$$Q(X, FY, FZ)+Q(FX, Y, FZ)=Q(Y, Z, X)+Q(Z, X, Y)-\\ -Q(Z, FX, FY)-Q(FY, Z, FX) \tag{4.34}$$

which, by taking into account (4.33) and (4.25) becomes

$$Q(X, FY, FZ)+Q(FX, Y, FZ)=-2Q(X, Y, Z). \tag{4.35}$$

Finally, (4.35) and (4.32) imply

$$Q(X, Y, Z)=0 \qquad \text{for } r>4 \tag{4.36}$$

which proves the first part of the theorem. The converse follows from a straightforward computation.

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Dipartimento di Matematica
Università di Cagliari
09100 – Cagliari, Italia

An. şt. Univ. „Al. I. Cuza" Iaşi
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ON EQUIVALENCE OF VARIATIONAL PROBLEMS AND ITS GEOMETRICAL BACKGROUND IN LAGRANGE SPACES

BY

MAGDALEN SZ. KIRKOVITS

1. Introduction. In the terminology of Carathéodory, two identical type problems $I^* = \int_{t_1}^{t_2} \mathcal{L}^* dt = \text{extr}\,!$ and $I = \int_{t_1}^{t_2} \mathcal{L} dt = \text{extr}\,!$ in the classical calculus of variations are said to be equivalent if their extremals are the same. If their respective Lagrangians $\mathcal{L}^*(x, y)$ and $\mathcal{L}(x, y)$ differ by a total derivative i.e. $\mathcal{L}^*(x, y) = \mathcal{L}(x, y) + dS(x)/dt$ then the variational problems are evidently equivalent in the sense of Carathéodory. In this case the Euler-Lagrange equations are identical for both problems

$$\mathcal{E}_i(\mathcal{L}^*) \equiv \mathcal{E}_i(\mathcal{L}), \quad i=1, 2, \ldots, n \tag{1.1}$$

where $\mathcal{E}_i$ is the following operator

$$\mathcal{E}_i = \frac{d}{dt}\frac{\partial}{\partial y^i} - \frac{\partial}{\partial x^i}. \tag{1.2}$$

A r t h u r M o ó r [2] has introduced *a kind of the conversion of the equivalent problem* in Finsler spaces. He considered the following relation

$$\mathcal{E}^i(\mathcal{L}^*(x, \dot{x})) \equiv \lambda(x)\ \mathcal{E}_i(\mathcal{L}(x, \dot{x}))\ ;\ \lambda(x) \neq 0\ ;\ \dot{x} : = dx/dt, \tag{1.3}$$

and obtained a *new problem of equivalence*.

In this paper we give a *generalization of Moór's problem in Lagrange spaces*. Further we prove the Theorems 4.1 and 4.2 which characterize the equivalence of two variational problems in Lagrange spaces. Namely: the variational problem in Lagrange space $L^n = (M, \mathcal{L})$ leads us to the equations (2.6) of geodesics. Then two variational problems in $L^{*n} = (M, \mathcal{L}^*)$ and $L^n = (M, \mathcal{L})$ are equivalent if the relation (3.1) holds identically. This condition is equivalent to the preservation of geodesics of L^{*n} and L^n and to the conformal property (3.2) (a) of L^{*n} and L^n. The most important result is given by the Theorem 4.2: "Two variational problems in L^{*n} and L^n are equivalent in the sense of Moór if and only if the transformation (4.8) holds."

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2. The variational problem in Lagrange space. Let us consider a Lagrange space $L^n = (M, \mathcal{L})$ where M is an n-dim. manifold, $\mathcal{L}(x, y)$ is a real differentiable function on the tangent bundle TM with the non-degenerate d-tensor field $g_{ij}=1/2\,\partial^2 L/\partial y^i\partial y^j$ [3].

Our variational problem is as follows [4]: Let $C: [0,1]\to M$ represent a curve on M between two fixed points $A(0)$ and $B(1)$. Consider the following variation of C

(2.1) $$t\to x^i(t)+\varepsilon v^i(t).$$

where ε is an infinitesimal and $v^i(0)=v^i(1)$. Hence we have

(2.2) $$I(\varepsilon)=\int_0^1 \mathcal{L}(x^i+\varepsilon v^i,\ dx^i/dt+\varepsilon\, dv^i/dt)\,dt.$$

The first necessary condition which C must satisfy in order that it affords an extreme value of the functional (2.2) is the following

(2.3) $$dI(\varepsilon)/d\varepsilon|_{\varepsilon=0}=0.$$

Since v^i is arbitrary we can express our condition in the Euler-Lagrange equations

(2.4) $$\varepsilon_i(\mathcal{L})\equiv\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial y^i}-\frac{\partial\mathcal{L}}{\partial x^i}=0;\qquad y=\frac{dx^i}{dt}.$$

After some calculation we get

(2.5) $$2g_{ij}\frac{d^2x^j}{dt^2}+\left(\frac{\partial^2\mathcal{L}}{\partial y^i\partial x^j}y^j-\frac{\partial\mathcal{L}}{\partial x^i}\right)=0.$$

Since det $(g_{ij})\neq 0$ we obtain

(2.6) $$d^2x^i/dt^2+2G^i(x,\ dx/dt)=0$$

where

(2.6) (a) $$G^i=\frac{1}{4}g^{ij}\left(\frac{\partial^2\mathcal{L}}{\partial y^j\partial x^k}y^k-\frac{\partial\mathcal{L}}{\partial x_j}\right).$$

(See [4]). The solutions of (2.6) are called *geodesics* of L^n.

By the help of G^i we can determine a nonlinear connection on TM [3]. Its coefficients are as follows

(2.6) (b) $$N^i_j=\partial G^i/\partial y^j.$$

These coefficients are determined only by the Lagrangian of L^n. In L^n we have also a canonical metrical connection $C\Gamma=(N^i_j,\ L^i_{jk},\ C^i_{jk})$ given by (2.6) (b) and by

(2.7) (a) $$L^i_{jk}=1/2\,g^{ih}\,(\delta g_{hk}/\delta x^j+\delta g_{jh}/\delta x^k-\delta g_{jk}/\delta x^h).$$

(2.7) (b) $$C^i_{jk}=1/2\,g^{ih}\,(\partial g_{hk}/\partial y^j+\partial g_{jh}/\partial y^k-\partial g_{jk}/\partial y^h).$$

where the operators $\delta/\delta x^k$ are determined by

(2.7.) (c) $$\delta/\delta x^k=\partial/\partial x^k-N^h_k\,\partial/\partial y^h.$$

It is clear that $C\Gamma$ has the following properties

(2.8) $$g_{ij|k}=0,\quad g_{ij}|_k=0.$$

The torsions of the connection $C\Gamma$ are

(2.9) (a) $T^i_{jk}=0$, (b) $S^i_{jk}=0$, (c) C^i_{jk},

(d) $R^i_{jk}=\delta_j N^i_k-\delta_k N^i_j$, (e) $P^i_{jk}=\partial N^i_j/\partial y^k-L^i_{jk}$ $(\delta_j=\delta/\delta x^j)$.

In the space L^n we have the tensor fields

(2.10) (a) $C_{ijk}=g_{jh}\,C^h_{ik}$, (b) $P_{ijk}=g_{ih}P^h_{jk}$

which are totally symmetric [3].

3. Geometrical background of the equivalent problem. Definition. *Two variational problems in* $L^{*n}=(M,\ \mathcal{L}^*)$ *and* $L^n=(M,\ \mathcal{L})$ (with the same base manifold) *are equivalent in the sense of Moór if the relations*

(3.1) $$\varepsilon_i(\mathcal{L}^*(x,\ y))=\lambda(x,\ y)\ \varepsilon_i(\mathcal{L}(x,\ y)),\quad \lambda(x,\ y)\neq 0$$

hold identically. We note that the proportionality factor λ depends not only on x^i as discussed by Moór in F_n, but on y^i too. Using this definition we have

Proposition 3.1. *The condition* (3.1) *is equivalent to the following two geometrical ones*

1. *The Lagrange spaces* L^{*n} *and* L^n *are in conformal correspondence*

(3.2) (a) $$g^*_{ij}(x,\ y)=\lambda(x,\ y)\ g_{ij}(x,\ y).$$

2. *The spaces* L^{*n} *and* L^n *are in projective correspondence.*

The proof is just a calculation.

Proposition 3.2. *The spaces* L^{*n} *and* L^n. *are in projective correspondence iff*

(3.2) (b) $$G^{*i}(x,\ y)=G^i(x,\ y).$$

Indeed, L^{*n} *and* L^n *have the same geodesics, after* (2.6) *iff* (3.2) (b) *holds.*

Proposition 3.3. *If the spaces* L^{*n} *and* L^n *have the following properties*

(3.3) (a) $g^*_{ij}(x,\ y)=\lambda(x,\ y)\ g_{ij}(x,\ y)$, (b) $G^{*i}(x,\ y)=G(x,\ y)$.

then we have

(3.4) (a) $$N^{*i}_j(x,\ y)=N^i_j(x,\ y).$$

(b) $$L^{*i}_{jk}(x,\ y)=L^i_{jk}(x,\ y)+(2\lambda(x,\ y))^{-1}\,(\delta^i_k\lambda_{|j}+\delta^i_j\lambda_{|k}-g_{jk}g^{ir}\lambda_{|r}).$$

(c) $$C^{*i}_{jk}(x,\ y)=C^i_{jk}(x,\ y)+(2\lambda(x,\ y))^{-1}\,(\delta^i_j\lambda|_k+\delta^i_k\lambda|_j-g_{jk}g^{ih}\lambda|_h).$$

where the "$_{|k}$" *and* "$|_k$" *denote the h-covariant and v-covariant derivatives with respect to* $C\Gamma$.

(d) $$P^{*i}_{jk}(x,\ y)=P^i_{jk}(x,\ y)-(2\lambda(x,\ y))^{-1}\,(\delta^i_k\lambda|_j+\delta^i_j\lambda|_k-g_{jk}g^{ir}\lambda|_r).$$

(e) $$R^{*i}_{jk}(x,\ y)=R^i_{jk}(x,\ y).$$

Proof. 1. Using (2.6) (b) and (3.3) (b) we directly get

(3.5) (a) $$N^{*i}_j = \partial G^{*i}/\partial y^j = \partial G^i/\partial y^j = N^i_j.$$

2. On account of (2.7) (a) and (3.3) (a) we obtain

(3.5) (b)
$$L^{*i}_{jk} = \frac{1}{2} g^{*ih}\left(\frac{\delta g^*_{hk}}{\delta x^j} + \frac{\delta g^*_{jh}}{\delta x^k} - \frac{\delta g^*_{jk}}{\delta x^h}\right) = \frac{1}{2\lambda(x, y)} g^{ih}\left(\frac{\delta(\lambda g_{hk})}{\delta x^j} + \frac{\delta(\lambda g_{jh})}{\delta x^k} - \frac{\delta(\lambda g_{jk})}{\delta x^h}\right) = L^i_{jk} + (2\lambda)^{-1}(\delta^i_k \lambda_{,j} + \delta^i_j \lambda_{,k} - g_{jk} g^{ih} \lambda_{,h}).$$

3. Consider (2.7) (b) and (3.3) (a). We get

(3.5) (c)
$$C^{*i}_{jk} = \frac{1}{2} g^{*ih}\left(\frac{\partial g^*_{hk}}{\partial y^j} + \frac{\partial g^*_{jh}}{\partial y^k} - \frac{\partial g^*_{jk}}{\partial y^h}\right) = \frac{1}{2\lambda(x, y)} g^{ih}\left(\frac{\partial(\lambda g_{hk})}{\partial y^j} + \frac{\partial(\lambda g_{jh})}{\partial y^k} - \frac{\partial(\lambda g_{jk})}{\partial y^h}\right) = C^i_{jk} + (2\lambda)^{-1}(\delta^i_j \lambda|_k + \delta^i_k \lambda|_j - g_{jk} g^{ir} \lambda|_r).$$

4. Taking into account (2.9) (c), (3.4) (a) and (3.4) (b) we can calculate

(3.5) (d)
$$P^{*i}_{jk} = \frac{\partial N^{*i}_j}{\partial y^k} - L^{*i}_{jk} = \frac{\partial N^i_j}{\partial y^k} - L^i_{jk} - \frac{1}{2\lambda(x, y)}(\delta^i_k \lambda_{,j} + \delta^i_j \lambda_{,k} - g_{jk} g^{ir} \lambda_{,r}) = P^i_{jk} - (2\lambda)^{-1}(\delta^i_k \lambda_{,j} + \delta^i_j \lambda_{,k} - g_{jk} g^{ir} \lambda_{,r}).$$

5. Finally let us consider (2.9) (d) and (3.4) (a). Hence we have

(3.5) (e) $$R^{*i}_{jk} = \delta_j N^{*i}_k - \delta_k N^{*i}_j = \delta_j N^i_k - \delta_k N^i_j = R^i_{jk}.$$ Q.E.D.

4. Transformation of the Lagrangians $\mathcal{L}^*$ and $\mathcal{L}$. Theorem 4.1. *If the relation (3.1) holds then the function $\lambda(x, y)$ is necessarily constant.*

Proof. From (3.4) (c) it follows

(4.1) $$C^*_{ijk} = \lambda(x, y)\, C_{ijk} + 1/2\,(g_{ij}\lambda|_k + g_{ik}\lambda|_j - g_{jk}\lambda|_i).$$

Because C^*_{ijk} and C_{ijk} are totally symmetric the last equality implies

(4.2) $$g_{ik}\lambda|_j - g_{jk}\lambda|_i = 0.$$

Contracting (4.2) by g^{ik} we find

(4.3) $$(n-1)\,\lambda|_j = 0.$$

Hence

(4.4) $$\lambda|_j = \partial\lambda/\partial y^j = 0.$$

Thus λ is independent of y^j.

Further on account of (3.4) (d) and (2.10) (b) we get

(4.5) $$P^*_{ijk} = \lambda(x)\, P_{ijk} - 1/2\,(g_{ij}\lambda_{,k} + g_{ik}\lambda_{,j} - g_{jk}\lambda_{,i}).$$

Since P^*_{ijk} and P_{ijk} are totally symmetric in i, j and k, the second term of the right side must be also totally symmetric. Thus we obtain

(4.6) $$g_{ik}\lambda_{,j} - g_{jk}\lambda_{,i} = 0.$$

Contracting (4.6) by g^{ik} we find

(4.6) (a) $$(n-1)\,\lambda_{,j} = 0.$$

Then

(4.7) $$\lambda_{,j} = \delta\lambda/\delta x^j = \partial\lambda/\partial x^j = 0.$$

Thus $\lambda = \mathrm{const}$. Q.E.D.

Theorem 4.2. *Two variational problems in $L^{*n} = (M, \mathcal{L}^*)$ and $L^n = (M, \mathcal{L})$ are equivalent in the sense of Moór if and only if between the Lagrangians $\mathcal{L}^*$ and $\mathcal{L}$ we have the transformation*

(4.8) $$\mathcal{L}^*(x, y) = \lambda\mathcal{L}(x, y) + \frac{\partial\varphi(x)}{\partial x^i} y^i + c,$$

where $\lambda = \mathrm{const.} \neq 0$; $c := \mathrm{const.}$, $y^i = dx^i/dt$.

Proof. A. If the variational problems in L^{*n} and L^n are equivalent by Moór, then the previous Theorem implies that $\lambda = \mathrm{const.}$ in (3.1). Applying the Proposition 3.1 we get $g^*_{ij}(x, y) = \lambda g_{ij}(x, y)$. In view of property of Lagrangians we obtain

(4.9) $$\partial^2(\mathcal{L}^* - \lambda\mathcal{L})/\partial y^k \partial y^l = 0.$$

Hence the function $\mathcal{L}^* - \lambda\mathcal{L}$ is necessarily linear in y^i

(4.10) $$\mathcal{L}^*(x, y) - \lambda\mathcal{L}(x, y) = A_i(x)\, y^i + U(x).$$

i.e.

(4.10) (a) $$\mathcal{L}^*(x, y) = \lambda\mathcal{L}(x, y) + A_i(x)\, y^i + U(x).$$

Moreover let us consider the condition (3.2) (b). By its meaning

(4.11) $$\frac{1}{4} g^{*ih}\left(\frac{\partial^2\mathcal{L}^*}{\partial y^h \partial x^k} y^k - \frac{\partial\mathcal{L}^*}{\partial x^h}\right) = \frac{1}{4} g^{ih}\left(\frac{\partial^2\mathcal{L}}{\partial y^h \partial x^k} y^k - \frac{\partial\mathcal{L}}{\partial x^h}\right).$$

Substituting (4.10) (a) in (4.11) and using (3.2) (a) we obtain

(4.12) $$(\partial A_h(x)/\partial x^k - \partial A_k(x)/\partial x^h)\, y^k - \partial U(x)/\partial x^h = 0.$$

Let us differentiate (4.12) with respect to y^k, then

(4.13) $$\partial A_h/\partial x^k - \partial A_k/\partial x^h = 0.$$

This means that

(4.14) (a) $$A_i(x) = \partial\varphi(x)/\partial x^i.$$

Thus A_i is a gradient. Substituting $\partial\varphi/\partial x^i$ in (4.12) we get

(4.14) (b) $$\partial U(x)/\partial x^h = 0.$$

Thus $U = \mathrm{const}$.

B. If the transformation (4.8) holds then, by direct calculation it results (3.1) with λ=const. $\neq 0$. Q.E.D.

Remarks. 1. In the proofs of the Theorems 4.1 and 4.2 we needed only geometrical methods, using the theory of Lagrange spaces from the excellent book of R. Miron and M. Anastasiei [4].

2. If $\lambda=1$ and L^n is a Finsler space the transformation (4.1) was discovered by Carathéodory.

3. If L^n is a Finsler space and λ is different to one then we obtain the Moór transformation.

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University of Sopron
Faculty of Forestry
Department of Mathematics
9401-Sopron, Pf. 132, Hungary

An. şt. Univ. „Al. I. Cuza" Iaşi
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ON THE CONSERVATION LAWS OF THE EINSTEIN EQUATIONS ON LAGRANGE SPACES

BY

ORIELLA AMICI and BIAGIO CASCIARO

Introduction. Recently in [5] a generalization of the theory of general relativity is introduced by R. Miron. This generalization uses a suitable pseudo-riemannian metric G of the tangent bundle TM of a manifold M. Generally, conservation laws are not verified by such theories. Hence it can be interesting to find equivalent forms of these laws and to give examples of pseudo-riemannian metrics of TM which verify them.

In this paper, by using the Bianchi's identities, we find such an equivalent form (cf. Theorem 1). Moreover, we test its usefulness by five examples. Two of them are completely new (cf. Theorems 9 and 11), while in Theorem 7 and Theorems 3 and 5 we find, in a simple way, the examples given in [7] and [1], respectively. We prevalently use definitions and properties of [7], then, for the sake of simplicity, we list them without giving the reference source.

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1. Preliminaries and conservation laws. Let M be an n-dimensional manifold[1], $\Pi : TM \to M$ its tangent bundle and $\Pi_T : T(TM) \to TM$ the tangent bundle of TM. We denote by (U, x^i) $(i, j, k, l, r, s, \ldots = 1, \ldots, n)$ a local chart on M and by $(\Pi^{-1}(U), x^i, y^a)$ $(a, b, c, d, e, f, \ldots = 1, \ldots, n)$ the local chart canonically induced by (U, x^i) on TM. Let N be a non-linear connection of TM, i.e. N is an n-dimensional distribution of TM such that $T_X(TM) = N_X \oplus V_X$ (direct sum), for each $X \in TM$, where V_X is the vertical subspace of $T_X(TM)$. Locally, N is determined by n^2 functions and we denote them by $N_j^a(x, y)$.

Moreover we put $\delta/\delta x^i = \partial/\partial x^i - N_i^a \partial/\partial y^a$.

Then $(\delta/\delta x^i, \partial/\partial y^a)$ is a local basis of the Lie algebra of the vector fields on TM, usually called adapted frame. We also denote by $(dx^i, \delta y^a)$ its dual basis, where $\delta y^a = dy^a + N_j^a dx^j$.

Now we fix a *d-connection (distinguished connection)* ∇ on TM. That means ∇ is a linear connection of TM such that

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[1] We work in the category of C^∞ differentiable manifolds.