1}^{(n)}}{\partial x_j} - \frac{\partial V_{j2}^{(n)}}{\partial x_j}\right|_{Q_A^{(n)}}^{(0)} +\right.\right.$$

$$\left.\left. + \left|\frac{\partial w_{j1}}{\partial x_j} - \frac{\partial w_{j2}}{\partial x_j}\right|_{Q_A^{(n)}}^{(0)}\right)\right\}.$$

Considering also that the derivatives of $W_i^{(n)}$ are bounded, since $W_i \in C_{(2+\lambda)}$, and the inequality (4.16) we get :

$$\sum_{i=1}^{2}|H_i| \leqslant L_4 \sum_{i=1}^{2}\left(|w_{i1} - w_{i2}|_{Q_A^{(n)}}^{(0)} + \sum_{j=1}^{2}\left|\frac{\partial w_{i1}}{\partial x_j} - \frac{\partial w_{i2}}{\partial x_j}\right|_{Q_A^{(n)}}^{(0)}\right).$$

and obviously

$$|H_i|_{Q_A^{(n)}}^{(\lambda)} \leqslant L_4|w_{i1} - w_{i2}|_{Q_A^{(n)}}^{(\lambda)}.$$

On the other hand :

$$|X_i|_{Q_A^{(n)}}^{(0)} = |b_{i1} - b_{i2}|_{Q_A^{(n)}}^{(0)} \leqslant \mu_i(|w_{i1} - w_{i2}|_{Q_A^{(n)}}^{(0)} + |V_{i1}^{(n)} - V_{i2}^{(n)}|_{Q_A^{(n)}}^{(0)}).$$

Using again (4.16), we have

$$|X_i|_{Q_A^{(n)}}^{(0)} = |b_{i1} - b_{i2}|_{Q_A^{(n)}}^{(0)} \leqslant |b_{i1} - b_{i2}|_{Q_A}^{(\lambda)} \leqslant L_5|w_{i1} - w_{i2}|_{Q_A^{(n)}}^{(\lambda)}, \quad L_5 = \mu_1(1 + K_6),$$

so that, finally, (4.19) leads to

$$\Sigma|X_i|_{Q_A^{(n)}}^{(2+\lambda)} \leqslant L|w_{i1} - w_{i2}|_{Q_A^{(n)}}^{(2+\lambda)},$$

L — a constant depending linearly on $\overline{k}_{ij}$, $\overline{\mu}_i$, L_2, L_3, L_4, L_5.

Imposing the condition $L<1$, it results that $\mathcal{T}_1$ is a contraction, and that the system

$$\frac{\partial W_i^{(n)}}{\partial \alpha} - \sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left(\overline{\overline{d}}_i(W_i^{(n)}, \overline{\overline{U}}_{ij}(V_{ij}^{(n)}, W_{ij}^{(n)}))\frac{\partial W_i^{(n)}}{\partial x_j}\right) - \overline{\overline{\Lambda}}_i(V_j^{(n)}, W_j)W^{(n)} = 0,$$

(4.19) $$W_i^{(n)}(x, \tau, 0) = b_i^{(n)}(x, \tau), \qquad W_i^{(n)}(x, \tau, \alpha)|_{|x|=n} = \max|b_1^{(n)}(x\tau)||_{|x|=n},$$

$$b_i^{(n)}(x, \tau) = \int_0^{\tau}\mu_i(x, \tau, \beta)W_i^{(n)}(x, \tau-\beta, \beta)d\beta + \int_{\tau}^{A}\mu_i(x, \tau, \beta)V_i^{(n)}(x, \tau, \beta-\tau)d\beta,$$

has a unique solution, belonging to $C(Q_A)$.

As in the case of the system (4.14), it also results that

$$W_i^{(n)} \in C_{(2+\lambda)}(Q^{(n)}).$$

Combining with the results of the first step, the stated theorem is completely proved.

5. **Existence theorem.** *The set of solutions* (V_i^n), (W_i^n) *is compact on compacts, and as a consequence a) one can choose from each sequence* $(V_1^{(n)})$, $(W_1^{(n)})$ *a subsequence converging to a continuous function, and b) these functions satisfy also (3.1).*

Proof. a) The functions $V_i^{(n)}$ are obviously uniformly bounded by M. Concerning the functions $W_i^{(n)}$, from the above considerations, it follows that $W_i^{(n)} \leqslant \overline{\mu}_i \overline{M}/(1+\overline{\mu}_i)$.

On the other hand, the derivatives of V_i^n and of $W_i^{(n)}$ also are uniformly bounded, since these functions belong to $C_{(2+\lambda)}(Q)$, and $|V_i^{(n)}|_{Q^{(n)}}^{(2+\lambda)}$, $|W_i|_{Q^{(n)}}^{(2+\lambda)}$ are uniformly bounded. As a consequence of Arzelà's theorem, the statement *a*) of the theorem follows.

Following the same ideas as in [14], we obtain

$$\iiiint_Q (V_i^{(n)}\rho_\tau - \sum_{j=1}^{2} d_i\rho_{x_j}V_{ix_j}^{(n)} + V_i^{(n)}\Lambda_i\rho)dx_1dx_2dt d\alpha = \iiint_{Q_A}\rho(x, 0, \alpha)u_{i0}^{(n)}dx_1dx_2d\alpha,$$

$$\iiiint_Q\left(w^{(n)}\sigma_\alpha - \sum_{j=1}^{2}d_i\sigma_{x_j}W_{ix_j}^{(n)} + W_i^{(n)}\sigma\Lambda_i\right)dx_1dx_2dt d\alpha = \iiint_{Q_T}\sigma(x, \tau, 0)b^{(n)}(x, \tau)dx_1dx_2d\tau,$$

$$b^{(n)}(x, \tau) = \int_0^{\tau}\mu_i(x, \tau, \beta)W_i^n(x, \tau-\beta, \beta)d\beta + \int_{\tau}^{A}\mu_i(x, \tau, \beta)V_i^{(n)}(x, \tau, \beta-\tau)d\beta,$$

where the index n runs over the subset of the natural numbers set, on which $\{V_i^{(n)}\}$ and $\{W_i^{(n)}\}$ are uniformly convergent. As $(V_{ix_j}^{(n)})(W_{ix_j}^{(n)})$ are bounded, one can choose a subset on which $\{V_{i_{n_j}}\}$, $\{W_{i_{n_j}}\}$ is weakly convergent, and so, by letting $n\to\infty$ and summing the first two relations we obtain (3.1).

6. **The Dependence of the Solutions on the Initial Data.** Consider two systems of the same form (1.1—4), with the same coefficients d_i, h_i, k_{ij}, μ_i but with two different initial functions $u_{i0}^{(k)} (k=1, 2)$. Denote by $u_i^{(k)}$ the corresponding generalized solutions of the two systems; then by the same transformations as those used in § 3, we arrive to the systems

$$\frac{\partial v_i^{(k)}}{\partial \tau} - \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(\overline{d}_i^{(k)} \frac{\partial v_i^{(k)}}{\partial x_j}\right) - \overline{\Lambda}_i^{(k)} v_i^{(k)} = 0,$$

$$\frac{\partial w_i^{(k)}}{\partial \alpha} - \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(\overline{\overline{d}}_1^{(k)} \frac{\partial w_i^{(k)}}{\partial x_j}\right) - \overline{\overline{\Lambda}}_i^{(k)} w_i^{(k)} = 0,$$

$$b_i^{(k)}(x, \tau) = \int_0^{\tau} \mu_i(x, \tau, \alpha) w_i^{(k)}(x, \tau-\alpha, \alpha)d\alpha + \int_{\tau}^{A} \mu_i(x, \tau, \alpha) v_i^{(k)}(x, \tau, \alpha-\tau)d\alpha,$$

$$u_i^{(k)}(x, 0, \alpha) = u_{i0}^{k}(x, \alpha), \text{ on } Q_n, \quad (k=1, 2).$$

As already proved, these systems have bounded solutions, with bounded first and second order derivatives. Denoting now

$$v_i^{(1)} - v_i^{(2)} = V_i e^{\omega_i \tau}, \quad \omega_i - \text{constant};$$

we have, by substracting the first corresponding equations

$$\frac{\partial V_i}{\partial \tau} - \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(\overline{d}_i^{(1)} \frac{\partial V_i}{\partial x_j}\right) + \overline{\Lambda}_i V_i - f_i = 0,$$

where f_i depends linearly on V_i, $\partial V_1/\partial x_j$, their maxima, and also linearly on the maximum of $(w_1^{(1)} - w_1^{(2)})e^{-\omega_i \tau}$. By choosing ω_i sufficiently large, it results that the maximum of V_1 on $Q^{(n)}$ is on the frontier, and therefore

$$\max_{Q_n} |V_i| \leqslant \max_{Q_n} |u_{i0}^{(1)} - u_{i0}^{(2)}|,$$

and hence

$$\max |v_i^{(1)} - v_i^{(2)}| \leqslant \max |u_{i0}^{(1)} - u_{i0}^{(2)}| e^{-\omega\tau} \leqslant \max |u_{i0}^{(1)} - u_{i0}^{(2)}|.$$

In the same manner one obtains

$$\max|w_i^{(1)} - w_i^{(2)}| \leqslant \max|b_i^{(1)} - b_i^{(2)}| \leqslant \max \overline{\mu}_i(|w_i^{(1)} - w_i^{(2)}| + |v_i^{(1)} - v_i^{(2)}|)$$

from which we get

$$\max_{Q} |w_i^{(1)} - w_i^{(2)}| \leqslant \frac{\overline{\mu}_i |v_{i0}^{(1)} - v_{i0}^{(2)}|}{1 - \overline{\mu}_i}.$$

The obvious consequence of these inequations, is the

Proposition. *The two generalized solutions $u_i^{(1)}$, $u_i^{(2)}$ of (3.1—4), obtained by the studied limiting process, corresponding to two initial functions $u_{i0}^{(1)}$, $u_{i0}^{(2)}$ satisfy the inequality*

$$|u_i^{(1)} - u_i^{(2)}| \leqslant \sup\left(1, \frac{\overline{\mu}_i}{1 - \overline{\mu}_i}\right) |u_{i0}^{(1)} - u_{i0}^{(2)}|.$$

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