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ADAPTED CONNECTIONS AND F-PROJECTIVE CURVATURE TENSOR ON NORMAL FRAMED MANIFOLDS

BY

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0. Introduction. C-projective curvature tensor fields on normal almost-contact manifolds have been obtained in [5] using some particular connec-tiors which are called „adapted" [see 4]; the existence of the torsion-free adap-ted connections characterizes completely the normality of almost contact manifolds. In this paper we study the adapted connections on framed mani-folds in order to introduce the Weyl F-projective curvature tensor field [cf. 5].

1. Adapted connections on framed manifolds. Let M be an n-dimensional manifold; we call M a framed manifold ([3], [8]) if there exists a field F of endomorphisms of the tangent spaces of constant rank r on M, $n-r$ differential forms $\eta^1, \eta^2, \ldots \eta^{n-r}$ and $n-r$ vector fields $\xi_1, \xi_2, \ldots, \xi_{n-r}$ such that

$$F^2 = -I + \eta^a \otimes \xi_a, \quad \eta^a((\xi_b) = \delta^a_b \qquad a, b = 1, 2, \ldots n-r \tag{1.1}$$

where I is the identity field. It follows from the definition that [3]

$$F\xi_a = 0, \quad \eta^a \circ F = 0 \quad a = 1, 2, \ldots, n-r. \tag{1.2}$$

Observe that F is an f-structure on M, i.e. F satisfies the algebraic condition $F^3 + F = 0$ and r is necessarily even [8]. Moreover the framed manifold (M, F, η^a, ξ_a) is said normal iff the tensor field N of type (1.2) defined by the equation

$$N(X, Y) = [F, F]\,(X, Y) + d\eta^a(X, Y)\,\xi_a, \tag{1.3}$$

where $[F, F]$ denotes the Nijenhuis tensor of F, vanishes on M.

In this case one can prove that

$$\begin{cases} L_{\xi_a}\eta^b = 0, & [\xi_a, \xi_b] = 0, \\ L_{\xi_a}F = 0, & d\eta_a(FX, Y) + d\eta_a(X, FY) = 0. \end{cases} \tag{1.4}$$

where L_X denotes the Lie derivative with respect to the vector field X [3]. Consider now the manifold $M \times R^{n-r}$ and a vector field $\bar{X}$ on this product, i.e.

$$\bar{X} = X + f^a\, \partial/\partial t^a.$$

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where X is tangent to M, t^a, $a=1, 2, \ldots, n-r$, denotes the coordinate of the a-th factor in R^{n-r} and f^a are functions on $M\times R^{n-r}$. It is easy to check that the (1,1) tensor field J given by

$$J(X+f^a\,\partial/\partial t^a)=FX-f^a\xi_a+\eta^a(X)\,\partial/\partial t^a \tag{1.5}$$

defines an almost complex structure on $M\times R^{n-r}$. The integrability of the almost complex structure J on $M\times R^{n-r}$ is a necessary and sufficient condition for the normality of the framed structure F on M.

It seems a quite interesting problem, to find different ways to characterize the normal framed manifolds. To do this, we consider the $(n-r)$-dimensional distribution V spanned by the vector fields ξ_a and the r-dimensional complementary distribution H on M; then $TM=H\oplus V$ and the corresponding projectors are

$$v=\eta^a\otimes\xi_a,\quad a=1, 2, \ldots, n-r, \text{ on } V; \qquad h=I-v=I-\eta^a\otimes\xi_a \text{ on } H \tag{1.6}$$

and

$$h=-F^2, \qquad hF=Fh=F, \qquad Fv=vF=0. \tag{1.7}$$

Let's now assume that a linear connection ∇ exists on M such that the following system holds:

$$\begin{aligned}(\nabla_X F)(Y)&=\sum_c\left\{\eta^c(Y)\,hX+\frac{1}{4}[d\eta^c(FX, hY)-d\eta^c(X, FY)]\,\xi_c\right\},\\ (\nabla_X\eta^a)(Y)&=\frac{1}{4}\{d\eta^a(X, Y)+d\eta^a(FX, FY)\},\ \nabla_X\xi_a=FX-\frac{1}{4}\sum_c d\eta^c(X, \xi_a)\xi_c,\end{aligned} \tag{1.8}$$

where $a=1, 2, \ldots n-r$ and $X, Y\in\mathfrak{X}(M)$. Our main purpose is to prove that the normality of the framed manifolds can be characterized by means of torsion-free connections ∇ satisfying the equations (1.8).

First of all, we can remark that if ∇ is such a connection on M, then the connection $\overline{\nabla}$ defined on $M\times R^{n-r}$ by the system

$$\begin{cases}\overline{\nabla}_X Y=\nabla_X Y+\sum_c\left\{\sum_b\eta^c(X)\,\eta^b(Y)-\frac{1}{4}[d\eta^c(FX, hY)-d\eta^c(X, FY)]\right\}\dfrac{\partial}{\partial t^c},\\ \overline{\nabla}_X\dfrac{\partial}{\partial t^a}=-X-\dfrac{1}{4}\sum_c d\eta^c(X, \xi_a)\dfrac{\partial}{\partial t^c},\\ \overline{\nabla}_{\partial/\partial t^a}Y=-Y+\partial Y/\partial t^a, \qquad \overline{\nabla}_{\partial/\partial t^a}\partial/\partial t^b=-\partial/\partial t^b, \qquad X, Y\in\mathfrak{X}(M)\end{cases} \tag{1.9}$$

satisfies the relation $\overline{\nabla}J=0$. Therefore, it seems quite natural to call "adapted" a connection ∇ on M satisfying (1.8). Let P be a tensor field of type (1,2) and $\dot\nabla$ a linear connection on M; then if ∇ is an adapted connection we can write

$$\nabla_X Y=\dot\nabla_X Y+P(X, Y) \qquad X, Y\in\mathfrak{X}(M). \tag{1.10}$$

We now introduce on M the tensor fields of type (2,2) Φ, Ψ, Θ given by [6]

$$\Phi=1/2\,(I\otimes I-F\otimes F),\quad \Psi=1/2\,(I\otimes I+F\otimes F),\quad \Theta=1/2\,(I\otimes I-h\otimes h), \tag{1.11}$$

and such that

$$\begin{aligned}&\Phi+\Psi=I\otimes I, \qquad \Phi^2=\Phi-1/2\,\Theta, \qquad \Psi^2=\Psi-1/2\,\Theta,\\ &\Phi\Psi=\Psi\Phi=\Phi\Theta=\Theta\Phi=\Psi\Theta=\Theta\Psi=\Theta^2=1/2\,\Theta,\end{aligned} \tag{1.12}$$

$$\begin{aligned}&(\Psi+\Theta)+(\Phi-\Theta)=I\otimes I, \quad (\Psi+\Theta)\,(\Phi-\Theta)=(\Phi-\Theta)\,(\Psi+\Theta)=0,\\ &(\Psi+\Theta)\,(\Psi+\Theta)=\Psi+\Theta, \quad (\Phi-\Theta)\,(\Phi-\Theta)=\Phi-\Theta.\end{aligned} \tag{1.13}$$

By using (1.11), (1.12), (1.13) we can prove the following.

Proposition 1.1. *Let (M, F, ξ_a, η^a), $a=1, 2, \ldots n-r$, be a framed manifold. Then the family of the adapted connections on M is obtained by the formula (1.10) where $\dot\nabla$ is a linear connection on M and P is the (1.2) tensor field*

$$\begin{aligned}P(X, Y)=\frac{1}{2}(\dot\nabla_X F)\,(FY)&+\sum_c\left\{\eta^c(Y)\,FX-\eta^c(Y)\,\dot\nabla_X\xi_c+\frac{1}{2}(\dot\nabla_X\eta^c)(Y)\,\xi_c+\right.\\ &\left.+\frac{1}{2}\eta^c(\dot\nabla_X\xi_a)\,\eta^a(Y)\,\xi_c-\frac{1}{4}[d\eta^c(X, Y)+d\eta^c(FX, FY)]\,\xi_c\right\}+\\ &+(\Phi-\Theta)\,Q(X, Y),\end{aligned} \tag{1.14}$$

for any tensor field Q of type (1.2). Thus we can conclude that there exists at least one adapted connection on any framed manifold (M, F, ξ_a, η^a).

The proof is similar to the one given in the case of almost-contact manifolds [see 4].

2. Torsion-free adapted connections on normal framed manifolds. As in the almost-contact case, there exist several important relations between the torsion tensor field T of an adapted connection ∇ and the tensor field N on M, given respectively by

$$T(X, Y)=\nabla_X Y-\nabla_Y X-[X, Y], \tag{2.1}$$

$$N(X, Y)=[FX, FY]-F[FX, Y]-F[X, FY]-h[X, Y]+d\eta^a(X, Y)\,\xi_a. \tag{2.2}$$

From the equations (1.8) we easily obtain the following expression for N

$$\begin{aligned}N(X, Y)=T(X, Y)-hT(FX, FY)+FT(X, FY)+FT(FX, Y)+\\ +\sum_a\eta^a(T(X, Y))\,\xi_a.\end{aligned} \tag{2.3}$$

It trivially follows from (2.3)

Proposition 2.1. *Let M be a framed manifold. If a torsion-free adapted connection exists on M, the framed structure of M is normal.*

On the other hand, taking into account the formula (2.3) we get

$$vN(X, Y)=2vT(X, Y); \tag{2.4}$$

$$hN(X, Y)=hT(X, Y)+FT(X, FY)+FT(FX, Y)-hT(FX, FY). \tag{2.5}$$

Recalling the formulas (1.12) and (1.13), after a long but simple computation (see [5]) we get also

$$
\begin{aligned}
&hN(X, Y)+hN(X, vY)+hN(vX, Y)+hN(vX, vY)= \\
&=4hT(X, Y)-(\Phi-\Theta)\ (T(X, hY)+FT(FX, Y)+2T(X, Y)+ \\
&+2T(vX, Y))+(\Phi-\Theta)\ (T(hX, Y)+(FT(X, FY)+2T(Y, X)+ \\
&+2T(vY, X)).
\end{aligned}
\tag{2.6}
$$

Thus, from (2.4), we see that the part from V of the torsion field T depends only on N, while (2.6) expresses hT as a function of the tensor field hN. Then if M is a normal framed manifold we have

$$vT(X, Y)=0 \tag{2.7}$$

$$
\begin{aligned}
&hT(X, Y)-1/4\ (\Phi-\Theta)\ (T(X, hY)+FT(FX, Y)+2T(X, Y)+ \\
&+2T(vX, Y))+1/4\ (\Phi-\Theta)\ (T(hX, Y)+FT(X, FY)+2T(Y, X)+ \\
&+2T(vY, X))=0
\end{aligned}
\tag{2.8}
$$

which imply that the connection $\tilde\nabla$ defined by the equation

$$
\begin{aligned}
\tilde\nabla_X Y=\nabla_X Y-1/4\ (\Phi-\Theta)\ [\,&T(X, hY)+FT(FX, Y)+2T(X, Y)+ \\
&+2T(vX, Y)]
\end{aligned}
\tag{2.9}
$$

is a torsion-free adapted connection on M. Thus, we can conclude with the following

Theorem 2.2. *Let (M, F, ξ^a, η_a), $a=1, 2, \dots, n-r$, be an n-dimensional framed manifold. M is normal if and only if a torsion-free adapted connection exists on M.*

3. F-projective transformations. In [5] the torsion-free adapted connections to normal almost-contact structures are used to introduce the concept of C-projective transformations without any restriction for the structure. The authors study the relations between two torsion-free adapted connection having the same C-flat paths and, by a procedure similar to the one adopted for real and for complex manifolds [7], they define a C-projective curvature tensor field on normal almost-contact manifolds.

Now we introduce the same notions on normal framed manifolds (M, F, ξ_a, η^a) considering torsion-free adapted connections ∇ on M; hence, by (1.4), the system (1.8) becomes

$$
\begin{aligned}
&(\nabla_X F)\ Y=\sum_c\left\{\eta^c(Y)\ hX+\frac{1}{2}\,d\eta^c(FX, Y)\ \xi_c\right\}. \\
&(\nabla_X\eta^a)\ (Y)=1/2\ d\eta^a(X, Y),\ \nabla_X\xi_a=FX, \qquad a=1, 2, \dots, n-r.
\end{aligned}
\tag{3.1}
$$

Let J be an open interval in R and let $\gamma: J\rightarrow R$ be a smooth path on M; we call γ *F-flat* with respect to the torsion-free adapted connection ∇ if the covariant derivative with respect to ∇ of the tangent vector field $\dot\gamma$ belongs to the subspace spanned by $\dot\gamma$ and $F\dot\gamma$; in other words, γ is an F-flat path if $\dot\gamma$ fulfills the equation

$$\nabla_{\dot\gamma}\dot\gamma=\lambda(t)\ \dot\gamma+\mu(t)\ F\dot\gamma \tag{3.2}$$

where λ and μ are smooth real functions along γ. We can see that, in this case, the subspace spanned by $\dot\gamma$ and $F\dot\gamma$ is not transported by parallelism along γ, but, due to the relation

$$\dot\gamma=h\dot\gamma+\eta^a(\dot\gamma)\ \xi_a \tag{3.3}$$

it is easy to prove that this is a subspace of constant dimension $n-r+1$, $n-r$, ..., 2, 1 or 0 along γ [5]. To go further it is necessary to study the relation between two torsion-free adapted connections ∇, $\tilde\nabla$ having the same F-flat paths; we shall call these connections *F-projectively related*. After a simple but tedious computation, similar to the one used in [2] we get the following

Proposition 3.1. *Two torsion-free adapted connections ∇, $\tilde\nabla$ on a normal framed manifold (M, F, ξ_a, η^a) are F-projectively related if and only if for an arbitrary 1-form α such that $\alpha(\xi_a)=0$ for any $a=1, 2, \dots, n-r$, we have*

$$\tilde\nabla_X Y=\nabla_X Y+P(X, Y) \tag{3.4}$$

where

$$P(X, Y)=\alpha(X)\ hY+\alpha(Y)\ hX-\beta(X)FY-\beta(Y)FX, \tag{3.5}$$

and $\beta=\alpha\circ F$.

The equations (3.4) and (3.5) represent an *F-projective transformation* on the normal framed manifold M.

4. F-projective curvature. In order to define an F-projective curvature tensor field, we must study the tensor fields obtained from the curvature tensor R of a torsion-free adapted connection ∇ which are invariant under F-projective transformations.

Before doing this, it is useful to give some formulas on R that we shall use later. First, by a straightforward computation, the system (3.1) gives

$$(\nabla_X d\eta^a)\ (Y, Z)=(L_X d\eta^a)\ (Y, Z)-d\eta^a(\nabla_Y X, Z)-d\eta^a(Y, \nabla_Y Z) \tag{4.1}$$

for any $a=1, 2, \dots, n-r$; in particular we have

$$(\nabla_X d\eta^a)\ (Y, \xi_c)=d\eta^a(FX, Y) \qquad a=1, 2, \dots, n-r. \tag{4.2}$$

Now, recalling the expression for the curvature tensor field of a connection ∇

$$R_{XY}Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X, Y]}Z,$$

by (3.1) and (4.1) we get

$$
\begin{aligned}
&R_{XY}\xi_c=\sum_a(\eta^a(Y)\ hX-\eta^a(X)\ hY), \\
&\eta^a(R_{XY}Z)=1/2\ (\nabla_Z d\eta^a)\ (X, Y) \text{ and } \eta^a(R_{Z\xi_c}Z)=1/2\ d\eta^a(FZ, X) \\
&\qquad a, c=1, 2, \dots, n-r,
\end{aligned}
$$

$$
\begin{aligned}
R_{XY}FZ-FR_{XY}Z=\sum_a\Big\{&\frac{1}{2}(\nabla_{FZ}d\eta^a)\ (X, Y)\ \xi_a+\frac{1}{2}d\eta^a(X, Z)\ hY- \\
&-\frac{1}{2}d\eta^a(Y, Z)\ hX-\left[\frac{1}{2}d\eta^a(Y, FZ)+\sum_b\eta^a(Z)\ \eta^b(Y)\right]FX+
\end{aligned}
\tag{4.3}
$$

$$+\left[\frac{1}{2}\,d\eta^a(X, FZ)+\sum_b \eta^a(X)\,\eta^b(Z)\right]FY\Bigg\}.$$

$$\eta^a(R_{FXFY}Z)=\eta^a(R_{XY}Z)-\frac{1}{2}\sum_b(\eta^b(X)\,d\eta^a(Y, FZ)-\eta^b(Y)\,d\eta^a(X, FZ)).$$

We remark that, due to the second relation in (4.3), the part from V of $R_{XY}Z$ depends on ∇; moreover, from the other relations, we see that $R_{XY}\xi_c$, $\eta^a(R_{X\xi_c}Z)$, $\eta^a(R_{FXFY}Z-R_{XY}Z)$ are F-projective invariant.

Next we can consider the Ricci tensor field S of the torsion-free adapted connection ∇; it is well known that S is the 2-times covariant tensor field given by the trace of the linear operator $X\to R_{XY}Z$, where Y, Z are fixed vector fields on M. From the equations (4.3), we get

$$S(X, \xi_c)=r\sum_a\eta^a(X), \quad S(X, FY)+\frac{r-1}{2}\sum_a d\eta^a(X, Y)=\mathrm{tr}(Z\to FR_{ZX}Y). \tag{4.4}$$

Now let's suppose that ∇ and $\tilde{\nabla}$ are two torsion-free adapted connections F-projectively related. If R and $\tilde{R}$ are the curvature tensor fields of ∇ and $\tilde{\nabla}$ respectively, then (3.4) and (3.5) imply

$$\begin{aligned}\tilde{R}_{XY}Z&=R_{XY}Z+\{A(X, Y)-A(Y, X)\}\,hZ+A(X, Z)hY-A(Y, Z)\,hX-\\&-\{A(X, FY)-A(Y, FX)\}\,FZ-A(X, FZ)\,FY+A(Y, FZ)\,FX+\\&+\sum_a\left\{\frac{1}{2}\alpha(X)\,d\eta^a(Y, Z)\,\xi_a-\frac{1}{2}\alpha(Y)\,d\eta^a(X, Z)\,\xi_a-\alpha(Z)\,d\eta^a(X, Y)\,\xi_a+\right.\\&\left.+\frac{1}{2}\beta(Y)\,d\eta^a(X, FZ)\,\xi_a-\frac{1}{2}\beta(X)\,d\eta^a(Y, FZ)\,\xi_a\right\}.\end{aligned} \tag{4.5}$$

where

$$A(X, Y)=(\nabla_X\alpha)(Y)-\alpha(X)\,\alpha(Y)+\beta(X)\beta(Y)+\Sigma(\beta(X)\,\eta^a(Y)+\beta(Y)\,\eta^a(X)) \tag{4.6}$$

in particular

$$A(X, \xi_c)=0 \text{ for any } c=1, 2, \dots n-r \text{ and } A(X, hY)=A(X, Y). \tag{4.7}$$

If we introduce the tensor field $D=\tilde{S}-S$, where $\tilde{S}$ and S are the Ricci tensor fields of $\tilde{\nabla}$ and ∇ respectively, by considering the trace of the operator $\tilde{R}_{XY}$ in (4.5), then (4.4) and the first Bianchi identity (see [5] in the almost-contact case) allow us to prove that

$$A(X, Y)=-\tilde{L}(X, Y)+L(X, Y). \tag{4.8}$$

where

$$L(X, Y)=\frac{1}{r+2}\,S(X, hY)+\frac{1}{r^2-4}\,[S(hX, Y)+S(hY, X)-S(FX, FY)-S(FY, FX)]+K\sum_a d\eta^a(X, FY). \tag{4.9}$$

where K denotes a constant which will be determined below. Thus we can write the following F-projective not trivial invariant depending on the curvature tensor R

$$\begin{aligned}W'_{XY}Z&=hR_{XY}Z+\{L(X, Y)-L(Y, X)\}\,hZ+L(X, Z)\,hY-L(Y, Z)\,hX-\\&-\{L(X, FY)-L(Y, FX)\}\,FZ-L(X, FZ)\,FY+L(Y, FZ)\,FX.\end{aligned} \tag{4.10}$$

Proposition 4.1. *The tensor field W given by*

$$\begin{aligned}W_{XY}Z&=W'_{XY}Z+\sum_{a,b}\{\eta^a(X)\,\eta^b(Z)\,hY-\eta^a(Y)\,\eta^b(Z)\,hX\}+\\&+\sum_a\left\{\frac{1}{2}d\eta^a(Y, Z)FX-\frac{1}{2}d\eta^a(X, Z)\,FY-d\eta^a(X, Y)\,FZ\right\}\end{aligned} \tag{4.11}$$

is F-projectively invariant. Moreover, if in (4.9) $K=1/(r+2)$, then the following relations hold

trace $(W_{XY})=0$, trace $(FW_{XY})=0$, trace $(X\to W_{XY}Z)=0$, trace $(X\to FW_{XY}Z)=0$

Proof. In fact, to study the F-projective flatness we must find an adapted connection $\tilde{\nabla}$ such that $\tilde{L}=0$. This means (see (4.6) and (4.8)) that we look for a 1-form α satisfying

$$(\nabla_X\alpha)(Y)=L(X, Y)+\alpha(X)\,\alpha(Y)-\beta(X)\,\beta(Y)-\beta(X)\sum_a\eta^a(Y)-\beta(Y)\sum_a\eta^a(X). \tag{4.12}$$

The complete integrability condition of this system of partial differential equations is expressed by

$$\begin{aligned}-\alpha(R_{XY}Z)&=(\nabla_XL)(Y, Z)-(\nabla_YL)(X, Z)+L(Y, FZ)\sum_a\eta^a(X)-\\&-[L(X, FY)-L(Y, FX)]\sum_a\eta^a(Z)-L(X, FZ)\sum_a\eta^a(Y)+[L(X, Y)-\\&-L(Y, X)]\,\alpha(Z)+[L(X, Z)+\sum_{a,b}\eta^a(X)\,\eta^b(Z)]\,\alpha(Y)-[L(Y, Z)+\\&+\sum_{a,b}\eta^a(Y)\,\eta^b(Z)]\,\alpha(X)-[L(X, FY)-L(Y, FX)+\sum_a d\eta^a(X, Y)]\beta(Z)-\\&-\left[L(Y, FZ)+\frac{1}{2}\sum_a d\eta^a(Y, Z)\right]\beta(X)-\left[L(X, FZ)+\frac{1}{2}d\eta^a(X, Z)\right]\beta(Y).\end{aligned} \tag{4.13}$$

From this equation together with (4.5) we obtain a new F-projective invariant tensor field W, given just by the formula (4.11), which is a tensor field of Weyl type if trace (W_{XY}), trace (FW_{XY}), trace $(X\to W_{XY}Z)$, trace $(X\to FW_{XY}Z)$ all vanish. Consequently, after long computations in local coordinates, it results that we must put in (4.9)

$$K=1/(r+2). \tag{4.14}$$

Therefore we take $K=1/(r+2)$ in order to consider W as the F-projective curvature tensor field associated to the torsion-free adapted connection ∇ on M. In what follows, we shall call ∇ *F-projectively flat* if its tensor field W vanishes.

Theorem 4.2. *Let ∇ be a torsion-free adapted connection on an n-dimensional normal framed manifold (M, F, ξ_a, η^a), $a=1, 2, \dots, n-r$. If $r>4$, ∇ is*

F-projectively flat iff is F-projectively related to a torsion-free adapted connection $\tilde{\nabla}$ *whose curvature tensor field satisfies the equation*

$$(4.15)\qquad \begin{aligned} h\tilde{R}_{XY}Z = \sum_a \Big\{ & \{d\eta^a(X, Y)\, FZ - \frac{1}{2} d\eta^a(Y, Z)\, FX + \frac{1}{2} d\eta^a(X, Z)\, FY + \\ & + \sum_b \eta^a(Y)\, \eta^b(Z)\, hX - \sum_b \eta^a(X)\, \eta^b(Z)\, hY \Big\}. \end{aligned}$$

Proof. If we suppose $W=0$, from the equation (4.11) we obtain

$$(4.16)\qquad \begin{aligned} hR_{XY}Z = {} & [L(Y, X) - L(X, Y)]\, hZ - [L(X, Z) + \sum_{a,b} \eta^a(X)\, \eta^b(Z)]\, hY + \\ & + [L(Y, Z) + \sum_{a,b} \eta^a(Y)\, \eta^b(Z)]\, hX + [L(X, FY) - L(Y, FX) + \\ & + \sum_a d\eta^a(X, Y)]FZ - \Big[L(Y, FZ) + \frac{1}{2} \sum_a d\eta^a(Y, Z)\Big]FX + \Big[L(X, FZ) + \\ & + \frac{1}{2} \sum_a d\eta^a(X, Z)\Big]FY, \end{aligned}$$

and, from (4.13) we get that, for $W=0$, the integrability condition becomes

$$(4.17)\qquad \begin{aligned} & Q(X, Y, Z) = (\nabla_X L)\ (Y, Z) - (\nabla_Y L)\ (X, Z) - [L(X, FY) - \\ & - L(Y, FX)]\sum_a \eta^a(Z) + L(Y, FZ) \sum_a \eta^a(X) - L(X, FZ) \sum_a \eta^a(Y) = 0. \end{aligned}$$

Thus, in order to prove the theorem we need to get $Q \equiv 0$ for $r > 4$. By applying the second Bianchi identity from (4.16) and (4.17) we obtain, after a long computation

$$(4.18)\qquad \begin{aligned} & -\{Q(X, Y, U) + Q(Y, U, X) + Q(U, X, Y)\}\, hZ + \{Q(U, X, FY) + \\ & + Q(Y, U, FX) + Q(X, Y, FU)\}\, FZ + Q(X, Y, FZ)\, FU + Q(Y, U, FZ)\, FX + \\ & + Q(U, X, FZ)\, FY - Q(Y, U, Z)\, hX - Q(U, X, Z)\, hY - Q(X, Y, Z)\, hU = 0, \end{aligned}$$

which, taking the trace with respect to Z, yelds

$$(4.19)\qquad \begin{aligned} & -(r+2)\ [Q(X, Y, U) + Q(Y, U, X) + Q(U, X, Y)] + \\ & + 2\sum_a [\eta^a(U) Q(X, Y, \xi_a) + \eta^a(X)\, Q(Y, U, \xi_a) + \eta^a(Y)\, Q(U, X, \xi_a)] = 0. \end{aligned}$$

Next, put $Z = \xi_a$ in (4.18) thus obtaining

$$(4.20)\qquad Q(Y, U, \xi_a)\, hX + Q(U, X, \xi_a) hY + Q(X, Y, \xi_a)\, hU = 0$$

for any ξ_a, $a = 1, 2, \dots n-r$, from which by taking the trace with respect to U and using the skew-symmetry of Q in the first two arguments, we get

$$(4.21)\qquad (r-2)\, Q(X, Y, \xi_a) + \sum_b \eta^b(X)\, Q(\xi_b, Y, \xi_a) + \sum_b \eta^b(Y)\, Q(X, \xi_b, \xi_a) = 0.$$

The relation

$$(4.22)\qquad (r-1)\, Q(X, \xi_c, \xi_a) + \sum_b \eta^b(X)\, Q(\xi_b, \xi_c, \xi_b) = 0$$

is immediate for $Y = \xi_c$ in (4.21). On the other hand, from the formulas (4.9) and (4.17) we get that $Q(\xi_b, \xi_c, \xi_a) = 0$, so that (4.22) implies

$$(4.23)\qquad Q(X, \xi_c, \xi_a) = 0 \qquad \text{for any } a, c = 1, 2, \dots n-r.$$

and (4.21) gives the equation

$$(4.24)\qquad Q(X, Y, \xi_a) = 0 \text{ for } r \neq 2 \text{ and for any } a = 1, 2, \dots n-r.$$

Taking into account this result (4.19) becomes

$$(4.25)\qquad Q(X, Y, U) + Q(Y, U, X) + Q(U, X, Y) = 0.$$

Now if we apply the endomorphism F to the relation (4.18) and take the trace with respect to Z we obtain

$$(4.26)\qquad Q(X, Y, FU) - Q(Y, U, FX) + Q(U, X, FY) = 0.$$

By using (4.26) and (4.25) the relation (4.18) gives

$$(4.27)\qquad \begin{aligned} & Q(Y, U, Z)\, hX + Q(U, X, Z)\, hY + Q(X, Y, Z)\, hU - Q(X, Y, FZ)\, FU - \\ & - Q(Y, U, FZ)\, FX - Q(U, X, FZ)\, FY = 0, \end{aligned}$$

which becomes

$$(4.28)\qquad Q(hX, \xi_a, Z) + rQ(\xi_a, X, Z) - Q(FX, \xi_a, FY) = 0$$

by putting $U = \xi_a$ and then taking the trace with respect to Y. The following two equalities

$$(4.29)\qquad Q(\xi_a, \xi_c, Z) = 0, \qquad (r-1)\, Q(X, \xi_a, Z) = -Q(FX, \xi_a, FZ)$$

follow immediately from (4.28) with $X = \xi_c$. If we substitute FX for X and FZ for Z in the second equation then

$$(4.30)\qquad (r-1)\, Q(FX, \xi_a, FZ) = -Q(X, \xi_a, Z).$$

From (4.29) and (4.30) we easily derive

$$(4.31)\qquad Q(X, \xi_a, Z) = 0 \text{ for } r \neq 2 \text{ and any } a = 1, 2, \dots n-r.$$

Thus we see from (4.31) and (4.24) that Q vanishes if one of its arguments is ξ_a, for any $a = 1, 2, \dots n-r$. On the other hand, if we again consider the relation (4.27) taking the trace with respect to U then we have

$$(4.32)\qquad (r-2)\, Q(X, Y, Z) + Q(FX, Y, FZ) + Q(X, FY, FZ) = 0.$$

Adding (4.32) to the relation obtained after substituting FX for X and FY for Y in the same equation we have

$$(4.33)\qquad Q(X, Y, Z) + Q(FX, FY, Z) = 0 \text{ for } r \neq 2.$$

By similar computation, we get from the formula (4.26),

$$Q(X, FY, FZ)+Q(FX, Y, FZ)=Q(Y, Z, X)+Q(Z, X, Y)-\\-Q(Z, FX, FY)-Q(FY, Z, FX) \tag{4.34}$$

which, by taking into account (4.33) and (4.25) becomes

$$Q(X, FY, FZ)+Q(FX, Y, FZ)=-2Q(X, Y, Z). \tag{4.35}$$

Finally, (4.35) and (4.32) imply

$$Q(X, Y, Z)=0 \qquad \text{for } r\geq 4 \tag{4.36}$$

which proves the first part of the theorem. The converse follows from a straightforward computation.

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ON EQUIVALENCE OF VARIATIONAL PROBLEMS AND ITS GEOMETRICAL BACKGROUND IN LAGRANGE SPACES

BY

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1. Introduction. In the terminology of Carathéodory, two identical type problems $I^*=\int_{t_1}^{t_2}\mathcal{L}^* dt=\text{extr}\,!$ and $I=\int_{t_1}^{t_2}\mathcal{L}dt=\text{extr}\,!$ in the classical calculus of variations are said to be equivalent if their extremels are the same. If their respective Lagrangians $\mathcal{L}^*(x, y)$ and $\mathcal{L}(x, y)$ differ by a total derivative i.e. $\mathcal{L}^*(x, y)=\mathcal{L}(x, y)+dS(x)/dt$ then the variational problems are evidently equivalent in the sense of Carathéodory. In this case the Euler-Lagrange equations are identical for both problems

$$\mathcal{E}_i(\mathcal{L}^*)\equiv\mathcal{E}_i(\mathcal{L}),\ i=1, 2, \ldots, n \tag{1.1}$$

where $\mathcal{E}_i$ is the following operator

$$\mathcal{E}_i=\frac{d}{dt}\frac{\partial}{\partial y^i}-\frac{\partial}{\partial x^i}. \tag{1.2}$$

A r t h u r M o ó r [2] has introduced *a kind of the conversion of the equivalent problem* in Finsler spaces. He considered the following relation

$$\mathcal{E}^i(\mathcal{L}^*(x, \dot x))\equiv\lambda(x)\ \mathcal{E}_i(\mathcal{L}(x, \dot x))\ ;\ \lambda(x)\neq 0\ ;\ \dot x\ :=dx/dt\,. \tag{1.3}$$

and obtained a *new problem of equivalence*.

In this paper we give a *generalization of Moór's problem in Lagrange spaces*. Further we prove the Theorems 4.1 and 4.2 which characterize the equivalence of two variational problems in Lagrange spaces. Namely: the variational problem in Lagrange space $L^n=(M, \mathcal{L})$ leads us to the equations (2.6) of geodesics. Then two variational problems in $L^{*n}=(M, \mathcal{L}^*)$ and $L^n=(M, \mathcal{L})$ are equivalent if the relation (3.1) holds identically. This condition is equivalent to the preservation of geodesics of L^{*n} and L^n and to the conformal property (3.2) (a) of L^{*n} and L^n. The most important result is given by the Theorem 4.2: "Two variational problems in L^{*n} and L^n are equivalent in the sense of Moór if and only if the transformation (4.8) holds."

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2. The variational problem in Lagrange space. Let us consider a Lagrange space $L^n=(M, \mathcal{L})$ where M is an n-dim. manifold, $\mathcal{L}(x, y)$ is a real