$$\overset{1}{T}_{ij}=0,\quad \overset{2}{T}_{bj}=0,\quad \overset{3}{T}_{jb}=0,\quad \overset{4}{T}_{ab}=0$$

*and the conservation laws hold.*

**Theorem 10.** *Let $G$ be an $(h, v)$-locally Minkowski metric. Moreover let us suppose that, with respect to each chart of an atlas $\mathcal{A}$ of $M$ the components $g_{ij}$ and $h_{ab}$ of $G$ depend both on $y$ only. We also suppose that $G^H$ is weakly regular and $N=\overset{\circ}{N}$. If $(U, x^i)$ is a chart of $\mathcal{A}$ then it results*

$$N^a_j=0,\quad L^i_{jk}=0,\quad 2\bar{C}^i_{jc}=g^{ih}\frac{\partial g_{hj}}{\partial y^c},\quad \bar{L}^a_{bk}=0,$$

$$2C^a_{bc}=h^{ad}\left\{\frac{\partial h_{ab}}{\partial y^c}+\frac{\partial h_{dc}}{\partial y^b}-\frac{\partial h_{bc}}{\partial y^d}\right\},$$

$$T^i{}_{jk}=0,\quad R^a{}_{jk}=0,\quad \overset{1}{P}{}^a{}_{jb}=0,\quad \overset{2}{P}{}^i{}_{jb}=\bar{C}^i_{jb},\quad S^a{}_{bc}=0,$$

$$R_j{}^i{}_{kl}=0,\quad \overset{2}{P}_j{}^i{}_{kc}=0,\quad \overset{1}{P}_b{}^a{}_{kc}=0.$$

$$S_b{}^a{}_{cd}=\partial C^a_{bc}/\partial y^d-\partial C^a_{bd}/\partial y^c+C^f_{bc}C^a_{fd}-C^f_{db}C^a_{fc}.$$

**Theorem 11.** *Under the same assumptions as in Theorem 10 we can write the Einstein equations of $\nabla$ as*

$$Sg_{ij}=-2k\overset{1}{T}_{ij},\quad \overset{2}{T}_{bj}=0,\quad S_{ab}-\frac{1}{2}Sh_{ab}=k\overset{4}{T}_{ab},\quad \overset{3}{T}_{jb}=0$$

*and the conservation laws are satisfied.*

*Remark* 1. If we replace the assumptions "$G^H$ is weakly regular and $N=\overset{\circ}{N}$" by "$G^V$ is weakly regular and $N=\overset{1}{N}$" into Theorems 10 and 11 they hold too.

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# SUBMERSIONS OF SEMI-INVARIANT SUBMANIFOLDS OF A SASAKIAN MANIFOLD

BY

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**1. Introduction.** The concept of CR-submanifold introduced by A. Bejancu has been intensively studied by many people (see [1] for more references). Recently, S. Kobayashi [7] studied submersions of CR-submanifolds of a Kaehler manifold onto an almost hermitian manifold. Some results on the geometry of submersions of CR-submanifolds were obtained in [6]. We remark that both [6] and [7] are dealing only with submersions of generic CR-submanifolds (i.e. $JD^\perp=TM^\perp$, where $D^\perp$ is the anti-holomorphic distribution of the CR-submanifold $M$ and $TM^\perp$ is the normal bundle to $M$). It is easily to see that every CR-product in a Kaehler manifold (see [1], [5]), in particular every CR-product in a complex space form is a submersion of CR-product onto an almost hermitian manifold. On the other hand, it is known [5] that there exist no proper generic CR-products in a complex space form $\tilde{M}(c)$ with $c\neq 0$. Thus it is naturally to study submersions of CR-submanifolds which are not necessarily generic. We note that the semi-invariant submanifolds of a Sasakian manifold have been studied in [2], [3], [4], [9], [10] and they are also called contact CR-submanifolds (see [11]) being extensions of CR-submanifolds [1] to submanifolds of Sasakian manifolds.

In the present paper we study the submersions of semi-invariant submanifolds of a Sasakian manifold onto an almost contact metric manifold without any assumption on the dimension of anti-invariant distribution $D^\perp$. We obtain basic properties of submersions of semi-invariant submanifolds of a Sasakian manifold and find necessary and sufficient conditions for such a submersion to be a normal semi-invariant submanifold or a semi-invariant product (Theorems 1, 2, 3).

**2. Preliminaries on semi-invariant submanifolds.** Let $\tilde{M}$ be a Sasakian manifold with the Sasakian structure $(\varphi, \xi, \eta, g)$ where $\varphi$ is a tensor field of type (1,1) on $\tilde{M}$, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is the Riemannian metric on $\tilde{M}$. These tensor fields are related by

$$(1)\qquad \begin{aligned}&\varphi^2X=-X+\eta(X)\,\xi\,;\ \varphi\xi=0\,;\ \eta\circ\varphi=0\,;\ \eta(\xi)=1\,;\ g(\varphi X, \varphi Y)=g(X, Y)-\\ &-\eta(X)\,\eta(Y)\,;\ (\tilde{\nabla}_X\varphi)\,Y=g(X, Y)\,\xi-\eta(Y)\,X\,;\ \tilde{\nabla}_X\xi=-\varphi X,\end{aligned}$$

for any vector fields $X$, $Y$ tangent to $\tilde{M}$, where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$.

Let $M$ be a Riemannian manifold isometrically immersed in $\tilde{M}$ and suppose the structure vector field $\xi$ of $\tilde{M}$ be tangent to $M$. We denote by $TM$ and $TM^{\perp}$ the tangent bundle to $M$ and respectively the normal bundle to $M$. We also denote by $\{\xi\}$ the distribution spanned by $\xi$ on $M$. The submanifold $M$ is called a semi-invariant submanifold (or a contact CR-submanifold) if it is endowed with the pair of distributions $(D, D^{\perp})$ satisfying the conditions (see [2])

(i) $TM = D \oplus D^{\perp} \oplus \{\xi\}$ and $D$, $D^{\perp}$, $\{\xi\}$ are orthogonal to each other;

(ii) the distribution $D$ is invariant by $\varphi$, i.e. we have $\varphi(D_x) = D_x$ for each $x \in M$;

(iii) the distribution $D^{\perp}$ is anti-invariant by $\varphi$, i.e. we have $\varphi(D_x^{\perp}) \subset T_x M^{\perp}$ for each $x \in M$.

A generic semi-invariant submanifold is a semi-invariant submanifold characterized by the condition $\varphi(D_x^{\perp}) = T_x M^{\perp}$, for each $x \in M$ (see [11]). We denote by the same symbol $g$ both Riemannian metrics on $\tilde{M}$ and $M$. For any vector bundle $H$ over $M$ we denote by $\Gamma(H)$ the module of all differentiable sections of $H$. The projection morphisms of $TM$ to $D$ and $D^{\perp}$ are denoted by $P$ and respectively $Q$. Then we have

(2) $$X = PX + QX + \eta(X)\,\xi \text{ for any } X \in \Gamma(TM).$$

The formulas of Gauss and Weingarten for the immersion of $M$ in $\tilde{M}$ are given by

(3) $$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \text{ and respectively}$$

(4) $$\tilde{\nabla}_X U = \nabla_X^{\perp} U - \tilde{A}_U X, \quad X, Y \in \Gamma(TM), \quad U \in \Gamma(TM^{\perp}),$$

where $\nabla$ is the Levi-Civita connection on $M$, $\nabla^{\perp}$ is the linear connection induced by $\tilde{\nabla}$ on $TM^{\perp}$, $h$ is the second fundamental form of $M$ and $\tilde{A}_U$ is the fundamental tensor of Weingarten with respect to the normal section $U$. We have

(5) $$g(h(X, Y), U) = g(\tilde{A}_U X, Y), \quad X, Y \in \Gamma(TM), \quad U \in \Gamma(TM^{\perp}).$$

Denote by $\nu$ the orthogonal complementary vector bundle of $\varphi(D^{\perp})$ in $TM^{\perp}$. Thus, the normal bundle to $M$ has the decomposition

(6) $$TM^{\perp} = \varphi(D^{\perp}) \oplus \nu \text{ and } g(\varphi D^{\perp}, \nu) = \{0\}.$$

For any $U \in \Gamma(TM^{\perp})$ we put

(7) $$U = nU + \nu U, \text{ where } nU \in \Gamma(\varphi D^{\perp}) \text{ and } \nu U \in \Gamma(\nu).$$

From (7) we have

(8) $$\varphi U = \varphi n U + \varphi \nu U, \quad U \in \Gamma(TM^{\perp}), \quad \varphi n U \in \Gamma(D^{\perp}), \quad \varphi \nu U \in \Gamma(\nu).$$

For any semi-invariant submanifold $M$ of the Sasakian manifold $\tilde{M}$, we define the tensor field $F$ of type (1.1) by $FX = \varphi PX$, $X \in \Gamma(TM)$. We also define the non-null normal bundle valued 1-form $u$ on $M$ by $uY = \varphi QY$, $Y \in \Gamma(TM)$. The triplet $(F, u, \eta)$ is called the semi-invariant structure on $M$. By means of this structure, we define the tensor field $S$ on $M$ by



(9) $$S(X, Y) = N_F(X, Y) - 2\varphi n(du(X, Y)) + 2d\eta(X, Y)\,\xi, \quad X, Y \in \Gamma(TM),$$

where $N_F$ is the Nijenhuis tensor field of $F$ and $du$, $d\eta$ are the exterior differentials of $u$ and $\eta$, respectively (see [4]).

The semi-invariant submanifold $M$ of the Sasakian manifold $\tilde{M}$ is called a *normal semi-invariant submanifold* if $S$ vanishes identically on $M$. [4].

We say that a semi-invariant submanifold $M$ of a Sasakian manifold $\tilde{M}$ is a *semi-invariant product* if the distribution $D \oplus \{\xi\}$ is involutive and locally $M$ is a Riemannian product $M_1 \times M_2$ where $M_1$ (resp. $M_2$) is a leaf of $D \oplus \{\xi\}$ (resp. $D^{\perp}$).

Concerning the semi-invariant submanifolds of a Sasakian manifold we recall for later use the following results:

**Lemma 1.** [3]. (i). *The distribution $D^{\perp}$ is always involutive.*

(ii). *The distribution $D \oplus \{\xi\}$ is involutive if and only if the second fundamental form satisfies*

(10) $$h(X, \varphi Y) - h(\varphi X, Y) \in \Gamma(\nu), \quad X, Y \in \Gamma(D).$$

**Lemma 2.** [4]. *Let $M$ be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then $M$ is a normal semi-invariant submanifold if and only if we have*

(11) $$h(X, \varphi Y) + h(\varphi X, Y) \in \Gamma(\nu), \quad X, Y \in \Gamma(D) \text{ and}$$

(12) $$h(X, U) \in \Gamma(\nu), \quad X \in \Gamma(D), \quad U \in \Gamma(D^{\perp}).$$

**Lemma 3.** [4]. *A normal semi-invariant submanifold is a semi-invariant product if and only if the distribution $D \oplus \{\xi\}$ is integrable.*

**Lemma 4.** [3]. *A leaf $M^{\perp}$ of the anti-invariant distribution $D^{\perp}$ is totally geodesic immersed in the semi-invariant submanifold $M$ if and only if we have*

(13) $$h(X, U) \in \Gamma(\nu), \quad X \in \Gamma(D), \quad U \in \Gamma(D^{\perp}).$$

**3. Submersions of semi-invariant submanifolds.** Let $M$ be a semi-invariant submanifold of the Sasakian manifold $\tilde{M}$ and $N$ be an almost contact metric manifold with the almost contact metric structure $(\varphi_*, \xi_*, \eta_*, g_*)$. Assume that there is a submersion $\pi : M \to N$ such that

(a) $D^{\perp} = \operatorname{Ker} \pi_*$, where $\pi_* : TM \to TN$ is the tangent mapping to $\pi$

(b) $\pi_* : D_x \oplus \{\xi_x\} \to T_{\pi(x)}N$ is an isometry for each $x \in M$ which satisfies: $\pi_* \circ \varphi = \varphi_* \circ \pi_*$; $\eta = \eta_* \circ \pi_*$; $\pi_*(\xi_x) = \xi_{*\pi(x)}$, where $T_{\pi(x)}N$ denotes the tangent space of $N$ at $\pi(x)$.

For the theory of submersions we follow B. O'Neill [8].

A vector field $X$ on $M$ is said to be *basic* if $X \in \Gamma(D \oplus \{\xi\})$ and $X$ is $\pi$-related to a vector field on $N$, i.e. there exists a vector field $X_* \in \Gamma(TN)$ such that $\pi_*(X_x) = X_{*\pi(x)}$ for each $x \in M$.

*Remark.* By the condition (b) of the definition of a submersion of a semi-invariant submanifold we have that the structure vector field $\xi$ is a basic vector field.

We recall the following Lemma from B. O'Neill [8].

**Lemma 5.** *Let $X$, $Y$ be basic vector fields on $M$. Then*

(i) $g(X, Y) = g_*(X_*, Y_*)\circ\pi$;

(ii) *the component* $P([X, Y]) + \eta([X, Y])\,\xi$ *of* $[X, Y]$ *is a basic vector field and corresponds to* $[X_*, Y_*]$, *i.e.* $\pi_*(P([X, Y]) + \eta([X, Y])\,\xi) = [X_*, Y_*]$;

(iii) $[U, X] \in \Gamma(D^\perp)$ *for any* $U \in \Gamma(D^\perp)$;

(iv) $P(\nabla_X Y) + \eta(\nabla_X Y)\,\xi$ *is a basic vector field corresponding to* $\nabla^*_{X_*} Y_*$, *where* $\nabla^*$ *denotes the Levi-Civita connection on* $N$.

In a similar way as in [7], for basic vector fields on $M$, we define the operator $\tilde{\nabla}^*$ corresponding to $\nabla^*$ by setting

$$(14)\qquad \tilde{\nabla}^*_X Y = P(\nabla_X Y) + \eta(\nabla_X Y)\,\xi,\quad X, Y \in \Gamma(D\oplus\{\xi\}).$$

By (iv) of Lemma 5, $\tilde{\nabla}^*_X Y$ is a basic vector field and we have

$$(15)\qquad \pi_*(\tilde{\nabla}^*_X Y) = \nabla^*_{X_*} Y_*.$$

Define the tensor field $C$ by

$$(16)\qquad \nabla_X Y = \tilde{\nabla}^*_X Y + C(X, Y),\quad C(X, Y) = Q(\nabla_X Y);\quad X, Y \in \Gamma(D\oplus\{\xi\}).$$

Then (see [7]) $C$ is skew-symmetric and satisfies:

$$(17)\qquad C(X, Y) = \frac{1}{2}Q([X, Y]),\quad X, Y \in \Gamma(D\oplus\{\xi\}).$$

Now we prove

**Proposition 1.** *Let* $\pi: M \to N$ *be a submersion of semi-invariant submanifold* $M$ *of the Sasakian manifold* $\tilde{M}$ *onto an almost contact metric manifold* $N$. *Then we have:*

$$(18)\qquad (\tilde{\nabla}^*_X\varphi)\,Y = g(X, Y)\,\xi - \eta(Y)\,X,$$

$$(19)\qquad C(X, \varphi Y) = \varphi n h(X, Y),$$

$$(20)\qquad \varphi C(X, Y) = nh(X, \varphi Y),$$

$$(21)\qquad \varphi\nu h(X, Y) = \nu h(X, \varphi Y),\ \textit{for any}\ X, Y \in \Gamma(D\oplus\{\xi\}).$$

*Proof.* For $X, Y \in \Gamma(D\oplus\{\xi\})$, by using (3), (7), (14) and (16) we get

$$(22)\qquad \tilde{\nabla}_X Y = \tilde{\nabla}^*_X Y + C(X, Y) + nh(X, Y) + \nu h(X, Y).$$

Hence

$$(23)\qquad \varphi\tilde{\nabla}_X Y = \varphi\tilde{\nabla}^*_X Y + \varphi C(X, Y) + \varphi n h(X, Y) + \varphi\nu h(X, Y).$$

Applying (22) to $\varphi Y \in \Gamma(D)$ we have

$$(24)\qquad \tilde{\nabla}_X\varphi Y = \tilde{\nabla}^*_X\varphi Y + C(X, \varphi Y) + nh(X, \varphi Y) + \nu h(X, \varphi Y),\quad X, Y \in \Gamma(D\oplus\{\xi\}).$$

By using $(\tilde{\nabla}_X\varphi)\,Y = \tilde{\nabla}_X\varphi Y - \varphi\tilde{\nabla}_X Y = g(X, Y)\,\xi - \eta(Y)\,X$ and (23), (24) we get

$$(25)\qquad \tilde{\nabla}^*_X\varphi Y + C(X, \varphi Y) + nh(X, \varphi Y) + \nu h(X, \varphi Y) - \varphi\tilde{\nabla}^*_X Y - \varphi C(X, Y) - \varphi n h(X, Y) - \varphi\nu h(X, Y) = g(X, Y)\,\xi - \eta(Y)\,X,\quad X, Y \in \Gamma(D\oplus\{\xi\}).$$



Then we obtain (18)–(21) from (25) by taking the components from $D\oplus\{\xi\}$, $D^\perp$, $\varphi D^\perp$ and $\nu$, respectively. From (15) and (18), by taking into account of the definition of a submersion of a semi-invariant submanifold and Lemma 5 we have

**Corollary 1.** *Let* $\pi: M \to N$ *be a submersion of semi-invariant submanifold* $M$ *of the Sasakian manifold* $\tilde{M}$ *onto an almost contact metric manifold* $N$. *Then* $(\nabla^*_{X_*}\varphi_*)Y_* = g_*(X_*, Y_*)\xi_* - \eta_*(Y_*)X_*$, $X_*, Y_* \in \Gamma(TN)$, *i.e.* $N$ *is also a Sasakian manifold.*

From (19), (20), (21) and the skew-symmetry of $C$, we also have

**Corollary 2.** *Let* $\pi: M \to N$ *be a submersion of semi-invariant submanifold* $M$ *of the Sasakian manifold* $\tilde{M}$ *onto an almost contact metric manifold* $N$. *Then*

$$(26)\qquad nh(X, \varphi Y) + nh(\varphi X, Y) = 0,$$

$$(27)\qquad nh(\varphi X, \varphi Y) = nh(X, Y),$$

$$(28)\qquad \nu h(\varphi X, \varphi Y) = -\nu h(X, Y),$$

$$(29)\qquad C(\varphi X, \varphi Y) = C(X, Y),\quad X, Y \in \Gamma(D\oplus\{\xi\}).$$

*Remark.* The relations (26) and (28) are respectively equivalent to

$$(30)\qquad h(X, \varphi Y) + h(\varphi X, Y) \in \Gamma(\nu)\ \text{and}$$

$$(31)\qquad h(X, \varphi Y) - h(\varphi X, Y) \in \Gamma(\varphi D^\perp),\quad X, Y \in \Gamma(D\oplus\{\xi\}).$$

Remark also that for obtain Corollary 2 we have used the relations $\tilde{\nabla}_X\xi = -\varphi X$, $h(X, \xi) = 0$ for any $X \in \Gamma(D\oplus\{\xi\})$ which are immediate consequences of (1) and (3).

From (26) and Lemma 2 we have

**Theorem 1.** *Let* $\pi: M \to N$ *be a submersion of semi-invariant submanifold* $M$ *of the Sasakian manifold* $\tilde{M}$ *onto an almost contact metric manifold* $N$. *Then* $M$ *is a normal semi-invariant submanifold of* $\tilde{M}$ *if and only if we have* $h(X, U) \in \Gamma(\nu)$, *for any* $X \in \Gamma(D)$, $U \in \Gamma(D^\perp)$.

Also, by using Lemmas 1.3 and Theorem 1 we obtain

**Theorem 2.** *Let* $\pi: M \to N$ *be a submersion of semi-invariant submanifold* $M$ *of the Sasakian manifold* $\tilde{M}$ *onto an almost contact metric manifold* $N$. *Then* $M$ *is a semi-invariant product if and only if the distribution* $D\oplus\{\xi\}$ *is integrable and* $h(X, U) \in \Gamma(\nu)$ *for any* $X \in \Gamma(D)$, $U \in \Gamma(D^\perp)$.

By using Lemma 4 and Theorem 1 we also have

**Theorem 3.** *Let* $\pi: M \to N$ *be a submersion of semi-invariant submanifold* $M$ *of the Sasakian manifold* $\tilde{M}$ *onto an almost contact metric manifold* $N$. *Then* $M$ *is a normal semi-invariant submanifold of* $\tilde{M}$ *if and only if the leaves of the anti-invariant distribution* $D^\perp$ *(i.e. the fibres of* $\pi$*) are totally geodesic submanifolds of the semi-invariant submanifold* $M$.

For a Sasakian manifold $\tilde{M}$, we define the $\varphi$-bisectional curvature with respect to two orthonormal vectors $X, Y$ from the contact distribution by

$$(32)\qquad \tilde{H}_B(X, Y) = \tilde{R}(X, \varphi X; \varphi Y, Y).$$

where $\tilde{R}$ denotes the curvature tensor of the Levi-Civita connection $\tilde{\nabla}$ on $\tilde{M}$ and $\tilde{R}(X, Y; Z, W) = g(\tilde{R}(X, Y)Z, W)$.

**Corollary 3.** *Let $\pi : M \to N$ be a submersion of semi-invariant submanifold $M$ of the Sasakian manifold $\tilde{M}$ onto an almost contact metric manifold $N$. If the fibres of $\pi$ are totally geodesic submanifolds of $M$, then the $\varphi$-bisectional curvature $\tilde{H}_B$ satisfies*

$$\tilde{H}_B(X, U) = 2\|h(X, U)\|^2 + 2\|\tilde{A}_{\varphi U} X\|^2 - 2g(nh(X, X), nh(U, U)) - 2, \tag{33}$$

*for all unit vectors $X \in \Gamma(D)$ and $U \in \Gamma(D^\perp)$.*

*Proof.* From hypotheses and Theorem 3 we get that $M$ is a normal semi-invariant submanifold of $\tilde{M}$. Then (33) follows from Lemma 4.3 in [10] using (32).

**Corollary 4.** *Let $\pi : M \to N$ be a submersion of semi-invariant submanifold $M$ of the Sasakian manifold $\tilde{M}$ onto an almost contact metric manifold $N$. Suppose the distribution $D \oplus \{\xi\}$ is integrable and the fibres of $\pi$ are totally geodesic submanifolds of $M$. Then we have*

$$\tilde{H}_B(X, Z) = 2\|h(X, Z)\|^2 - 2 \tag{34}$$

*for all unit vectors $X \in \Gamma(D)$, $Z \in \Gamma(D^\perp)$.*

*Proof.* From hypotheses, Lemma 4 and Theorem 2 we obtain that $M$ is a semi-invariant product of the Sasakian manifold $\tilde{M}$. Then (34) follows by Lemma 3.1 in [9].

Now, for $X \in \Gamma(D \oplus \{\xi\})$ and $U \in \Gamma(D^\perp)$, define the operator $A$ by

$$\nabla_X U = Q(\nabla_X U) + A_X U. \tag{35}$$

Since $[U, X] \in \Gamma(D^\perp)$ and taking into account that $\mathcal{V}(\nabla_X U) = \mathcal{V}(\nabla_U X) = 0$, we obtain

$$A_X U = P(\nabla_X U) = P(\nabla_U X). \tag{36}$$

The operators $A$ and $C$ are related by (see [7]):

$$g(A_X U, Y) = -g(U, C(X, Y)), \quad X, Y \in \Gamma(D \oplus \{\xi\}), \quad U \in \Gamma(D^\perp). \tag{37}$$

By using (35) and (37) we obtain that the curvature tensors $R$, $R^*$ of the connections $\nabla$, $\nabla^*$ on $M$ and $N$ respectively are related by [7]

$$\begin{aligned} R(X, Y; Z, W) &= R^*(X_*, Y_*; Z_*, W_*) + g(C(X, Z), C(Y, W)) - \\ &- g(C(Y, Z), C(X, W)) + 2g(C(X, Y), C(Z, W)), \\ &X, Y, Z, W \in \Gamma(D \oplus \{\xi\}). \end{aligned} \tag{38}$$

The curvature tensors $\tilde{R}$ and $R$ of the connections $\tilde{\nabla}$ and $\nabla$ on $\tilde{M}$ and $M$ respectively are related by the equation of Gauss

$$\begin{aligned} \tilde{R}(X, Y; Z, W) &= R(X, Y; Z, W) - g(h(X, W), h(Y, Z)) + \\ &+ g(h(X, Z), h(Y, W)), \end{aligned} \tag{39}$$



$X, Y, Z, W \in \Gamma(TM)$. For $X, Y \in \Gamma(D)$, from (39) by using (1), (7), (21) and Corollary 2 we get:

$$\begin{aligned} \tilde{R}(X, \varphi X; \varphi Y, Y) &= R(X, \varphi X; \varphi Y, Y) - \|nh(X, \varphi Y)\|^2 - \\ &- \|nh(X, Y)\|^2 + 2\|\nu h(X, Y)\|^2. \end{aligned} \tag{40}$$

On the other hand, from (38) by using (19), (20), Corollary 2 and the skew-symmetry of $C$ we obtain

$$\begin{aligned} R(X, \varphi X; \varphi Y, Y) &= R^*(X_*, \varphi_* X_*; \varphi_* Y_*, Y_*) - \|nh(X, \varphi Y)\|^2 - \\ &- \|nh(X, Y)\|^2 - 2g(nh(X, X), nh(Y, Y)), \quad X, Y \in \Gamma(D). \end{aligned} \tag{41}$$

Thus (40) and (41) give

$$\begin{aligned} \tilde{R}(X, \varphi X; \varphi Y, Y) &= R^*(X_*, \varphi_* X_*; \varphi_* Y_*, Y_*) - 2\|nh(X, \varphi Y)\|^2 - \\ &- 2\|nh(X, Y)\|^2 + 2\|\nu h(X, Y)\|^2 - 2g(nh(X, X), nh(Y, Y)), \\ &X, Y \in \Gamma(D). \end{aligned} \tag{42}$$

Setting $Y = X$ in (42) and taking into account of Corollary 2 we obtain that the $\varphi$-sectional curvatures of $\tilde{M}$ and $N$ are related by

$$\tilde{H}(X) = H^*(X_*) - 4\|nh(X, X)\|^2 + 2\|\nu h(X, X)\|^2, \quad X \in \Gamma(D), \ \|X\| = 1. \tag{43}$$

*Remark.* If we suppose $M$ be a generic semi-invariant submanifold of the Sasakian manifold $\tilde{M}$, then we have $nh(X, X) = h(X, X)$, $\nu h(X, X) = 0$ and (43) becomes

$$\tilde{H}(X) = H^*(X_*) - 4\|h(X, X)\|^2, \quad X \in \Gamma(D), \ \|X\| = 1. \tag{44}$$

The relation (44) is similar to the relation obtained by S. Kobayashi for the submersions of generic CR-submanifolds (see (4.12) in [7]).

Thus, combining Corollary 1 and (43) we obtain

**Theorem 4.** *Let $\pi : M \to N$ be a submersion of semi-invariant submanifold $M$ of the Sasakian manifold $\tilde{M}$ onto an almost contact metric manifold $N$. Then*

(i) *$N$ is also a Sasakian manifold,*

(ii) *the $\varphi$-sectional curvatures $\tilde{H}$ and $H^*$ of $\tilde{M}$ and respectively $N$ are related by (43).*

**Corollary 5.** *Let $\pi : M \to N$ be a submersion of semi-invariant submanifold $M$ of the Sasakian manifold $\tilde{M}$ onto an almost contact metric manifold $N$. If the distribution $D \oplus \{\xi\}$ is integrable then we have*

$$\tilde{H}(X) = H^*(X_*) + 2\|\nu h(X, X)\|^2, \quad X \in \Gamma(D), \ \|X\| = 1. \tag{45}$$

*In particular if $M$ is a semi-invariant product then (45) holds.*

*Proof.* It follows from (43), Lemma 1 and (26) of Corollary 2.

**Corollary 6.** *Let $\pi : M \to N$ be a submersion of semi-invariant submanifold $M$ of the Sasakian manifold $\tilde{M}$ onto an almost contact metric manifold $N$. Suppose* *the distribution $D\oplus\{\xi\}$ is integrable. Then the leaves of $D\oplus\{\xi\}$ are totally geodesic immersed in $\tilde{M}$ if and only if we have $\tilde{H}(X)=H^*(X)$ for any $X\in\Gamma(D)$. In particular, if $\tilde{M}(k)$ is a Sasakian space form of constant $\varphi$-sectional curvature $k$ and $\pi: M\to N$ is a submersion of semi-invariant submanifold $M$ of $\tilde{M}(k)$ onto the almost contact metric manifold $N$ such that the distribution $D\oplus\{\xi\}$ be integrable, then the leaves of $D\oplus\{\xi\}$ are totally geodesic immersed in $\tilde{M}(k)$ if and only if $N$ is also a Sasakian space form of the same constant $\varphi$-sectional curvature $k$.*

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# A UNIQUENESS THEOREM IN THE NON-LINEAR DYNAMICS OF ELASTIC MATERIALS WITH VOIDS

BY

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**1. Introduction.** In this note we present a uniqueness result in the context of the finite-deformation theory for *elastic* bodies with *voids*. We consider the dynamic problem of this theory, early introduced by Nunziato & Cowin [1] in order to describe the mechanical behavior of *porous* materials having (non-linear) elastic response in the skeletal structure. The interstices, or *voids*, in this structure are assumed to be lacking in matter and anything else of energetic relevance.

In the field of application of this theory, which of course generalizes the classical elasticity, should fall various kinds of geological materials, such as rock and soils, or porous manufactured materials, such as ceramics and pressed powders. We refer to [1], cf. also [2], for more details and physical insights into the basic concepts of the theory.

Some general theorems for the dynamic problem of elastic bodies with voids have been established, in the linear response case, by Cowin & Nunziato (uniqueness and *weak* stability [3]) and by Ieşan (reciprocity and variational characterization [4]). Chandrasekharaiah [5] also treats uniqueness in the context of a particular formulation.

The non-linear case has been recently tackled by Ieşan in [6], where, among other things, a theorem of uniqueness is proved; this result, inspired by [7], is however only of *local* type.

Our aim is thus to prove a non-local uniqueness theorem for (regular) solutions to the non-linear initial-boundary value problem of the theory at hand. Of course, we do not expect nor propose to find conditions for global uniqueness in the solution space; nevertheless, it should be possible to select subspaces of this in which uniqueness may hold good.

The equations governing the problem are stated in sect. 2, where the main assumptions are also formulated; all these are employed in sect. 3 to derive an energy equality and then to prove the uniqueness theorem.

**2. Basic equations and assumptions.** According to [1, sect. 2, 3, appendix], the motion of an elastic body with voids $\mathcal{B}$ subject to finite deformations, is governed, in the Lagrangian formulation, by the following local balances of momentum and equilibrated force:(*)

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(*) Throughout the paper, we shall use small, small boldface and capital boldface letters to denote scalars, vectors and tensors of order not less than two, respectively.