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SUMAR

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RINGS "WITH FEW ZERO-DIVISORS"

BY

DORIAN SPULBER

§0. The problem to define and to characterize the rings which are not domains, but which keep some of the properties of the Prüfer domains, eventually modified, was considered by many authors. D a v i s [3] defines *P-rings*, B u t t s and S m i t h [2] the so called *"Prüfer rings"*. L a r s e n [5] proves the equivalence between the characterizations given by D a v i s and those given by B u t t s and S m i t h, a little improved, and, also, the equivalence to the characterization given by R i c h m a n [9] by means of flatness, but all these taking place for *rings "with few zero-divisors"* defined by D a v i s [3]. We shall complete the L a r s e n 's theorem recalled above. In [5], L a r s e n extends the R i c h m a n 's considerations on the flat overrings to the rings "with few zero-divizors". In the next section, we shall prove some similar results for *CPZ-rings*, a class of rings defined in our paper [10], which contains also as particular cases both the rings "with few zero-divisors" and the *CP-rings* [8,6]. In connection with the characterization of Prüfer rings, the notion of quasivaluation ring defined by D a v i s for rings "with few zero-divisors" is used. We shall show that the definition is kept generally, without the restriction of being "with few zero-divisors", and we shall give a theorem of characterization of these rings following B o u r b a k i ([1], ch. 6).

In that follows the rings will be commutative unitary rings; the ring morphisms will be unitary. All notions related to the localization of rings are taken from G a b r i e l [4] or P o p e s c u [7].

If $\varphi: R \to S$ is a ring morphism, F_φ will denote the localizing system formed by the ideals A of R with the property $\varphi(A)\,S=S$. Let R be a ring; we shall call an overring of R, a ring S with the property $R\subset S\subset R_{F_i}$, where R_{F_i} is the total quotient ring of R. We denote $R_{F_i}=Q$. An ideal of R generated only by regular elements is called an *r-ideal* of R.

§1. Theorem 1.1. *Let R be a ring and S be an overring of R. The following statements are equivalent with respect to S:*

1) *Every element x being regular in Q is in S or its inverse x^{-1} is in S.*
2) *The regular principal ideals of S are linearly ordered by the inclusions.*
3) *The r-ideals of S are linearly ordered by inclusion.*

Proof. 1)⇒2): Let a, b be two regular elements of S. Assume that $(a)\not\subset(b)$ (by (a) we denote the principal ideal generated by a), hence $a\notin(b)$. That means $ab^{-1}\notin S$, therefore $a^{-1}b\in S$, hence $b\in(a)$ and $(b)\subset(a)$.

2) $\Rightarrow$3) : Let A and B be two r-ideals; if $A\not\subset B$, then there exists at least a regular generator $x\in A$ so that $x\notin B$, hence $x\notin(y)$, and, by 2), $(y)\subset(x)$, therefore $B\subset A$.

3) $\Rightarrow$1) : Let x be a regular element in Q; then $x=ab^{-1}$ with a, b regular elements in S. If $(a)\subset(b)$, then $x\in S$; if $(b)\subset(a)$, then $x^{-1}\in S$.

A ring verifying one of the equivalent conditions of the previous theorem is called a *quasivaluation ring*.

A ring which has a finite number of maximal ideals formed only by zero-divisors is called by D a v i s [3] a *ring "with few zero-divisors"*. In the same paper, D a v i s defines also the quasivaluation ring as a ring "with few zero-divisors" verifying the condition 1) of the theorem 1.1. We can easily see that the D a v i s' definition for the quasivaluation ring is obtained by particularizing our definition.

Corollary 1.2. *Let R be a quasivaluation ring. Every overring S of R is also a quasivaluation ring.*

Proof. Indeed, let $R\subset S\subset Q$; if x is regular in Q and $x\notin S$, then $x\notin R$; but then $x^{-1}\in R$, therefore $x^{-1}\in S$.

§2. C. M. R e i s and T. M. V i s v a n a t h a n [8] define the property:

(CP): *Let R be a ring. If A is an ideal of R an $\{P_i\}_{i\in I}$ is a set of prime ideals of R so that $A\subset\bigcup_{i\in I}P_i$, then $A\subset P_{i_0}$ for some $i_0\in I$.*

We consider the following property:

(CPZ): *Let R be a ring. If A is an ideal of R and $\{M_i\}_{i\in I}$ is a set of maximal ideals of R not belonging to the localizing system F_r (the localizing system of the regular ideals), with possible exception of at most a finite set of prime ideals such that $A\subset\bigcup_{i\in I}M_i$, then $A\subset M_{i_0}$ for some $i_0\in I$* [10].

A ring verifying the property **(CPZ)** is called a *CPZ-ring*.

Corollary 2.1. *If R is a CP-ring, then R is a CPZ-ring.*

Corollary 2.2. ([1]). *A ring "with few zero-divisors" is a CPZ-ring.*

Theorem 2.3. *Let R be a ring. The following statements on R are equivalent:*

1) *R is a CPZ-ring.*

2) *Let F be a localizing system of R such that $F=\bigcap_i F_{M_i}$, with M_i a family of maximal ideals of R which do not belong to the localizing system F_r, with the possible exception of the finite set of prime ideals. Then $F=F_S$ for some closed multiplicative system S of R.*

Proof. 1)$\Rightarrow$2) : If $F=\bigcap_i F_{M_i}$ and we denote $S=R\setminus(\bigcup_i M_i)$, then S is a closed multiplicative system in R and $F_S\subset F$. If A is an ideal of R such that $A\notin F_S$, then $A\cap S=\varnothing$ and $A\subset\bigcup_i M_i$, hence there exists an i_0 such that $A\subset M_{i_0}$. That means $A\notin F_{M_{i_0}}$, therefore $A\notin F$, hence $F_S=F$.

2)$\Rightarrow$1) : Let $\{M_i\}$ be a family of maximal ideals which do not belong to F_r, with the possible exception of a finite number of prime ideals of R. Denote $F=\bigcap F_{M_i}$. By 2), $F=F_{S'}$ for some closed multiplicative system S'. But $M_i\notin F$ for every i, hence $M_i\cap S'=\varnothing$; therefore $S'\subset S$ with $S=R\setminus(\bigcup_i M_i)$, hence $F_{S'}\subset F_S$. On the other hand, $M_i\cap S=\varnothing$ for every i, therefore $F_S\subset F_{M_i}$,

hence $F_S\subset F_{S'}$. As $A\notin F_{S'}$, we have $A\notin\bigcap_i F_{M_i}$, that is $A\notin F_{M_i}$ for some i_0, which is equivalent to $A\subset M_{i_0}$.

Let P a prime ideal in a ring R, $S(P)$ be the closed multiplicative system of regular elements of $R\setminus P$, and $F_{S(P)}$ be the associated localizing system; denote by $R_{S(P)}$ the localization ring $R_{F_{S(P)}}$. If $\varphi: R\to T$ is a ring morphism and A is an ideal of R, then $A^e=\varphi(A)\,T$ is the extended ideal of A, and if B is an ideal in T, then $B^c=\varphi^{-1}(B)$ is the contraction of B (see [1]).

Proposition 2.4. *Let R be a CPZ-ring and P be a regular prime ideal of R. Then $PR_{S(P)}$ is the unique regular maximal ideal of $R_{S(P)}$.*

Proof. Let $B\subset R_{S(P)}$ be a regular ideal; then $B=B^{ce}$, where $\varphi: R\to R_{S(P)}$ is the canonical morphism. As $B^c\cap S(P)=\varnothing$, we have $B^c\subset P$; otherwise, $B^c\not\subset P$ implies $B^c\not\subset M_i$ for every M_i a maximal ideal formed only by zero-divisors in R such that $B^c\not\subset(\bigcup_i M_i)$; this implies the existence of an element $x\in B^c\cap(R\setminus((\bigcup_i M_i)\cup P))$, hence $x\in S(P)$ and $B^c\cap S(P)\neq\varnothing$, which contradicts our hypotheses. Hence $B^c\subset P$ and $B^{ce}\subset P^e=PR_{S(P)}$.

§3. Now, we state a theorem which completes the theorem 5 (L a r s e n [5]) for rings "with few zero-divisors".

The inclusion $R\subset Q$ is called completely *epimorphic* [12, 13] if for each overring T of R, the canonical inclusion $R\subset T$ is an epimorphism in the category of rings.

Theorem 3.1. *Let R be a ring "with few zero-divisors". The following statements are equivalent:*

1) *The inclusion $R\subset Q$ is completely epimorphic.*

2) *Every overring of R is integrally closed in Q (R is a P-ring [3]).*

3) *For each maximal ideal M in R, $R_{S(M)}$ is a quasivaluation ring.*

4) *Every overring T is R-flat.*

5) *For each overring T, the set I of the localizing systems $F\subset F_r$ such that $T=R_F$ is nonvoid and there exists an element F' in I such that for any ideal B in T, $B\neq T$, $B\cap R\notin F'$* [11].

Proof. 1)$\Rightarrow$2) : Is a consequence of the Lemma 3 [13].

2)$\Rightarrow$3) Is the Theorem 3, §2 [3].

3)$\Rightarrow$4) : Let M be a maximal ideal in T and $P=M\cap R$; let us prove that $R_{S(P)}=T_{S(M)}$, therefore T is R-flat. Let M' be a maximal ideal in R which includes the ideal P. Then there exist the inclusions: $R_{S(M')}\subset R_{S(P)}\subset T_{S(M)}$. Let $x\in T_{S(P)}$ and $x\notin R_{S(P)}$. Then $x^{-1}\in R_{S(P)}$, since $R_{S(P)}$ is an overring of the quasivaluation ring $R_{S(M')}$ and therefore is itself a quasivaluation ring. This implies that $x^{-1}\in T_{S(M)}$, hence x, $x^{-1}\notin MT_{S(M)}$, therefore $x^{-1}\notin PR_{S(P)}$. We obtain that $x^{-1}\in(R_{S(P)}\setminus PR_{S(P)})$, hence x is inversable and hence $x\in R_{S(P)}$, which contradicts the hypothesis. By Lemma C, §2 [3], it follows that $R_{S(P)}=T_{S(M)}$.

Remark. The implication 3)$\Rightarrow$4) is given by L a r s e n [5]; the above proof, different from that given by L a r s e n, seems to be simpler.

4)$\Rightarrow$5) : Let T be a R-flat overring. The localizing system $F=\{A\subset R\setminus AT=T\}$ has the properties: $T=R_F$ and for every ideal $B\neq T$, $B\cap R\notin F$.

5)$\Rightarrow$1) : Let F' be the localizing system given by 5). The inclusion $R\subset T$ is a flat morphism. But this is valid for every overring T; it results that the inclusions $R\subset Q$ is completely epimorphic.

R E F E R E N C E S

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Department of Mathematics
University of Iaşi
6600 — Iasi, Romania

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THE CONSISTENCY AND CONSERVATIVE EXTENSION PROBLEM FOR THE IMPREDICATIVE EXTENSION OF VBG: MODIFICATIONS OF ZF EQUICONSISTENT WITH ZF^{KM}

BY

S. M. KIM

In [3], W. Marek and A. Mostowski set forth an interesting extension of ZF (Zermelo-Fraenkel set theory), denoted as ZF^{KM}; ZF^{KM} is the set of formulas Φ of the language of ZF set theory such that the relativisation Φ^V of Φ to the universe of sets is provable in KM (Kelley-Morse theory, i.e., the impredicative extension of von Neumann-Bernays-Gödel set-theory (VBG)). Then the system ZF^{KM} is axiomatisable, but no axiomatisation (in the language of set theory) is known. They conjectured that ZF^{KM} consists of sentences as true as those of ZF set theory. This reminds us of the fact that ZF is equivalent to VBG and hence ZF is equiconsistent with VBG [6] [7] and [4] [5], respectively. Now it is natural to ask how ZF^{KM} proof-theoretically and/or model-theoretically compares with ZF or some other modifications of ZF. First however, we comment on notations. For the most part, our notation will be that commonly employed in set theory. In general, α, β, γ, ... denote ordinals. lim (α) stands for "α is a limit ordinal". fn (f) means that f is a function. Denote dom (f) for domain of f. If a is a set, $p(a)$ denotes its power set. con (ZF) stands for "ZF is consistent". We refer the reader to [2] for any notion we do not cover.

Lemma 1.

$$\mathrm{con}\,(ZF^{KM}) \leftrightarrow \mathrm{con}\,(KM).$$

Let us denote the second order ZF set theory by ZF_{II}. If we define ZF'_{II} to be the ZF_{II} plus the formula $(\forall A\ \forall B(\forall a(A(a) \equiv B(a)) \rightarrow A = B))$ then we obtain

Theorem 2.

$$\mathrm{con}\,(KM) \leftrightarrow \mathrm{con}\,(ZF'_{II}).$$

Proof. (1) Proof theoretical equivalence

$$(q(a, \ldots) \wedge \forall b(q(b, \ldots) \rightarrow q(b, \ldots)) \wedge (\neg q(b, \ldots) \rightarrow \neg q(b, \ldots))) \rightarrow$$

$$\rightarrow \exists p(Pa \wedge \forall b((q(b, \ldots) \rightarrow Pb) \wedge (\neg q(b, \ldots) \rightarrow \neg Pb))) \rightarrow$$

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