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EQUIVALENCE OF MULTIVALUED INTEGRO-DIFFERENTIAL SYSTEMS

BY

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1. **Introduction.** A relation of the form

$$\dot{x}(t) \in G(t, x(t))$$

is called a multivalued differential equation or a contingent equation, where $x = x(t)$ is an Euclidean vector function defined on an interval, and $G(t, x)$ is a function which associates with every point (t, x) from a certain region of $(n+1)$-Euclidean space a set $G(t, x)$ of n-Euclidean space. It is known that there exists a connection between control problems and the theory of contingent equations. At first, the theory was concerned with suitable smooth convex orientor fields $G(t, x)$. In case when $G(t, x)$ is non-convex, T. Wazewski introduced a generalization of trajectories and investigated important properties of the control systems. Later, A.F. F i l i p p o v showed the existence of solutions for a non-convex orientor field $G(t, x)$ which satisfies Lipschitz condition in (t, x) and achieved some further investigations. The case when $G(t, x)$ satisfies Carathéodory type condition were investigated by many other authors. The study of the control problem of the contingent functional-differential equation under the assumption that a Carathéodory type condition is satisfied has been considered by N. K i k u c h i [9]. Such an equation is obtained by eliminating the control function from the control system described by a functional-differential equation. Existence results, qualitative properties, existence of continuous or measurable selectors of the functional differential contingent equations under more general form have been studied extensively in [2], [6], [7], [10], [11]. In recent years, considerable attention has been paid to the development of the qualitative theory for integral and integro-differential inclusions, see, for example, [3], [4], [14]. The aim of the present paper is to extend part of the results of [1] and [15] concerning the asymptotic equivalence of two systems of integro-differential equations and of [13] concerning the asymptotic equivalence of a linear differential system and its multivalued perturbed differential system. Our techniques are a combination of the well-known Hölder's inequality and a fixed point theorem for multifunctions.

2. **Notations and definitions.** In the sequel we shall use the following notations: $\varphi(t)$ and $\psi(t)$ are positive continuous functions on $J = [t_0, \infty)$; $C_c(J, R^n)$-the set of all continuous functions $z : J \to R^n$ with the topology

of uniform convergence on bounded intervals, i.e., it is induced by the denumerable family of seminorms given by $|z|_n = \sup\{|z(t)| : t_0 \leq t \leq t_0 + n\}$, where $|\cdot|$ is any convenient vector or matrix norm. Under this topology $C_c(J, R^n)$ is a locally convex, metrizable, complete, vector space and therefore, we shall use the denumerable sequences;

$B_\psi = B_\psi(J, R^n)$-the set of all continuous functions $z : J \to R^n$ such that $\sup\{|\psi^{-1}(t)|z(t)|| : t \in J\} < \infty$. We shall say that such a function is ψ-bounded on J and we shall denote $\|z\|_\psi = \sup\{\psi^{-1}(t)|z(t)| : t \in J\}$. Then B_ψ with norm $\|\cdot\|$ is a Banach space; $B_{\psi,\rho} = \{z(t) \in B_\psi : \|z\| \leq \rho\}$;

$L_{p,\psi}(J, R^n)$-the set of all $z : J \to R^n$ such that $\psi^{-1}(t)|z(t)| \in L_p(J, R^n)$, $\|z\|_{p,\psi} = \|\psi^{-1}(t)z(t)\|_p$.

If X is a linear topological space and $E \subset X$, then $cf(E)$ denotes the family of all convex and closed subsets of E; if E is a subset of R^n, we define $|E| = \sup\{|e| : e \in E\}$ and it is called the norm of E.

As regards to the set B_ψ we note that it is a Banach space with respect to the norm $\|\cdot\|_\psi$ and that its topology is stronger than the topology of $C_c(J, R^n)$. Moreover, the convergence in B_ψ implies the uniform convergence on every bounded interval of J. Indeed, if $a > 0$, we have $\sup\{|x(t)| : t_0 \leq t \leq t_0 + a\} \leq \|x\|_\psi \sup\{\psi(t) ; t_0 \leq t \leq t_0 + a\}$.

Let $x(t)$ be a continuous function on J with values in R^n. We define the mapping $x_t = x(\cdot) : [t_0, t] \to R^n$, $t \in J$, by $x_t(s) = x(s)$ for $s \in [t_0, t]$. The symbol x_t can be interpreted as the segment of the function $x(s)$ defined by letting $s \in [t_0, t]$. We note that for $x_t \in C([t_0, t], R^n)$ its norm is defined as $\|x_t\| = \sup\{|x(s)| : s \in [t_0, t]\}$. If $x(t) \in B_\psi$, where $\psi(t) > 0$ is a non-decreasing function on J, then between $\|x_t\|$ and $\|x\|_\psi$ there exists the following relation $\|x\|_\psi = \sup\{\psi^{-1}(t)\|x_t\| : t \in J\}$, i.e., $\|x_t\| \in B_\psi$ if and only if $x(t) \in B_\psi$ [15]. For all ordered pairs (t, x_t) with $t \in J$ and $x \in C_c(J, R^n)$, consider a multifunction $F = F(t, x(\cdot)) : J \times C([t_0, t], R^n) \to \Omega(R^n)$, where $\Omega(R^n)$ denotes the set of all nonempty compact subsets of R^n. For fixed $x \in C_c(J, R^n)$ the multifunction $F(t, x(\cdot))$ is said to be continuous (measurable) in t if $F(t, x(\cdot))$ is a continuous (measurable) multifunction of t, $t \in J$. For fixed $t \in J$ the multifunction $F(t, x(\cdot))$ is said to be continuous (upper semicontinuous) in x, if for every $\varepsilon > 0$ there exists a positive δ, such that the relation $\mathrm{dist}(F(t, x(\cdot)), F(t, y(\cdot))) < \varepsilon$ $(F^\varepsilon(t, x(\cdot)) \supset F(t, y(\cdot)))$ holds for every y such that $\|x_t - y_t\| < \delta$, $y \in C([t_0, t], R^n)$, where the distance between two sets is in the Hausdorff's sense and for $E \subset R^n$, we denote by E^ε the ε-neighbourhood of E. We shall assume throughout this paper that F satisfies the following hypotheses:

(*H*-1). for each $(t, x_t) \in J \times C([t_0, t], R^n)$, $F(t, x(\cdot))$ is convex;

(*H*-2). for each $t \in J$, $F(t, x(\cdot))$ is upper semicontinuous in x;

(*H*-3). for each measurable function $x : J \to R^n$ there exists a measurable function $g^x : J \to R^n$ such that $g^x(t) \in F(t, x(\cdot))$ a.e. on J. For $z(t) \in B_\psi$, or for a given measurable and essentially bounded on J function $z(t)$, we shall denote by $M(z(t))$ the set of all measurable selectors from $F(t, z(\cdot))$. Under the hypotheses (*H*-1) and (H-3), $M(z(t))$ is nonempty and convex.

We consider the integro-differential system

$$y'(t) = A(t)y(t) + \int_{t_0}^{t} B(t, s)y(s)ds + f(t), \qquad t \in J, \tag{2.1}$$

and its perturbed integro-differential multivalued system

$$x'(t) \in A(t)x(t) + \int_{t_0}^{t} B(t, s)x(s)ds + f(t) + F(t, x(\cdot)), \quad t \in J, \tag{2.2}$$

where $x, y, f : J \to R^n$ are n-dimensional functions, A and B are $n \times n$ matrix functions from $J \times J$ to R^{n^2} and F is the multifunction above defined. We can say that the operator multifunction F is of type Volterra, such typical operators being

$$F(t, x(\cdot)) \equiv F(t, x_t) = F(t, x(t)), \quad F(t, x_t) = F\left((t, x(t), \int_{t_0}^{t} k(t, s, x(s)\right) ds,$$

particularly, $F(t, x_t) = \int_{t_0}^{t} C(t, s)h(s, x(s))ds$, where $h : J \times R^n \to 2^{R^n}$ is a multifunction and $C(t, s)$ a matrix function [14], or such that x_t belongs to $C([-r, t_0], R^n)$ and is defined by $x_t(s) = x(t + s)$, [7], i.e., inclusions with deviating argument.

The purpose of this note is to give sufficient conditions for ψ-asymptotic equivalence and for (ψ, p)-integral equivalence, below defined, for (2.1) and (2.2) on J. To this end, we introduce the fundamental solution matrix (or resolvent kernel) $r(t, s)$ associated with $A(t)$ and $B(t, s)$ (see, [5]) as the solution of initial value problem

$$\partial r(t, s)/\partial t = A(t)r(t, s) + \int_{s}^{t} B(t, u)r(u, s)\, du, \quad r(s, s) = I, \quad t_0 \leq s \leq t, \tag{2.3}$$

where I is the $n \times n$ identity matrix. If $s > t \geq t_0$, we define $r(t, s) \equiv 0$. In what follows, we will assume that $A(t)$ and $B(t, s)$ are locally integrable for $t \in J$ and $t_0 \leq s \leq t$ respectively, and that $f(t)$ is a continuous function on J. From here (see, [12]), it is easy to see that (2.3) is equivalent to

$$r(t, s) = I + \int_{t_0}^{t} \left[A(u) + \int_{u}^{t} B(\sigma, u)d\sigma \right] r(u, s)du,$$

and therefore $r(t, s)$ exists and it is continuous in (t, s) for $t_0 \leq s \leq t$. Moreover, if $x(t)$ is a solution of (2.2), then there exists a function $g^x(t) \in F(t, x(\cdot))$ a.e. on J which is measurable and locally integrable, such that

$$x(t)=y(t)+\int_{t_0}^{t} r(t,s)g^{\tau}(s)ds, \quad t\in J, \tag{2.4}$$

where $y(t)$ is a solution of (2.1) given by

$$y(t)=r(t,t_0)y_0+\int_{t_0}^{t} r(t,s)f(s)ds, \quad y(t_0)=y_0. \tag{2.5}$$

Following J.M. Cushing [5], we will assume that there exists an $n\times n$ matrix $P(t)$, locally integrable for $t\in J$ in terms of which we define two matrices

$$V(t,s)=r(t,s)-r(t,\tau_0)P(s), \ \tau_0\leqslant s\leqslant t; \quad W(t,s)=r(t,\tau_0)P(s), \ \tau_0\leqslant t\leqslant s, \tag{2.6}$$

where $\tau_0\geqslant t_0$, by means of which the solution $x(t)$ of (1.2) may be given as

$$x(t)=y(t)+\int_{\tau_0}^{t} V(t,s)g^{\tau}(s)ds-\int_{t}^{\infty} W(t,s)g^{\tau}(s)ds, \quad t\geqslant\tau_0, \tag{2.7}$$

and this can be directly verified if we take into account that $r(t,s)\equiv 0$ if $s>t\geqslant t_0$. In the special case of differential systems ($B(t,s)\equiv 0$) it turns out that $P(s)=P_2Y^{-1}(s)$ so that $V(t,s)=Y(t)P_1Y^{-1}(s)$, $W(t,s)=Y(t)P_2Y^{-1}(s)$, where $Y(t)$ is the fundamental matrix for the linear homogeneous system and P_1, P_2 are supplementary projections into R^n. This approach has been used by J.M. Cushing [5] to the study of admisibility and M.A.B Boudourides [1] and the author [15] to the study of asymptotic equivalence of integro-differential systems.

In this paper we shall be concerned with those properties of systems (2.1) and (2.2) given by the following definitions:

Definition 2.1. *Let $\psi: J\to R$ be a positive continuous function. We say that (2.1) and (2.2) are ψ-asymptotically equivalent on J if, for each solution $y(t)$ of (2.1) existing on J, there exists a solution $x(t)$ of (2.2) defined on $\bar{J}\subseteq J$, such that*

$$\lim_{t\to\infty}\psi^{-1}(t)|x(t)-y(t)|=0, \tag{2.8}$$

and conversely, for each solution $x(t)$ of (2.2) existing on J, there exists a solution $y(t)$ defined on $\bar{J}\subseteq J$, such that (2.8) holds.

Definition 2.2. *Let ψ be as in Definition 2.1 and $p>0$. We say that (2.1) and (2.2) are (ψ, p)-integrally equivalent on J if, for each solution $y(t)$ of (2.1) existing on J, there exists a solution $x(t)$ of (2.2) defined on $\bar{J}\subseteq J$ such that*

$$\psi^{-1}(t)|x(t)-y(t)|\in L_p(J), \tag{2.9}$$

and conversely, for each solution $x(t)$ of (2.2) existing on J, there exists a solution $y(t)$ of (2.1) defined on $\bar{J}\leqslant J$, such that (2.9) is true.

Since the results which follow are formulated in terms of arbitrary functions $\varphi(t)$ and $\psi(t)$, they offer greater versality in obtaining various asymptotic properties for specific integro-differential and multivalued integro-differential systems. To establish our results, we need the following two lemmas:

Lemma 2.1. (D. Bushaw, quoted and proved in [13], Theorem 2.2). *A relation $\mathcal{R}$ on a metric space X to comp (Y), where Y is a metric space, such that $\mathcal{R}(x)\neq\varnothing$ for each $x\in X$ is upper semicontinuous at $x_0\in X$ if and only if it is sequentially semicontinuous at x_0, i.e., if for every sequence $\{(x_n, y_n)\}$ in $\mathcal{R}$ such that $x_n\to x_0$, there exists a subsequence of $\{y_n\}$ that converges to some $y_0\in\mathcal{R}(x_0)$.*

Lemma 2.2. ([13], Corollary 2.8). *Let A be a closed, bounded, convex subset of a locally convex topological vector space B. If $T: A\to \mathrm{cf}(A)$ is upper semicontinuous and $\overline{T(A)}$ is compact, then there exists $x\in A$ such that $x\in T(x)$.*

We note that Lemma 2.2 is a consequence of the K. Fan's fixed point theorem for multifunctions.

3. **Main results.** We now consider the ψ-asymptotic equivalence of the systems (2.1) and (2.2.)

Theorem 3.1. *Let $\varphi(t)$ and $\psi(t)$ be positive continuous functions on J. Assume that the following conditions are satisfied: a) there exist constant K, q, with $K>0$ and $1<q<\infty$, such that for all $t\in J$, $\int_{t_0}^{t}|\psi^{-1}(t)\,V(t,s)\varphi(s)|^q ds+$*

$+\int_{t}^{\infty}|\psi^{-1}(t)\,W(t,s)\varphi(s)|^q ds\leqslant K^q$; b) *there exists $\omega: J\times[0,\infty)\to[0,\infty)$ such that $\omega(t,u)$ is monotone nondecreasing in u for each fixed $t\in J$ and integrable on compact subintervals of J for each fixed $u\geqslant 0$; c) for any constant $c\geqslant 0$,*

$\int_{t_0}^{\infty}\omega^p(s,c)\,ds<\infty$, where $p^{-1}+q^{-1}=1$;

d) *for each $x(t)\in C_c(J,R^n)$, $|F(t,x(\cdot))|\leqslant\varphi(t)\omega(t,\psi^{-1}(t)||x_t||)$, a.e. on J;*

e) *for every $\tau\in J$, $\lim\int_{t_0}^{\tau}|\psi^{-1}(t)V(t,s)\varphi(s)|^q\,ds=0$ as $t\to\infty$.*

Then, there is a ψ-asymptotic equivalence between the ψ-bounded solutions $y(t)$ of (2.1) and ψ-bounded solutions $x(t)$ of (2.2).

Proof. The proof of this theorem is similar to those from [13] and [14] and uses K. Fan's fixed point theorem for multifunctions (see, [13],, Theorem 2.7 and Corollary 2.8) given by Lemma 2.2. To this end let $\tau_0\geqslant t_0$ be temporarily fixed and let us consider the operator T defined for $x(t)\in B_\psi$ by the relation

$$T(x(t))=\Big\{z\in C_c(J,R^n);\ z(t)=y(t)+\int_{\tau_0}^{t}V(t,s)g^{\tau}(s)ds- -\int_{t}^{\infty}W(t,s)g^{\tau}(s)ds,\quad g^{\tau}(t)\in M(x(t))\Big\}, \tag{3.1}$$

where $y(t)$ is a solution of (2.1).

First, we note that for $x(t)\in B_\psi$ with $||x||_\psi=\rho_0$, by (H-1) and (H-3) it follows that $M(x(t))$ is nonempty and convex. Taking into account b) and d), for $g^{\tau}(t)\in M(x(t))$ we have $|g^{\tau}(t)|\leqslant\varphi(t)\omega(t,\rho_0)$ a.e. on J, and therefore by c) we get

$$\|g^x\|_{p,\varphi}=\Big[\int_{\tau_0}^{\infty}|\varphi^{-1}(s)g^x(s)|^p ds\Big]^{1/p}<\Big[\int_{t_0}^{\infty}\omega^p(s,\rho_0)ds\Big]^{1/p}<\infty,$$

i.e., $g^x\in L_{p,\varphi}(J, R^n)$.

From here, from the continuity of $\varphi(t)$ and a) it follows that the operator T defined by (3.1) is well defined. Let now $y(t)$ be a ψ-bounded solution of (2.1) on J with $\|y\|_\psi\leqslant\rho$, for some $\rho>0$. From (3.1), using Hölder's inequality and hypotheses a) and c) it follows that T maps B_ψ into 2^{B_ψ}. Moreover, for each $x(t)\in B_{\psi,2\rho}$ and $z(t)\in T(x(t))$ we have

$$|\psi^{-1}(t)z(t)|\leqslant\rho+\Big[\int_{\tau_0}^{t}|\psi^{-1}(t)V(t,s)\varphi(s)|^q ds\Big]^{1/q}\Big[\int_{\tau_0}^{t}\omega^p(s,2\rho)ds\Big]^{1/p}+$$

$$\Big[\int_{t}^{\infty}|\psi^{-1}(t)\,W(t,s)\,\varphi(s)|^q ds\Big]^{1/q}\Big[\int_{t}^{\infty}\omega^p(s,2\rho)ds\Big]^{1/p}\leqslant\rho+2K\Big[\int_{\tau_0}^{t}\omega^p(s,2\rho)ds\Big]^{1/p},$$

and if we choose $\tau_0\geqslant t_0$ so large that $\int_{\tau_0}^{\infty}\omega^p(s,2\rho)ds<[\rho/2K]^p$, it follows $|\psi^{-1}(t)z(t)|\leqslant\leqslant 2\rho$, for all $t\in J$, i.e., $z(t)\in B_{\psi,2\rho}$. Therefore, $T(B_{\psi,2\rho})\subseteq B_{\psi,2\rho}$ from which it follows the uniform boundedness of the set $T(B_{\psi,2\rho})$. To verify the equicontinuity of the family of functions $T(B_{\psi,2\rho})$, consider $x(t)\in B_{\psi,2\rho}$ and $z(t)\in\in T(x(t))$. Since $y(t)$ is a solution of (2.1) and

$$z(t)=y(t)+\int_{\tau_0}^{t}V(t,s)g^x(s)ds-\int_{t}^{\infty}W(t,s)g^x(s)ds,$$

for some $g^x(t)\in M(x(t))$, it follows that $z(t)$ satisfies the equation

$$z'(t)=A(t)z(t)+\int_{\tau_0}^{t}B(t,s)\,z(s)ds+f(t)+g^x(t),\ \text{a.e. on } J.$$

From here, for $t_1,t_2\geqslant\tau_0$ $(t_1\leqslant t_2)$, we have

$$|z(t_1)-z(t_2)|=\Big|\int_{t_1}^{t_2}z'(s)ds\Big|\leqslant\int_{t_1}^{t_2}|A(s)|\,|z(s)|ds+\int_{t_1}^{t_2}\Big[\int_{t_0}^{s}|B(s,u)|\,|z(u)|du\Big]ds+$$

$$+\int_{t_1}^{t_2}|f(s)|ds+\int_{t_1}^{t_2}|g^x(s)|ds.$$

Now, $z(t)\in T(x(t))$ and $x(t)\in B_{\psi,2\rho}$ implies that $|z(t)|\leqslant 2\rho\psi(t)$ and $|g^x(t)|\leqslant\leqslant\varphi(t)\omega(t,2\rho)$ a.e. on J. Hence

$$|z(t_1)-z(t_2)|\leqslant 2\rho\Big\{\int_{t_1}^{t_2}|A(s)\psi(s)|ds+\int_{t_1}^{t_2}\Big[\int_{t_0}^{s}|B(s,u)|\psi(u)du\Big]ds\Big\}+$$

$$+\int_{t_1}^{t_2}|f(s)|ds+\int_{t_1}^{t_2}\varphi(s)\omega(s,2\rho)ds. \tag{3.2}$$

Since $A(t)$ and $\omega(t,2\rho)$ are integrable on compact subsets of J, $B(t,s)$ locally integrable for $t\in J$ and $t_0\leqslant s\leqslant t$ and φ, ψ and f are continuous, it follows from (3.2) that $T(B_{\psi,2\rho})$ is equicontinuous on $[\tau_0,\infty)$. Hence $\overline{T(B_{\psi,2\rho})}$ is a compact subset of $C_c(J, R^n)$. On the other hand, it is obviously that $B_{\psi,2\rho}$ is a closed set in $C_c(J, R^n)$.

Now we are going to prove that T is an upper semicontinuous multifunction on B_ψ. To this end, we shall use Lemma 2.1 proving that T is sequentially upper semicontinuous on B_ψ and, partially, a similar argument of that one used in [14].

Let now take $x_n(t)$, $x(t)\in B_\psi$, $n=1,2,\dots$ and let us assume that $x_n(t)\to\to x(t)$ as $n\to\infty$ in B_ψ, i.e., $\psi^{-1}(t)x_n(t)\to\psi^{-1}(t)x(t)$ uniformly on J. Therefore, there exists $\rho_0>0$ such that $x_n(t)$, $x(t)\in B_{\psi,\rho_0}$, $n=1,2,\dots$ If $h_n(t)\in T(x_n(t))$, $n=1,2,\dots$, then there exists $g_n^x(t)\in M(x_n(t))$, $n=1,2,\dots$, such that

$$h_n(t)=y(t)+\int_{\tau_0}^{t}V(t,s)g_n^x(s)ds-\int_{t}^{\infty}W(t,s)g_n^x(s)ds,$$

and

$$\|g_n^x\|_{p,\varphi}=\Big[\int_{\tau_0}^{\infty}|\varphi^{-1}(s)g_n^x(s)|^p ds\Big]^{1/p}\leqslant\Big[\int_{t_0}^{\infty}\omega^p(s,\rho_0)ds\Big]^{1/p}.$$

Thus, the set $\{g_n^x(t)\}$ is bounded in $L_{p,\varphi}(J, R^n)$ and therefore it is weakly precompact in $L_{p,\varphi}(J, R^n)$. Hence, there exists a subsequence $\{g_{n_j}^x\}$ of the sequence $\{g_n^x\}$ which weakly converges to some $g(t)\in L_{p,\varphi}(J, R^n)$. From the fact that x_{n_j} converges to x and from the assumption (H-2) it follows that for each $\varepsilon>0$ and for almost every fixed $t\in J$ there exists a natural number $m=m(\varepsilon,t)$ such that for $n_j\geqslant m$, $F(t,x_{n_j}(\,.\,))\subset F^\varepsilon(t,x(\,.\,))$. Hence $g_{n_j}^x(t)\in F^\varepsilon(t,x(\,.\,))$. Further, by M a z u r's theorem ([16], p. 173), there exists a subsequence of $\{g_{n_j}^x\}$, $n_j\geqslant m$, denoted by $\{g_{j_k}^x\}$, $j_k>m$, such that $\|\sigma_k-g\|_{p,\varphi}\to 0$ as $k\to\infty$, where $\sigma_k(t)=\sum_{s=1}^{k}\alpha_{j_s}g_{j_s}^x$, $\alpha_{j_s}\geqslant 0$, $\sum_{s=1}^{k}\alpha_{j_s}=1$. Taking into account that $F^\varepsilon(t,x(\,.\,))$ is a convex set, we have $\sigma_k(t)\in F^\varepsilon(t,x(\,.\,))$, $k=1,2,\dots$ From here, by F. Riesz' theorem, it follows the existence of such a subsequence of $\{\sigma_k(t)\}$ which converges to $g(t)$ a.e. on J. Taking $\varepsilon\to 0$, we have $g(t)\in F(t,x(\,.\,))$ and for this reason, we denote it by $g(t)\equiv g^x(t)$. Because $g(t)\equiv g^x(t)\in L_{p,\varphi}(J, R^n)$, the function

$$h(t)=y(t)+\int_{\tau_0}^{t}V(t,s)g^x(s)ds-\int_{t}^{\infty}W(t,s)g^x(s)ds,\ \ t\in J,$$

is well defined and $h(t)\in T(x(t))$. In view of the fact that $\{g_{n_j}^x\}$ converges weakly to $g(t)\equiv g^x(t)$ and that $V(t,.)$ and $W(t,.)$ belong to $L_{q,\varphi}(J, R^n)$, we get that the sequence

$$h_{n_j}(t)=y(t)+\int_{\tau_0}^{t}V(t,s)g_{n_j}^x(s)ds-\int_{t}^{\infty}W(t,s)g_{n_j}^x(s)ds,\ j=1,2,\dots$$

converges to $h(t)$ on J. Indeed, we have

$$h_{n_j}(t)-h(t)=\int_{\tau_0}^{t}V(t,s)[g_{n_j}^x(s)-g^x(s)]ds-\int_{t}^{\infty}W(t,s)[g_{n_j}^x(s)-g^x(s)]ds\to 0\ \text{ as } j\to\infty.$$

If we choose $y\in B_{\psi,\rho}$ and $x_n(t)$, $x(t)\in B_{\psi,2\rho}$, then $T(B_{\psi,2\rho})\subset B_{\psi,2\rho}$ is precompact in $C_c(J, R^n)$ and because $h_{n_j}(t)\in T(B_{\psi,2\rho})\subset\overline{T(B_{\psi,2\rho})}$, $j=1,2,\dots$ with $\overline{T(B_{\psi,2\rho})}$ compact in $C_c(J, R^n)$, it follows the existence of a subsequence of $\{h_{n_j}\}$ which converges in $C_c(J, R^n)$ to a function $\tilde{h}(t) \in \overline{T(B_{\psi, 2\rho})}$ uniformly on every compact subinterval of J. Thus, we get that $h(t) = \tilde{h}(t) \in T(x(t))$ and this proves the upper semicontinuity of T. To prove that $T(x(t))$ is a convex closed set in $C_c(J, R^n)$, as we have mentioned above, $M(x(t))$ is a nonempty convex set and therefore $T(x(t))$ is also a nonempty set and, in addition, $T(x(t)) \subset B_{\psi, 2\rho}$. Then, if $h_n(t) \in T(x(t))$, $n = 1, 2, \ldots$ is a sequence convergent to $h(t)$ in $C_c(J, R^n)$, by an analogous argument to that from the proof of the upper semicontinuity of T, we get that $h(t) \in T(x(t))$ which means that $T(x(t))$ is a closed set. Hence, because $B_{\psi, 2\rho}$ is closed bounded and convex subset of the locally convex topological vector space $C_c(J, R^n)$, $T : B_{\psi, 2\rho} \to cf(B_{\psi, 2\rho})$ is upper semicontinuous and $\overline{T(B_{\psi, 2\rho})}$ is compact, it follows that there exists $x(t) \in B_{\psi, 2\rho}$ such that $x(t) \in T(x(t))$, i.e., $x(t)$ is a solution of (2.2) with $\|x\|_\psi \leqslant 2\rho$ and there exists $g^x(t) \in M(x(t))$ satisfying

$$x(t) = y(t) + \int_{\tau_0}^{t} V(t, s) g^x(s) ds - \int_{t}^{\infty} W(t, s) g^x(s) ds.$$

Further, we have

$$|\psi^{-1}(t)[x(t) - y(t)]| \leqslant \int_{\tau_0}^{t} |\psi^{-1}(t) V(t, s)| \varphi(s) \omega(s, 2\rho) ds +$$

$$(3.3) \qquad + \int_{t}^{\infty} |\psi^{-1}(t) W(t, s)| \varphi(s) \omega(s, 2\rho) ds \leqslant \int_{t_0}^{t} |\psi^{-1}(t) V(t, s) \varphi(s)| \omega(s, 2\rho) ds +$$

$$+ K \left[\int_{t}^{\infty} \omega^p(s, 2\rho) ds \right]^{1/p}.$$

To verify (2.8), we shall use c) and e). Fix $\varepsilon > 0$ and select $t_1 \geqslant \tau_0 \geqslant t_0$ such that, on the one hand, $\int_{t_0}^{t_1} |\psi^{-1}(t) V(t, s) \varphi(s)|^q ds < (\varepsilon/2)^q \left[\int_{t_0}^{\infty} \omega^p(s, 2\rho) ds \right]^{-q/p}$ for $t \geqslant t_1$ and, on the other hand, $\int_{t}^{\infty} \omega^p(s, 2\rho) ds < (\varepsilon/4K)^p$ for $t \geqslant t_1$. Then, the first term on the right hand of (3.4) can be estimated by

$$\int_{t_0}^{t} |\psi^{-1}(t) V(t, s) \varphi(s)| \omega(s, 2\rho) ds \leqslant \left[\int_{t_0}^{t_1} |\psi^{-1}(t) V(t, s) \varphi(s)|^q ds \right]^{1/q} \cdot \left[\int_{t_0}^{t_1} \omega^p(s, 2\rho) ds \right]^{1/p} +$$

$$+ \left[\int_{t_1}^{t} |\psi^{-1}(t) V(t, s) \varphi(s)^q ds \right]^{1/q} \left[\int_{t_1}^{t} \omega^p(s, 2\rho) ds \right]^{1/p} <$$

$$< (\varepsilon/2) + (\varepsilon/4).$$

Hence, $\psi^{-1}(t) |x(t) - y(t)| < \varepsilon$, and thus, we conclude that $\lim \psi^{-1}(t) |x(t) - y(t)| = 0$ as $t \to \infty$, which means that (2.8) holds. Conversely, let $x(t)$ be a ψ-bounded solution of (2.1). It means that there exists $g^x(t) \in M(x(t))$ such that $x(t)$ satisfies

$$x'(t) = A(t) x(t) + \int_{\tau_0}^{t} B(t, s) x(s) ds + f(t) + g^x(t), \text{ a.e. on } J.$$

Define for $t \in J$,

$$y(t) = x(t) - \int_{\tau_0}^{t} V(t, s) g^x(s) ds + \int_{t}^{\infty} W(t, s) g^x(s) ds,$$

where $g^x(t) = x'(t) - A(t) x(t) - \int_{\tau_0}^{t} B(t, s) x(s) ds - f(t) \in F(t, x(\cdot))$, a.e. on J. An easy calculation shows that $y(t)$ is a bounded solution of (2.1) and that (2.8) holds.

In what follows two results concerning the (ψ, p)-integral equivalence will be given.

Theorem 3.2. *Let all assumptions of Theorem 3.1, except e), be satisfied. If, instead of e), the condition*

$$\text{e}') \quad \int_{t_0}^{t} |\psi^{-1}(t) V(t, s) \varphi(s)|^q ds + \int_{t}^{\infty} |\psi^{-1}(t) W(t, s) \varphi(s)|^q ds \in L_1(J)$$

is satisfied, then there is a (ψ, q)-integral equivalence between the ψ-bounded solutions $y(t)$ of (2.1) and ψ-bounded solutions $x(t)$ of (2.2).

Proof. Using the assumption c) of Theorem and the assumption e'), we have

$$2^{-q} [\psi^{-1}(t) |x(t) - y(t)|]^q \leqslant \left[\int_{\tau_0}^{t} |\psi^{-1}(t) V(t, s) \varphi(s)| \omega(s, 2\rho) ds \right]^q +$$

$$+ \left[\int_{t}^{\infty} |\psi^{-1}(t) W(t, s) \varphi(s)| \omega(s, 2\rho) ds \right]^q \leqslant \left[\int_{t_0}^{t} |\psi^{-1}(t) V(t, s) \varphi(s)|^q ds \right] \cdot$$

$$\left[\int_{t_0}^{t} \omega^p(s, 2\rho) ds \right]^{q/p} + \left[\int_{t}^{\infty} |\psi^{-1}(t) W(t, s) \varphi(s)|^q ds \right] \cdot \left[\int_{t}^{\infty} \omega^p(s, 2\rho) ds \right]^{q/p} \leqslant$$

$$\leqslant \left[\int_{t_0}^{t} \omega^p(s, 2\rho) ds \right]^{q/p} \left[\int_{t_0}^{\infty} |\psi^{-1}(t) V(t, s) \varphi(s)|^q ds + \int_{t}^{\infty} |\psi^{-1}(t) W(t, s) \varphi(s)|^q ds \right] \in L_1(J).$$

Theorem 3.3. *Let the assumptions a), b), d) of Theorem 3.1, except c) and e), be satisfied; if, instead of c) and e), the conditions*

c') for any constant $c \geqslant 0$, $\int_{t_0}^{\infty} s \omega^p(s, c) ds < \infty$;

$$\text{e}') \quad \int_{t_0}^{t} |\psi^{-1}(t) V(t, s) \varphi(s)|^q ds \in L_1(J), \; p^{-1} + q^{-1} = 1, \; q \geqslant p,$$

are satisfied, them the conclusion of Theorem 3.2 is again valid.

Proof. Indeed, c') implies that $\int_{t}^{\infty} \omega^p(s, 2\rho) ds \in L_r(J)$, $(\forall)$ $r \geqslant 1$ ([8], Lemma 2). Taking into account the other hypotheses we get $2^{-q} [\psi^{-1}(t) |x(t) - y(t)|]^q \leqslant$

$$\leqslant \int_{\tau_0}^{t} |\psi^{-1}(t) V(t, s) \varphi(s)|^q ds \cdot \left[\int_{\tau_0}^{t} \omega^p(s, 2\rho) ds \right]^{q/p} +$$

$$+ \int_{t}^{\infty} |\psi^{-1}(t) W(t, s) \varphi(s)|^q ds \left[\int_{t}^{\infty} \omega^p(s, 2\rho) ds \right]^{q/p} \leqslant$$

$$\leqslant \left[\int_{t_0}^{\infty} \omega^p(s, 2\rho) ds \right]^{q/p} \int_{t_0}^{t} |\psi^{-1}(t) V(t, s) \varphi(s)|^q ds + K^q \left[\int_{t}^{\infty} \omega^p(s, 2\rho) ds \right]^{q/p},$$

from which it follows that (2.1) and (2.2) are (ψ, q)-integrally equivalent.

Remarks 3.1. For $q = \infty$ (hence $p = 1$) Therems 3.1 remains valid if conditions a) and e) are replaced by a′) $\sup\{|\psi^{-1}(t)V(t,s)\varphi(s)|\ ;\ t_0 \leqslant s \leqslant t\} + \sup\{|\psi^{-1}(t)W(t,s)\varphi(s)|\ ;\ t \leqslant s < \infty\} \leqslant K$, for each $t \in J$;

e″) $\lim\limits_{t\to\infty} \sup\{|\psi^{-1}(t)V(t,s)\varphi(s)|\ ;\ t_0 \leqslant s \leqslant \tau\} = 0$ for every $\tau \in J$.

3.2. For $q=1$ (hence $p=\infty$), Theorem 3.1 remains valid if conditions b) and c) are replaced by

b″) $\omega(t, u)$ is nondecreasing in u for fixed $t \in J$ and it is measurable and bounded on J for each fixed $u \in [0, \infty)$;

c″) $\lim\limits_{t\to\infty} \sup\{\omega(s, c)\ ;\ s \geqslant t\} = 0$, for each $c \geqslant 0$,

and the other conditions remain unchanged. We only note that the upper semicontinuity of T can be proved as in the proof of Theorem 2 [14] using a result on the weak precompacity in $L_1(J)$.

3.3. In particular case of differential inclusions ($B(t, s) \equiv 0$) it turns out that $P(s) = P_2Y^{-1}(s)$ so that $V(t, s) = Y(t)P_1Y^{-1}(s)$, $W(t, s) = Y(t)P_2Y^{-1}(s)$, where $Y(t)$ is the fundamental matrix for the linear homogeneous system and P_1, P_2 are supplementary projections into R^n, i.e., $P_1^2 = P_1$, $P_2^2 = P_2$, $P_1 + P_2 = I$. This case has been studied by S e k W u i S e a h [13] for $F(t, x_t) \equiv F(t, x(t))$.

3.4. The case of integro-differential equations with $F(t, x_t)$ a single valued function using the two matrices $V(t, s)$ and $W(t, s)$ has been studied in [1] and [15].

3.5. Analogous results to those from Remarks 3.1—3.3 can be obtained for Theorems 3.2 and 3.3.

If we take $\varphi \equiv \psi \equiv 1$, then the previous theorems reduce to the following corollaries:

Corollary 3.1. *Assume that:*

a$_1$) *there exist constant* K, q *with* $K > 0$ *and* $1 < q < \infty$, *such that for all*

$$t \in J,\quad \int_{t_0}^{t} |V(t, s)|^q\, ds + \int_{t}^{\infty} |W(t, s)|^q\, ds \leqslant K^q;$$

b$_1$) *there exist* $\omega : J \times [0, \infty) \to [0, \infty)$ *satisfying* b) *from Theorem 3.1;*

c$_1$) *for any constant* $c \geqslant 0$, $\omega(t, c)$ *satisfies* c) *from Theorem 3.1;*

d$_1$) *for each* $x(t) \in C_c(J, R^n)$, $|F(t, x(\cdot))| \leqslant \omega(t, \|x_t\|)$ *a.e. on* J;

e$_1$) *for every* $\tau \in J$, $\lim\limits_{t\to\infty} \int_{t_0}^{\tau} |V(t, s)|^q\, ds = 0$.

Then, there exists a 1-asymptotic equivalence between bounded solutions $y(t)$ *of* (2.1) *and bounded solutions* $x(t)$ *of* (2.2).

This Corollary reduces to Corollary 3.2 from [13] when $B(t, s) \equiv 0$ and to a particular case of Theorem 2.1 from [15] when F is a single-valued function. We note that analogous corollaries to 3.1 can be given to the Theorem 3.2 and 3.3.

Corollary 3.2. *Let all assumptions of Corollary 3.1, except* e$_1$), *be satisfied. If, instead of* e$_1$), *the condition*

$$\text{e}_2)\qquad \int_{t_0}^{t} |V(t, s)|^q\, ds + \int_{t}^{\infty} |W(t, s)|^q\, ds \in L_1(J)$$

is satisfied, then there exists and (1, q)-integral equivalence between bounded solutions $y(t)$ *of* (2.1) *and bounded solutions* $x(t)$ *of* (2.2).

Corollary 3.3. *Let the assumptions* a$_1$), b$_1$), d$_1$) *of Corollary 3.1, except of* c$_1$) *and* e$_1$), *be satisfied. If, instead of* c$_1$) *and* e$_1$), *the conditions*

c$_2$) *for any constant* $c \geqslant 0$, $\int_{t_0}^{\infty} s\, \omega^p(s, c)\, ds < \infty$;

e$_2$) $\int_{t_0}^{t} |V(t, s)|^q\, ds \in L_1(J)$, $p^{-1} + q^{-1} = 1$, $q \geqslant p$,

are satisfied, then the conclusion of Corollary 3.2 is again valid.

As a special case of Theorem 3.1 with $p = 1$ we give next the following:

Theorem 3.4. *Let* $\psi(t)$, $\alpha(t)$ *and* $\beta(t, s)$ *be positive functions for* $t \in J$, *respectively for* $t \geqslant s \geqslant t_0$, *with* $\lim\limits_{t\to\infty} \psi^{-1}(t) = 0$ *and* $\beta(t, s)$ bounded on $t_0 \leqslant s \leqslant t < \infty$. *Assume that*

a) *there exists* $C > 0$ *such that*

$$|V(t, s)\alpha^{-1}(s)| \leqslant C\beta(t, s) \ \textit{for}\ t_0 \leqslant s \leqslant t \ \textit{and}\ |W(t, s)\alpha^{-1}(s)| \leqslant C \ \textit{for}\ t_0 \leqslant t \leqslant s < \infty;$$

b) *there exists* $\tilde{\omega} : J \times [0, \infty) \to [0, \infty)$ *such that* $\tilde{\omega}(t, u)$ *is monotone nondecreasing in* u *for each fixed* $t \in J$ *and* $\tilde{\omega}(t, c\psi(t))$ *is integrable on compact subintervals of* J *for each* $c \geqslant 0$;

c) *for each* $c \geqslant 0$, $\int_{t_0}^{\infty} \alpha(s)\tilde{\omega}(s, c\psi(s))\, ds < \infty$;

d) *for each* $x(t) \in C_c(J, R^n)$, $|F(t, x)(\cdot))| \leqslant \tilde{\omega}(t, \|x_t\|)$ *a.e. on* J;

e) *for each* $c \geqslant 0$, $\lim\limits_{t\to\infty} \int_{t_0}^{t} \beta(t, s)\alpha(s)\tilde{\omega}(s, c\psi(s))\, ds = 0$.

Then, there exists a ψ*-asymptotic equivalence between the* ψ*-bounded solutions* $y(t)$ *of* (2.1) *and* ψ*-bounded solutions* $x(t)$ *of* (2.2).

Proof. Let $\tau_0 \geqslant t_0$ temporarly fixed, and $y(t)$ be a ψ bounded solution of (2.1) with $\|y\|_\psi \leqslant \rho$ for some $\rho > 0$. If we take $\varphi(t) = \alpha^{-1}(t)$ and $\omega(t, u) = \alpha(t)\tilde{\omega}(t, \psi(t)u)$ for $t \geqslant t_0$, then for each $t \geqslant \tau_0$, $\omega(t, u)$ is monotone nondecreasing in u and for each $u \in J$, $\omega(t, u)$ is integrable on compact subintervals of $[\tau_0, \infty)$ according to b). From c), we have $\int_{t_0}^{\infty} \omega(s, c)\, ds < \infty$ for $c \geqslant 0$. Then, condition a) and the assumptions on $\psi(t)$ and $\beta(t, s)$ imply that there exists $K > 0$ such that

$$\sup\{|\psi^{-1}(t)V(t, s)\varphi(s)|\ ;\ \tau_0 \leqslant s \leqslant t\} + \sup\{|\psi^{-1}(t)W(t, s)\varphi(s)|\ ;\ t \leqslant s < \infty\} \leqslant K.$$

Using the same arguments as in the proof of Theorem 3.1 we can conclude that there exists a solution $x(t)$ of (2.2) with $\|x\|_\psi \leqslant 2\rho$ and for some $g^x(t) \in M(x(t))$, we have

$$|x(t) - y(t)| \leqslant \int_{t_0}^{t} |V(t, s)\alpha^{-1}(s)|\alpha(s)\tilde{\omega}(s, 2\rho\psi(s))ds + \int_{t}^{\infty} |W(t, s)\alpha^{-1}(s)|\alpha(s)\tilde{\omega}(s, 2\rho\psi(s))\, ds \leqslant$$

$$\leqslant C\Big[\int_{t_0}^{t} \beta(t, s)\alpha(s)\tilde{\omega}(s, 2\rho\psi(s))\, ds + \int_{t}^{\infty} \alpha(s)\tilde{\omega}(s, 2\rho\psi(s))\, ds\Big].$$

In view of conditions c) and e), for any $\varepsilon > 0$ there exists $t_1 > \tau_0$ such that for $t \geqslant t_1$

$$\int_{t_0}^{t} \beta(t, s)\alpha(s)\tilde{\omega}(s, 2\rho\psi(s))ds<(\varepsilon/2C) \text{ and } \int_{t}^{\infty} \alpha(s)\tilde{\omega}(s, 2\rho\psi(s))ds<(\varepsilon/2C).$$

Hence, for $t \geqslant t_1$, $|x(t)-y(t)|<\varepsilon$ and since $\varepsilon>0$ is arbitrary, it follows that $\lim_{t\to\infty} |x(t)-y(t)|=0$. The proof of the converse is exactly the same as for Theorem 3.1.

Remark 3.5. In particular case of differential inclusions (see Remark 3.3) Theorem 3.4 reduces to Theorem 3.3 [13].

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ON PSEUDO-KAEHLERIAN FINSLER SPACES

BY

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Osculating Riemann metrics associated to Finsler metrics were considered previously by O. Varga [12] and A. Moor [11]. Also T. Maebashi [8] has investigated subspaces of Finsler spaces admitting a preferential direction ; yet his results are true for the case $R^i_{jk}=\delta N^i_j/\delta x^k-\delta N^i_k/\delta x^j=0$ only. Moreover, G. Asanov [3] has shown that given a 4-dimensional manifold with full parallelism structure $\{S_i\}_{1\leqslant i\leqslant 4}$ (which stands for space-time), the osculating Riemann metric (at the preferential direction $S=\sum_{i=1}^{4} S_i$) associated to Berwald-Moore metric of the space represents an approximation of the Finsler metric (up to terms of second order in the standard deviations of the supporting elements from the preferential direction S) and thus the basic structure of the real space-time is Finslerian. Clearly the cases considered in [8], [11], [12] as well as in [2], [3] are subject to topological restrictions (such as the vanishing of Euler-Poincaré characteristic) even if not mentioned.

Our purpose is to examine osculating objects at a preferential direction of a differentiable manifold associated to several finslerian geometrical objects such as finslerian almost complex structures, finslerian almost hermitian structures, etc. Attention is devoted to the integrable case, i.e. $R^i_{jk}=0$, indicated by M. Matsumoto-L. Tamassy [10] to be of particular interest; the main result is that a pseudo-Kaehlerian Finsler space (M, E, J) induces locally osculating Kaehlerian structures on M.

A result of T. Maebashi [8] is reconsidered in §4.

1. The notations and conventions throughout the present paper are a combined version of those from B. T. M. Hassan [7] and M. Matsumoto [9].

Let M be a n-dimensional real C^∞-differentiable manifold and let j be the natural imbedding of M into the total space $T(M)$ of the tangent bundle over M. Let $\tilde{M}=T(M)\setminus j(M)$ and $\pi:\tilde{M}\to M$ be the natural projection ; moreover we denote by $\pi^{-1}TM$ the bundle induced by $T(M)$ and π. A finslerian almost complex structure on M is a bundle morphism $J:\pi^{-1}TM\to\pi^{-1}TM$ such that $J^2=-I$, where I denotes the identity transformation on $\pi^{-1}TM$.

Let (U, x^i) be a local coordinate system on M and $(\tilde{U}, x^i, y^i)$ the naturally induced coordinates on $\tilde{M}$, where $\tilde{U}=\pi^{-1}(U)$. Let $\bar{\partial}_i$ be the natural lift of the locally defined tangent vector fields $\partial/\partial x^i$, i.e. $\bar{\partial}_i(\tilde{x})=(\tilde{x}, \partial/\partial x^i|_{\pi\tilde{x}})$, $\tilde{x}\in\tilde{M}$. We put $J\bar{\partial}_i=F^j_i(x, y)\bar{\partial}_j$; then $F^j_i F^k_j=-\delta^k_i$. It is easily seen that a differentiable manifold carrying a finslerian almost complex structure should be even-dimensional. Note that any finslerian almost complex structure on a