B. If the transformation (4.8) holds then, by direct calculation it results (3.1) with λ=const. $\neq 0$. Q.E.D.

Remarks. 1. In the proofs of the Theorems 4.1 and 4.2 we needed only geometrical methods, using the theory of Lagrange spaces from the excellent book of R. Miron and M. Anastasiei [4].

2. If $\lambda=1$ and L^n is a Finsler space the transformation (4.1) was discovered by Carathéodory.

3. If L^n is a Finsler space and λ is different to one then we obtain the Moór transformation.

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University of Sopron
Faculty of Forestry
Department of Mathematics
9401-Sopron, Pf. 132, Hungary

An. șt. Univ. „Al. I. Cuza" Iași
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ON THE CONSERVATION LAWS OF THE EINSTEIN EQUATIONS ON LAGRANGE SPACES

BY

ORIELLA AMICI and BIAGIO CASCIARO

Introduction. Recently in [5] a generalization of the theory of general relativity is introduced by R. Miron. This generalization uses a suitable pseudo-riemannian metric G of the tangent bundle TM of a manifold M. Generally, conservation laws are not verified by such theories. Hence it can be interesting to find equivalent forms of these laws and to give examples of pseudo-riemannian metrics of TM which verify them.

In this paper, by using the Bianchi's identities, we find such an equivalent form (cf. Theorem 1). Moreover, we test its usefulness by five examples. Two of them are completely new (cf. Theorems 9 and 11), while in Theorem 7 and Theorems 3 and 5 we find, in a simple way, the examples given in [7] and [1], respectively. We prevalently use definitions and properties of [7], then, for the sake of simplicity, we list them without giving the reference source.

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1. Preliminaries and conservation laws. Let M be an n-dimensional manifold[1], $\Pi : TM \to M$ its tangent bundle and $\Pi_* : T(TM) \to TM$ the tangent bundle of TM. We denote by (U, x^i) $(i, j, k, l, r, s, \ldots = 1, \ldots, n)$ a local chart on M and by $(\Pi^{-1}(U), x^i, y^a)$ $(a, b, c, d, e, f, \ldots = 1, \ldots, n)$ the local chart canonically induced by (U, x^i) on TM. Let N be a non-linear connection of TM, i.e. N is an n-dimensional distribution of TM such that $T_X(TM) = N_X \oplus V_X$ (direct sum), for each $X \in TM$, where V_X is the vertical subspace of $T_X(TM)$. Locally, N is determined by n^2 functions and we denote them by $N^a_j(x, y)$.

Moreover we put $\delta/\delta x^i = \partial/\partial x^i - N^a_i\, \partial/\partial y^a$.

Then $(\delta/\delta x^i, \partial/\partial y^a)$ is a local basis of the Lie algebra of the vector fields on TM, usually called adapted frame. We also denote by $(dx^i, \delta y^a)$ its dual basis, where $\delta y^a = dy^a + N^a_j dx^j$.

Now we fix a *d-connection (distinguished connection)* ∇ on TM. That means ∇ is a linear connection of TM such that

* Partially supported by M.P.I. 60% Italy.

[1] We work in the category of C^∞ differentiable manifolds.

$$(1)\qquad \begin{aligned} &\nabla_{\delta/\delta x^k}\delta/\delta x^j = L^i_{jk}(x,y)\,\delta/\delta x^i, \quad \nabla_{\partial/\partial y^c}\delta/\delta x^j = \bar{C}^i_{jc}(x,y)\,\delta/\delta x^i, \\ &\nabla_{\delta/\delta x^k}\partial/\partial y^b = \bar{L}^a_{bk}(x,y)\,\partial/\partial y^a, \quad \nabla_{\partial/\partial y^c}\partial/\partial y^b = C^a_{bc}(x,y)\,\partial/\partial y^a. \end{aligned}$$

Moreover we denote by $|$ and $|$ the so-called *h-* and *v-covariant derivative associated to* ∇, respectively. The torsion tensor field T of ∇ generally can not vanish and its components with respect to an adapted frame are

$$(2)\qquad \begin{aligned} &T^i{}_{jk} = L^i_{jk} - L^i_{kj}\,; \quad R^a{}_{jk} = \delta N^a_j/\delta x^k - \delta N^a_k/\delta x^j\,; \\ &\overset{1}{P}{}^a{}_{jb} = \partial N^a_j/\partial y^b - \bar{L}^a_{bj}\,; \quad \overset{2}{P}{}^i{}_{jb} = \bar{C}^i_{jb}\,; \quad S^a_{bc} = C^a_{bc} - C^a_{cb} \end{aligned}$$

In the sequel we suppose that $T^i{}_{jk} = S^a{}_{bc} = 0$, i.e. ∇ has zero *h(hh)-torsion* and *v(vv)-torsion tensor fields* (cf. [2]).

Moreover we recall that $R^a{}_{jk}$ vanishes if and only if N is an integrable distribution.

Let R be the curvature tensor field of ∇. In the sequel we use the following components of R, only

$$(3)\qquad \begin{aligned} &R_j{}^i{}_{kl} = \delta L^i_{jk}/\delta x^l - \delta L^i_{jl}/\delta x^k + L^h_{jk}L^i_{hl} - L^h_{jl}L^i_{hk} + \bar{C}^i_{je}R^e{}_{kl} \\ &\overset{2}{P}{}_j{}^i{}_{kc} = \partial L^i_{jk}/\partial y^c - \delta\bar{C}^i_{jc}/\delta x^k + L^h_{jk}\bar{C}^i_{hc} - L^i_{hk}\bar{C}^h_{jc} + \partial N^d_k/\partial y^c\,\bar{C}^i_{jd}, \\ &\overset{1}{P}{}_b{}^a{}_{kc} = \partial\bar{L}^a_{bk}/\partial y^c - \delta C^a_{bc}/\delta x^k + \bar{L}^d_{bk}C^a_{dc} - \bar{L}^a_{dk}C^d_{bc} + \partial N^d_k/\partial y^c\,C^a_{bd}, \\ &S_b{}^a{}_{cd} = \partial C^a_{bc}/\partial y^d - \partial C^a_{bd}/\partial y^c + C^f_{bc}C^a_{fd} - C^f_{bd}C^a_{fc}. \end{aligned}$$

with respect to an adapted frame.

We also denote by Ric the Ricci tensor field of ∇ whose components are

$$(4)\qquad R_{ij} = R_i{}^h{}_{jh}, \quad \overset{1}{P}_{bj} = \overset{1}{P}{}_b{}^a{}_{ja}, \quad -\overset{2}{P}_{jb} = -\overset{2}{P}{}_j{}^h{}_{hb}, \quad S_{ab} = S_a{}^c{}_{bc}.$$

We shall use the following identities verified by the above components of R

$$(5)\qquad R_j{}^i{}_{kl} = -R_j{}^i{}_{lk}, \quad S_b{}^a{}_{cd} = -S_b{}^a{}_{dc}$$

$$(6)\qquad R_j{}^i{}_{kl|s} + R_j{}^i{}_{sk|l} + R_j{}^i{}_{ls|k} = -\overset{2}{P}{}_j{}^i{}_{kc}R^c{}_{sl} - \overset{2}{P}{}_j{}^i{}_{sc}R^c{}_{lk} - \overset{2}{P}{}_j{}^i{}_{lc}R^c{}_{ks}$$

$$(7)\qquad S_b{}^a{}_{cd}|_f + S_b{}^a{}_{fc}|_d + S_b{}^a{}_{df}|_c = 0.$$

Now we suppose that TM is endowed with an *(h, v)-metric* G, i.e. G is a pseudo-riemannian metric such that N is orthogonal to the vertical distribution. This locally implies

$$(8)\qquad G = g_{ij}(x,y)\,dx^i\otimes dx^j + h_{ab}(x,y)\,\delta y^a\otimes\delta y^b.$$

Moreover

$$G^H = g_{ij}(x,y)\,dx^i\otimes dx^j \text{ and } G^V = h_{ab}(x,y)\,\delta y^a\otimes\delta y^b$$

are called *h-metric* and *v-metric* of G, respectively.

We also suppose that ∇ is compatible with G. That means $\nabla G = 0$ or equivalently

$$(9)\qquad g_{ij|k} = 0, \quad g_{ij}|_c = 0, \quad h_{ab|k} = 0, \quad h_{ab}|_c = 0$$

Then the components of ∇, with respect to an adapted frame are:

$$(10)\qquad \begin{aligned} &2L^i_{jk} = g^{ih}\left\{\frac{\delta g_{hj}}{\delta x^k} + \frac{\delta g_{hk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h}\right\}, \qquad 2\bar{C}^i_{jc} = g^{ih}\frac{\partial g_{hj}}{\partial y^c}, \\ &2\bar{L}^a_{bk} = \frac{\partial N^a_k}{\partial y^b} + h^{ac}\left\{\frac{\delta h_{bc}}{\delta x^k} - \frac{\partial N^d_k}{\partial y^c}h_{ab}\right\}, \quad 2C^a_{bc} = h^{ad}\left\{\frac{\partial h_{db}}{\partial y^c} + \frac{\partial h_{dc}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d}\right\} \end{aligned}$$

Moreover we put

$$(11)\qquad \begin{aligned} &R_{ijkl} = g_{jh}R_i{}^h{}_{kl}, \quad \overset{2}{P}_{ijka} = g_{jh}\overset{2}{P}{}_i{}^h{}_{ka}, \\ &\overset{1}{P}_{abkd} = h_{bf}\overset{1}{P}{}_a{}^f{}_{kd}, \quad S_{abcd} = h_{bf}S_a{}^f{}_{cd}. \end{aligned}$$

Then it results

$$(12)\qquad \begin{aligned} &R_{ijkl} + R_{jikl} = 0, \quad \overset{2}{P}_{ijka} + \overset{2}{P}_{jika} = 0, \\ &\overset{1}{P}_{abkd} + \overset{1}{P}_{bakd} = 0, \quad S_{abcd} + S_{bacd} = 0 \end{aligned}$$

Hence we have

$$(13)\qquad R_{jk} = -R_{ijks}g^{is}, \quad S_{ab} = -S_{cabd}h^{cd}, \quad \overset{2}{P}_{ic} = -\overset{2}{P}_{jikc}g^{jk}.$$

Moreover the scalar curvature $\tilde{\mathcal{R}}$ of ∇ is the sum of the following two functions

$$(14)\qquad R = R_{ij}g^{ij}, \qquad S = S_{ab}h^{ab}.$$

Furthermore, with these notations, the Einstein equation of ∇

$$(15)\qquad \text{Ric} - 1/2\,\tilde{\mathcal{R}}G = k\tilde{T}$$

can be written as

$$(16)\qquad \begin{aligned} &R_{ij} - 1/2\,(R+S)g_{ij} = k\overset{1}{T}_{ij}, \quad \overset{1}{P}_{bj} = k\overset{2}{T}_{bj}, \\ &S_{ab} - 1/2\,(S+R)h_{ab} = k\overset{4}{T}_{ab}, \quad \overset{2}{P}_{jb} = -k\overset{3}{T}_{jb}, \end{aligned}$$

where k is a real number and $\overset{1}{T}_{ij}$, $\overset{2}{T}_{bj}$, $\overset{3}{T}_{jb}$ and $\overset{4}{T}_{ab}$ are the components, with respect to an adapted coframe of the tensor field $\tilde{T}$ on TM.

Finally the conservation laws are given by

$$(17)\qquad \left(R^i{}_j - \frac{1}{2}(R+S)\delta^i_j\right)_{|i} + \overset{1}{P}{}^a{}_j|_a = 0, \quad \left(S^a{}_b - \frac{1}{2}(R+S)\delta^a_b\right)\Big|_a - \overset{2}{P}{}^i{}_{b|i} = 0,$$

where $R^i{}_j = g^{ih}R_{hj}$, $S^a{}_b = h^{ac}S_{cb}$, $\overset{1}{P}{}^a{}_j = h^{ac}\overset{1}{P}_{cj}$, $\overset{2}{P}{}^i{}_b = g^{ij}\overset{2}{P}_{jb}$ and δ^i_j, δ^a_b are the Kronecker symbols.

Under the above assumptions we have

Theorem 1. *The conservation laws* (17) *are equivalent to*

$$\text{(18)}\qquad \frac{\delta S}{\delta x^j} = S_{|j} - 2\overset{1}{P}{}^a{}_j|_a + 2\overset{2}{P}{}^i{}_c R^c{}_{ij} + \overset{2}{P}{}_s{}^i{}_{jc} R^c{}_{ki} g^{sk}.$$
$$\frac{\partial R}{\partial y^a} = R|_a = -2\overset{2}{P}{}^i{}_{a|i}.$$

Proof. Contracting i with l in (6) and taking into account (4) it follows

$$R_{jk|s} + R_j{}^i{}_{sk|i} + R_j{}^i{}_{is|k} = -\overset{2}{P}{}_j{}^i{}_{kc} R^c{}_{si} - \overset{2}{P}{}_j{}^i{}_{sc} R^c{}_{ik} - \overset{2}{P}{}_{jc} R^c{}_{ks}.$$

By using (5) the above identity becomes

$$\text{(19)}\qquad R_{jk|s} + R_j{}^i{}_{sk|i} - R_{js|k} = \overset{2}{P}{}_j{}^i{}_{kc} R^c{}_{is} + \overset{2}{P}{}_j{}^i{}_{sc} R^c{}_{ki} + \overset{2}{P}{}_{jc} R^c{}_{sk}$$

being $R^a{}_{jk} = -R^a{}_{kj}$ too. Moreover, contracting both sides of (19) with g^{jk} and using (9) and (14) we obtain

$$R_{|s} + (R_j{}^i{}_{sk}\, g^{jk})_{|i} - R_{js|k}\, g^{jk} = \overset{2}{P}{}_j{}^i{}_{kc} R^c{}_{is}\, g^{jk} + \overset{2}{P}{}_j{}^i{}_{sc} R^c{}_{ki} g^{jk} + \overset{2}{P}{}_{jc} R^c{}_{sk} g^{jk}$$

Furthermore (11) and (13) imply

$$R_{|s} - 2R^h{}_{s|h} = \overset{2}{P}{}_j{}^i{}_{kc} R^c{}_{is} g^{jk} + \overset{2}{P}{}_j{}^i{}_{sc} R^c{}_{ki} g^{jk} + \overset{2}{P}{}_{jc} R^c{}_{sk} g^{jk}.$$

Then, by (13) it follows

$$\text{(20)}\qquad \frac{1}{2} R_{|s} - R^h{}_{s|h} = \overset{2}{P}{}^i{}_c R^c{}_{si} + \frac{1}{2} \overset{2}{P}{}_j{}^i{}_{sc} R^c{}_{ki} g^{jk}.$$

Replacing (20) in the first identity of (17) it results

$$\overset{2}{P}{}^i{}_c R^c{}_{ji} + \frac{1}{2} \overset{2}{P}{}_s{}^i{}_{jc} R^c{}_{ki} g^{sk} - \frac{1}{2} S_{|j} + \overset{1}{P}{}^a{}_j|_a = 0.$$

Consequently the first identity of (18) is proved. The second one follows in an analogous way.

Corollary 1. *If N is an integrable distribution (18) become*

$$S_{|j} = 2\overset{1}{P}{}^a{}_j|_a, \quad R|_a = -2\overset{2}{P}{}^i{}_{a|i}$$

2. Examples. Theorem 1 can be useful to verify whether the Einstein equations (16) satisfy the conservation laws (17) or not.

We will give some examples which show the way in which it can be done. For this purpose we need some further definitions.

Definition 1. *G is called h-Riemannian (v-Riemannian) if $g_{ij}(x, y)$ ($h_{ab}(x, y)$) depend on $x \in M$, only. G is called (h, v)-Riemannian metric if G is an h- and v-Riemannian metric.*

It is evident that if G is an h-Riemannian (v-Riemannian) metric then G^H (G^V) defines in an obvious way a pseudo-riemannian metric on M which we denote by $g(h)$.

Definition 2. *G is called h-locally Minkowski (v-locally Minkowski) if each point $(x_0, y_0) \in TM$ has acanonical coordinate neighbourhood in which $g_{ij}(x, y)$ ($h_{ab}(x, y)$) depend on y only. G is called (hv)-locally Minkowski metric if G is an h- and v-locally Minkowski metric.*

Definition 3. *G is called Riemannian-locally Minkowski metric if G is an h-Riemannian and v-locally Minkowski metric. G is called locally Minkowski-Riemannian metric if G is an h-locally Minkowski and v-Riemannian metric.*

Definition 4. *G^H is called weakly regular if the matrix $\|\tilde{g}_{ij}\|$ defined by*

$$2\tilde{g}_{ij} = \partial^2 \varepsilon / \partial y^i \partial y^j \qquad \varepsilon = g_{ij}(x, y)\, y^i y^j$$

is non degenerate.

In this case by putting

$$\text{(21)}\qquad \overset{0}{N}{}^a_j = \frac{\partial \overset{0}{G}{}^a}{\partial y^j}, \quad 4\overset{0}{G}{}^a = \tilde{g}^{ij} \left(\frac{\partial^2 \varepsilon}{\partial y^j \partial x^r} y^r - \frac{\partial \varepsilon}{\partial x^j} \right) \delta^a_i,$$

we obtain a non-linear connection $\overset{0}{N}$.

In an analogous way

Definition 5. *G^V is called weakly regular if the matrix $\|\tilde{h}_{ab}\|$ defined by*

$$2\tilde{h}_{ab} = \partial^2 \varepsilon' / \partial y^a \partial y^b, \qquad \varepsilon' = h_{ab}(x, y)\, y^a y^b$$

is non-degenerate.

Then

$$\text{(22)}\qquad \overset{1}{N}{}^a_j = \partial \overset{1}{G}{}^a / \partial y^j. \quad 4\overset{1}{G}{}^a = \tilde{h}^{ab} \left(\frac{\partial^2 \varepsilon'}{\partial y^b \partial x^c} y^c - \frac{\partial \varepsilon'}{\partial x^b} \right)$$

defines a non-linear connection $\overset{1}{N}$ of TM.

If G is an h-Riemannian metric then G^H is weakly regular and it results

$$\overset{0}{N}{}^a_j(x, y) = \overset{0}{\Gamma}{}^i_{jk}(x)\, y^k \delta^a_i.$$

where $\overset{0}{\Gamma}{}^i_{jk}$ are the local components of the *Levi-Civita connection* $\overset{0}{\nabla}$ defined on M by g. Moreover we denote by $\overset{0}{\mathcal{R}}$, $\overset{0}{R}{}_j{}^i{}_{kl}$ and $\overset{0}{R}{}_{jk}$ the scalar curvature and the local components of the curvature and Ricci tensor fields of $\overset{0}{\nabla}$, respectively. In an analogous way, if G is a v-Riemannian metric G^V is weakly regular and we have

$$\overset{1}{N}{}^a_j(x, y) = \overset{1}{\Gamma}{}^i_{jk}(x)\, y^k \delta^a_i$$

where $\overset{1}{\Gamma}{}^i_{jk}$ are the local coefficients of the Levi-Civita connection $\overset{1}{\nabla}$ defined on M by h. Then $\overset{1}{R}{}_j{}^i{}_{kl}$, $\overset{1}{R}{}_{jk}$ and $\overset{1}{\mathcal{R}}$ will be the local components of the curvature and Ricci tensor fields and the scalar curvature of $\overset{1}{\nabla}$, respectively.

By these notations, using (10), (2), (3), (4) and (14) we get

Theorem 2. *If G is an (h, v)-Riemannian metric and $N = \overset{\circ}{N}$, then the local coefficients of ∇ are*

$$L^i_{jk} = \overset{\circ}{\Gamma}{}^i_{jk}, \quad \bar{C}^i_{jc} = 0, \quad C^a_{bc} = 0,$$

$$2L^a_{bk} = \overset{\circ}{\Gamma}{}^i_{jk}\delta^a_i\delta^j_b + h^{ac}\left\{\frac{\partial h_{bc}}{\partial x^k} - \overset{\circ}{\Gamma}{}^i_{jk}\delta^d_i\delta^j_c h_{bd}\right\}.$$

Furthermore it results

$$R^a{}_{jk} = \overset{\circ}{R}_r{}^i{}_{jk}\, y^r\delta^a_i, \quad \overset{1}{P}{}^a{}_{jb} = \overset{\circ}{\Gamma}{}^i_{kj}\delta^a_i\delta^k_b - \bar{L}^a_{bk}, \quad \overset{2}{P}{}^i{}_{jb} = 0.$$

Finally we have

$$R_j{}^i{}_{kr} = \overset{\circ}{R}_j{}^i{}_{kr}, \quad \overset{1}{P}_b{}^a{}_{kc} = 0, \quad \overset{2}{P}_j{}^i{}_{kc} = 0, \quad S_b{}^a{}_{cd} = 0,$$

$$R_{jk} = \overset{\circ}{R}_{jk}, \quad \overset{1}{P}_{bk} = 0, \quad \overset{2}{P}_{jc} = 0, \quad S_{bc} = 0, \quad R = \overset{\circ}{\mathcal{R}}, \quad S = 0.$$

Consequently by Theorem 1 we have

Theorem 3. *Let G be an (h, v)-Riemannian metric and $N = \overset{\circ}{N}$. The Einstein equations of ∇ can be written as*

$$\overset{\circ}{R}_{ij} - \frac{1}{2}\overset{\circ}{\mathcal{R}} g_{ij} = k\overset{1}{T}_{ij}, \quad \overset{2}{T}_{bj} = 0, \quad \overset{\circ}{\mathcal{R}} h_{ab} = -2k\overset{4}{T}_{ab}, \quad \overset{3}{T}_{jb} = 0,$$

and the conservation laws hold.

In an analogous way we have

Theorem 4. *If G is an (h, v)-Riemannian metric and $N = \overset{1}{N}$ then the local components of ∇ are*

$$L^i_{jk} = \overset{\circ}{\Gamma}{}^i_{jk}, \quad \bar{C}^i_{jc} = 0, \quad \bar{L}^a_{bk} = \overset{1}{\Gamma}{}^i_{jk}\delta^a_i\delta^j_b, \quad C^a_{bc} = 0.$$

Furthermore it results

$$R^a{}_{jk} = \overset{1}{R}_r{}^i{}_{jk}\, y^r\delta^a_i, \quad \overset{1}{P}{}^a{}_{jb} = 0, \quad \overset{2}{P}{}^i{}_{jb} = 0.$$

Finally we get

$$R_j{}^i{}_{kr} = \overset{\circ}{R}_j{}^i{}_{kr}, \quad \overset{1}{P}_b{}^a{}_{kc} = 0, \quad \overset{2}{P}_j{}^i{}_{kc} = 0, \quad S_b{}^a{}_{cd} = 0,$$

$$R_{jk} = \overset{\circ}{R}_{jk}, \quad \overset{1}{P}_{bk} = 0, \quad \overset{2}{P}_{jc} = 0, \quad S_{bc} = 0, \quad R = \overset{\circ}{\mathcal{R}}, \quad S = 0$$

Theorem 5. *Let G be an (h, v)-Riemannian metric and $N = \overset{1}{N}$. The Einstein equations of ∇ can be written as*

$$\overset{\circ}{R}_{ij} - \frac{1}{2}\overset{\circ}{\mathcal{R}} g_{ij} = k\overset{1}{T}_{ij}, \quad \overset{2}{T}_{bj} = 0, \quad \overset{\circ}{\mathcal{R}} h_{ab} = -2k\overset{4}{T}_{ab}, \quad \overset{3}{T}_{jb} = 0$$

and the conservation laws hold.

Theorems 3 and 5 determine quite different examples. In fact under the assumptions of Theorem 3 the orbits are the geodesics of g, while under the

assumptions of Theorem 5 the orbits are geodesics both of g and h. Moreover, the Einstein equations of ∇ obtained in Theorem 3 coincide with the classical ones since the d-tensor having as components $\overset{4}{T}_{ab} = -\frac{1}{2k}\overset{\circ}{\mathcal{R}} h_{ab}$ does not play any role.

Theorem 6. *Let G be a Riemannian-locally Minkowski metric having G^V weakly regular and $N = \overset{1}{N}$. If (U, x^i) is a chart of M on which the components h_{ab} of G^V depend on y only, then it results*

$$N^a_j = 0, \quad L^i_{jk} = \overset{\circ}{\Gamma}{}^i_{jk}, \quad \bar{C}^i_{jc} = 0, \quad \bar{L}^a_{bk} = 0,$$

$$2C^a_{bc} = h^{ad}\{\partial h_{db}/\partial y^c + \partial h_{dc}/\partial y^b - \partial h_{bc}/\partial y^d\}.$$

Moreover ∇ has zero torsion (i.e. ∇ is the Levi-Civita connection of G) and we have:

$$R_j{}^i{}_{kl} = \overset{\circ}{R}_j{}^i{}_{kl}, \quad \overset{2}{P}_j{}^i{}_{kc} = 0, \quad \overset{1}{P}_b{}^a{}_{kc} = 0,$$

$$S_b{}^a{}_{cd} = \frac{\partial C^a_{bc}}{\partial y^d} - \frac{\partial C^a_{bd}}{\partial y^c} + C^f_{bc}C^a_{fd} - C^f_{bd}C^a_{fc}.$$

Theorem 7. *Let G be a Riemannian-locally Minkowski metric having G^V weakly regular and $N = \overset{1}{N}$. Then the Einstein equations of ∇ are:*

$$\overset{\circ}{R}_{ij} - \frac{1}{2}(\overset{\circ}{\mathcal{R}} + S)g_{ij} = k\overset{1}{T}_{ij}, \qquad \overset{2}{T}_{bj} = 0$$

$$S_{ab} - \frac{1}{2}(S + \overset{\circ}{\mathcal{R}})h_{ab} = k\overset{4}{T}_{ab}, \qquad \overset{3}{T}_{jb} = 0$$

and the conservation laws are satisfied.

Theorem 8. *Let G be a locally Minkowski-Riemannian metric having G^H weakly regular and $N = \overset{\circ}{N}$. If (U, x^i) is a chart of M on which the components g_{ij} of G^H depend on y only then we get:*

$$N^a_j = 0, \quad L^i_{jk} = 0, \quad 2\bar{C}^i_{jc} = g^{ih}\frac{\partial g_{hj}}{\partial y^c}, \quad 2\bar{L}^a_{bk} = h^{ac}\frac{\partial h_{bc}}{\partial x^k}, \quad C^a_{bc} = 0.$$

Moreover it results

$$T^i{}_{jk} = 0, \quad R^a{}_{jk} = 0, \quad \overset{1}{P}{}^a{}_{jb} = -\bar{L}^a_{bj}, \quad \overset{2}{P}{}^i{}_{jb} = \bar{C}^i_{jb}, \quad S^a{}_{bc} = 0.$$

Finally we have

$$R_j{}^i{}_{kl} = 0, \quad \overset{2}{P}_j{}^i{}_{kc} = 0, \quad \overset{1}{P}_b{}^a{}_{kc} = 0, \quad S_b{}^a{}_{cd} = 0.$$

Theorem 9. *Let G be a locally Minkowski-Riemannian metric having G^H weakly regular and $N = \overset{\circ}{N}$. Then the Einstein equations of ∇ are*

$$\overset{1}{T}_{ij}=0,\quad \overset{2}{T}_{bj}=0,\quad \overset{3}{T}_{jb}=0,\quad \overset{4}{T}_{ab}=0$$

and the conservation laws hold.

Theorem 10. *Let G be an (h, v)-locally Minkowski metric. Moreover let us suppose that, with respect to each chart of an atlas $\mathcal{A}$ of M the components g_{ij} and h_{ab} of G depend both on y only. We also suppose that G^H is weakly regular and $N=\overset{\circ}{N}$. If (U, x^i) is a chart of $\mathcal{A}$ then it results*

$$N^a_j=0,\quad L^i_{jk}=0,\quad 2\bar{C}^i_{jc}=g^{ih}\frac{\partial g_{hj}}{\partial y^c},\quad L^a_{bk}=0,$$

$$2C^a_{bc}=h^{ad}\left\{\frac{\partial h_{db}}{\partial y^c}+\frac{\partial h_{dc}}{\partial y^b}-\frac{\partial h_{bc}}{\partial y^d}\right\},$$

$$T^i{}_{jk}=0,\quad R^a{}_{jk}=0,\quad \overset{1}{P}{}^a{}_{jb}=0,\quad \overset{2}{P}{}^i{}_{jb}=\bar{C}^i_{jb},\quad S^a{}_{bc}=0,$$

$$R_j{}^i{}_{kl}=0,\quad \overset{2}{P}_j{}^i{}_{kc}=0,\quad \overset{1}{P}_b{}^a{}_{kc}=0.$$

$$S_b{}^a{}_{cd}=\partial C^a_{bc}/\partial y^d-\partial C^a_{bd}/\partial y^c+C^f_{bc}C^a_{fd}-C^f_{bd}C^a_{fc}.$$

Theorem 11. *Under the same assumptions as in Theorem 10 we can write the Einstein equations of ∇ as*

$$Sg_{ij}=-2k\overset{1}{T}_{ij},\quad \overset{2}{T}_{bj}=0,\quad S_{ab}-\frac{1}{2}Sh_{ab}=k\overset{4}{T}_{ab},\quad \overset{3}{T}_{jb}=0$$

and the conservation laws are satisfied.

Remark 1. If we replace the assumptions "G^H is weakly regular and $N=\overset{\circ}{N}$" by "G^V is weakly regular and $N=\overset{1}{N}$" into Theorems 10 and 11 they hold too.

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Department of Mathematics
Via G. Fortunato 2 Bari, Italia

An. şt. Univ. „Al. I. Cuza" Iaşi
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SUBMERSIONS OF SEMI-INVARIANT SUBMANIFOLDS OF A SASAKIAN MANIFOLD

BY

NECULAI PAPAGHIUC

1. Introduction. The concept of CR-submanifold introduced by A. Bejancu has been intensively studied by many people (see [1] for more references). Recently, S. Kobayashi [7] studied submersions of CR-submanifolds of a Kaehler manifold onto an almost hermitian manifold. Some results on the geometry of submersions of CR-submanifolds were obtained in [6]. We remark that both [6] and [7] are dealing only with submersions of generic CR-submanifolds (i.e. $JD^\perp=TM^\perp$, where $D^\perp$ is the anti-holomorphic distribution of the CR-submanifold M and $TM^\perp$ is the normal bundle to M). It is easily to see that every CR-product in a Kaehler manifold (see [1], [5]), in particular every CR-product in a complex space form is a submersion of CR-product onto an almost hermitian manifold. On the other hand, it is known [5] that there exist no proper generic CR-products in a complex space form $\tilde{M}(c)$ with $c\neq 0$. Thus it is naturally to study submersions of CR-submanifolds which are not necessarily generic. We note that the semi-invariant submanifolds of a Sasakian manifold have been studied in [2], [3], [4], [9], [10] and they are also called contact CR-submanifolds (see [11]) being extensions of CR-submanifolds [1] to submanifolds of Sasakian manifolds.

In the present paper we study the submersions of semi-invariant submanifolds of a Sasakian manifold onto an almost contact metric manifold without any assumption on the dimension of anti-invariant distribution $D^\perp$. We obtain basic properties of submersions of semi-invariant submanifolds of a Sasakian manifold and find necessary and sufficient conditions for such a submersion to be a normal semi-invariant submanifold or a semi-invariant product (Theorems 1, 2, 3).

2. Preliminaries on semi-invariant submanifolds. Let $\tilde{M}$ be a Sasakian manifold with the Sasakian structure (φ, ξ, η, g) where φ is a tensor field of type (1,1) on $\tilde{M}$, ξ is a vector field, η is a 1-form and g is the Riemannian metric on $\tilde{M}$. These tensor fields are related by

$$\begin{aligned}(1)\qquad &\varphi^2X=-X+\eta(X)\,\xi;\ \varphi\xi=0;\ \eta\circ\varphi=0;\ \eta(\xi)=1;\ g(\varphi X, \varphi Y)=g(X, Y)-\\ &-\eta(X)\,\eta(Y);\ (\tilde{\nabla}_X\varphi)\,Y=g(X, Y)\,\xi-\eta(Y)\,X;\ \tilde{\nabla}_X\xi=-\varphi X.\end{aligned}$$

for any vector fields X, Y tangent to $\tilde{M}$, where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$.