the distribution $D\oplus\{\xi\}$ is integrable. Then the leaves of $D\oplus\{\xi\}$ are totally geodesic immersed in $\tilde{M}$ if and only if we have $\tilde{H}(X) = H^*(X_*)$ for any $X \in \Gamma(D)$. In particular, if $\tilde{M}(k)$ is a Sasakian space form of constant φ-sectional curvature k and $\pi : M \to N$ is a submersion of semi-invariant submanifold M of $\tilde{M}(k)$ onto the almost contact metric manifold N such that the distribution $D\oplus\{\xi\}$ be integrable, then the leaves of $D\oplus\{\xi\}$ are totally geodesic immersed in $\tilde{M}(k)$ if and only if N is also a Sasakian space form of the same constant φ-sectional curvature k.

REFERENCES

1. Bejancu, A. — *Geometry of CR-Submanifolds*, D. Reidel Publishing Company, Dordrecht, 1986.
2. Bejancu A., Papaghiuc, N. — *Semi-invariant submanifolds of a Sasakian manifold*, An. șt. Univ. Iași, tom. XXVII, s. I a (1981), f. 1, 163 — 170.
3. Bejancu A., Papaghiuc N. — *Semi-invariant submanifolds of a Sasakian space form*, Coloquium Mathematicum, XLVIII (1984), f. 2, 229 — 240.
4. Bejancu A., Papaghiuc, N. — *Normal semi-invariant submanifolds of a Sasakian manifold*, Matematiciki vesnik, 35 (1983), 345 — 356.
5. Chen B. Y. — *CR-submanifolds of a Kaehler manifold*, I, J. Differential Geometry, 16 (1981), 305 — 322.
6. Deshmukh, S., Ali, S., Hussain, S. I. — *Submersions of CR-Submanifolds of a Kaehler manifold*, to appear.
7. Kobayashi, S., — *Submersions of CR-submanifolds*, Tohoku Math. J., 39 (1987), 95—100.
8. O'Neill, B. — *The fundamental equations of a submersion*, Michigan Math. J., 13 (1966), 459 — 469.
9. Papaghiuc, N. — *Semi-invariant products in Sasakian manifolds*, An. șt. Univ. Iași, tom XXX, s. Ia (1984), f. 2, 69 — 78.
10. Papaghiuc, N. — *On normal semi-invariant submanifolds in Sasakian space forms*, An. șt. Univ. Iași, tom XXXI, s. Ia (1985), 69 — 76.
11. Yano, K., Kon M. — *Contact CR-submanifolds*, Kodai Math. J., 5 (1982), 238 — 252.

Received 8. III. 1988

Polytechnic Institute of Iași
Department of Mathematics,
Iași, R.S. Romania

An. șt. Univ. „Al. I. Cuza" Iași
Matematica, 35 (1989)

A UNIQUENESS THEOREM IN THE NON-LINEAR DYNAMICS OF ELASTIC MATERIALS WITH VOIDS

BY

EDOARDO SCARPETTA

1. Introduction. In this note we present a uniqueness result in the context of the finite-deformation theory for *elastic* bodies with *voids*. We consider the dynamic problem of this theory, early introduced by Nunziato & Cowin [1] in order to describe the mechanical behavior of *porous* materials having (non-linear) elastic response in the skeletal structure. The interstices, or *voids*, in this structure are assumed to be lacking in matter and anything else of energetic relevance.

In the field of application of this theory, which of course generalizes the classical elasticity, should fall various kinds of geological materials, such as rock and soils, or porous manufactured materials, such as ceramics and pressed powders. We refer to [1], cf. also [2], for more details and physical insights into the basic concepts of the theory.

Some general theorems for the dynamic problem of elastic bodies with voids have been established, in the linear response case, by Cowin & Nunziato (uniqueness and *weak* stability [3]) and by Ieșan (reciprocity and variational characterization [4]). Chandrasekharaiah [5] also treats uniqueness in the context of a particular formulation.

The non-linear case has been recently tackled by Ieșan in [6], where, among other things, a theorem of uniqueness is proved; this result, inspired by [7], is however only of *local* type.

Our aim is thus to prove a non-local uniqueness theorem for (regular) solutions to the non-linear initial-boundary value problem of the theory at hand. Of course, we do not expect nor propose to find conditions for global uniqueness in the solution space; nevertheless, it should be possible to select subspaces of this in which uniqueness may hold good.

The equations governing the problem are stated in sect. 2, where the main assumptions are also formulated; all these are employed in sect. 3 to derive an energy equality and then to prove the uniqueness theorem.

2. Basic equations and assumptions. According to [1, sect. 2, 3, appendix], the motion of an elastic body with voids $\mathcal{B}$ subject to finite deformations, is governed, in the Lagrangian formulation, by the following local balances of momentum and equilibrated force:(*)

(*) Throughout the paper, we shall use small, small boldface and capital boldface letters to denote scalars, vectors and tensors of order not less than two, respectively.

$$\mathrm{Div}\,\mathbf{T}+\rho\mathbf{b}=\rho\ddot{\mathbf{v}},$$

(1) $$\mathrm{Div}\,\mathbf{h}+g+\rho l=\rho(k\dot{\psi})^{\cdot},\ \text{in}\ \Omega_T\equiv\Omega\times(0,\ T],$$

where $\Omega\equiv$ the interior of $\varkappa(\mathcal{B})\subset\mathbf{R}^3$ and $T\in\mathbf{R}^+$.

In such equations, $\varkappa$ is a fixed reference configuration which maps $\mathcal{B}$ into a smooth bounded domain $(\bar{\Omega})$ of the physical point space (identified with $\mathbf{R}^3$); $\mathbf{T}$ denotes the first Piola-Kirchhooff stress tensor, $\mathbf{h}$ the equilibrated stress vector and g the intrinsic equilibrated body force — these are the dynamical variables of the theory needing a constitutive equation. Moreover, ρ is the *bulk* mass density, k the equilibrated inertia, $\mathbf{b}$ the external body force and l the extrinsic equilibrated body force. Finally, $\mathbf{v}$ and ψ are the independent kinematical variables: the displacement vector field and the change in volume fraction with respect to $\varkappa$ (where, for convenience, the distribution of voids is supposed uniform) [1,3]. This pair $(\mathbf{v},\ \psi)$ characterizes the *motion* of $\mathcal{B}$.

In view of the classical interpretation of system (1), we shall assume throughout that:

$$\rho\in C^0(\bar{\Omega})\,;\ \mathbf{b},\ l,\ g\in C^{0,0}(\bar{\Omega}_T)\,;$$

$$\mathbf{T},\,\mathbf{h}\in C^{1,0}(\bar{\Omega}_T)\,;\ k\in C^{0,1}(\bar{\Omega}_T)\,;\ \ \mathbf{v},\ \psi\in C^{1,2}(\bar{\Omega}_T).$$

Denote now with Lin the class of all second-order tensors. Concerning the constitutive equations for (1), we shall consider $\mathbf{T}$, $\mathbf{h}$ and g as *arbitrary* functions of $\nabla\mathbf{v}$, ψ, $\nabla\psi$(*)

(2) $$\mathbf{T}=\hat{\mathbf{T}}(\nabla\mathbf{v},\ \psi,\ \nabla\psi),\ \ \mathbf{h}=\hat{\mathbf{h}}(\nabla\mathbf{v},\ \psi,\ \nabla\psi),\ \ g=\hat{g}(\nabla\mathbf{v},\ \psi,\ \nabla\psi),$$

and assume $\hat{\mathbf{T}},\ \hat{\mathbf{h}},\ \hat{g}\in C^2(\mathrm{Lin}\times\mathbf{R}\times\mathbf{R}^3)$ with $\nabla_i\hat{\mathbf{T}},\ \nabla_i\hat{\mathbf{h}},\ \nabla_i\hat{g}\in C^{0,1}(\Omega_T)$ and $\nabla_i\nabla_j\hat{\mathbf{T}},\ \nabla_i\nabla_j\hat{\mathbf{h}},\ \nabla_i\nabla_j\hat{g}\in C^{0,0}(\Omega_T)$ ($i,j=1,2,3$ and ∇_i stands for the partial gradient with respect to i-th argument in (2)).

A thermodynamic treatment of the question, in which what above could be regarded as the isothermal case, would show that a (smooth) internal energy $\varepsilon=\hat{\varepsilon}(\nabla\mathbf{v},\ \psi,\ \nabla\psi,\ \ldots$ thermal variables...) can be introduced such that

(3) $$\hat{\mathbf{T}}=\nabla_1\hat{\varepsilon},\ \ \hat{\mathbf{h}}=\nabla_3\hat{\varepsilon},\ \ \hat{g}=-\nabla_2\hat{\varepsilon}.$$

In this connection, see [1]. Along with (2), we bear in mind that for the equilibrated inertia also is given a (kinematical) constitutive equation: $k=\hat{K}(\psi)$ [1,3]. We assume $\hat{K}\in C^3(\mathbf{R})$ with $\partial^n\hat{K}/\partial\psi^n\in C^{0,0}(\Omega_T)$ for $n=1,2,3$.

To equations (1) the following system of initial-boundary conditions of the mixed type is appended:

* We do not account for inelastic surface effects associated with changes in the volume of the voids, that would require in $(2)_3$ a dependence on $\dot{\psi}$ also [8].

$$\mathbf{v}=\mathbf{v}_0,\ \ \dot{\mathbf{v}}=\dot{\mathbf{v}}_0,\ \ \psi=\psi_0,\ \ \dot{\psi}=\dot{\psi}_0\ \text{in}\ \Omega\times\{0\},$$

(4) $$\mathbf{T}\mathbf{n}=\mathbf{t}_\Sigma\ \text{in}\ \partial_1\Omega\times(0,\ T],\ \ \mathbf{h}\cdot\mathbf{n}=h_\Sigma\ \text{in}\ \partial_3\Omega\times(0,\ T],$$

$$\mathbf{v}=\mathbf{v}_\Sigma\ \text{in}\ \partial_2\Omega\times(0,\ T],\ \ \psi=\psi_\Sigma\ \text{in}\ \partial_4\Omega\times(0,\ T],$$

where the right-hand members denote smooth prescribed fields; together with $\mathbf{b}$ and l (see above), these are the *data* of the problem. Moreover, $\partial_i\Omega$ and $\partial_{i+1}\Omega$ $(i=1,3)$ stand for complementary and disjoint subsets of $\partial\Omega$, and $\mathbf{n}$ is the outward unit normal to this.
The loads $\mathbf{b}$, l and $\mathbf{t}_\Sigma$, h_Σ are said of *dead* type, i.e., not depending on the spatial position of the material particles of $\mathcal{B}$.

A montion $(\mathbf{v},\ \psi)$ verifying equations (1), (4) through (2), for some assignment of the data, is referred to as a *regular solution* of the problem.

In the class of the motions of $\mathcal{B}$ we now define a particular subset; to this end, let $\mathcal{H}$ denote the set of all pairs of smooth displacement fields $\mathbf{u}$ and change in volume fraction fields φ such that

$$\mathbf{u}=\mathbf{0}\ \text{on}\ \partial_2\Omega\times(0,\ T],\ \ \varphi=0\ \text{on}\ \partial_4\Omega\times(0,\ T].$$

We shall call $F(\mathcal{B})$ the class of those motions $(\mathbf{v},\ \psi)$ of $\mathcal{B}$ for which the following coercive inequality holds for some $\lambda>0$ and $\forall\ (\mathbf{u},\ \varphi)\in\mathcal{H}$:

$$\int_\Omega[\mathbf{A}\nabla\mathbf{u}:\nabla\mathbf{u}+\mathbf{B}\nabla\varphi\cdot\nabla\varphi+c\varphi^2+2\mathbf{A}'\nabla\varphi:\nabla\mathbf{u}+2\mathbf{B}'\varphi:\nabla\mathbf{u}+2\mathbf{c}'\varphi\cdot\nabla\varphi]\,d\Omega\geq$$

$$\geq\lambda\int_\Omega[|\nabla\mathbf{u}|^2+|\nabla\varphi|^2+\varphi^2]\,d\Omega,$$

where $\mathbf{A}\equiv\nabla_1\hat{\mathbf{T}}\,|_{(\mathbf{v},\psi)}$ (a fourth-order tensor), $\mathbf{B}\equiv\nabla_3\hat{\mathbf{h}}\,|_{(\mathbf{v},\psi)}$ (second-order), $c\equiv-\nabla_2\hat{g}\,|_{(\mathbf{v},\psi)}$, $\mathbf{A}'\equiv\nabla_3\hat{\mathbf{T}}\,|_{(\mathbf{v},\psi)}$ (third-order), $\mathbf{B}'\equiv\nabla_2\hat{\mathbf{T}}\,|_{(\mathbf{v},\psi)}$ (second-order), $\mathbf{c}'\equiv\nabla_2\hat{\mathbf{h}}\,|_{(\mathbf{v},\psi)}$.
In component form, $(\mathbf{A}\nabla\mathbf{u})_{ij}=A_{ijkl}(\nabla\mathbf{u})_{kl}$, $(\mathbf{B}\nabla\varphi)_i=B_{ij}(\nabla\varphi)_j$,

$$(\mathbf{A}'\nabla\varphi)_{ij}=A'_{ijk}(\nabla\varphi)_k,\ \ \mathbf{D}:\mathbf{E}=D_{ij}E_{ij}\ \forall\mathbf{D},\mathbf{E}\in\mathrm{Lin};\ \text{moreover,}\ A_{ijkl}=$$

$$\partial T_{ij}/\partial(\nabla\mathbf{v})_{kl},\ \ B_{ij}=\partial h_i/\partial(\nabla\psi)_j,\ \ A'_{ijk}=\partial T_{ij}/\partial(\nabla\psi)_k,$$

$$B'_{ij}=\partial T_{ij}/\partial\psi,\ \ c'_i=\partial h_i/\partial\psi.$$

Cf. [6] for a similar assumption in the same context; we only remark that in the linear theory, where $\hat{\mathbf{T}}$, $\hat{\mathbf{h}}$, $\hat{g}$ are *linear* in their arguments, $\mathbf{A}$, $\mathbf{B}$, c, $\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{c}'$ are calculated in the reference configuration, and the left side of the above inequality reduces to the *potential energy* of $\mathcal{B}$ along the motion $(\mathbf{u},\ \varphi)$.

Note finally that, in view of (3), it is $(\nabla_1\hat{g}\,|_{(\mathbf{v},\psi)})_{ij}=-B'_{ij}$,

$$(\nabla_1\hat{\mathbf{h}}\,|_{(\mathbf{v},\psi)})_{ijk}=A'_{jki},\ \ (\nabla_3\hat{g}\,|_{(\mathbf{v},\psi)})_i=-c'_i\ \ \text{and moreover}$$

$$A_{ijkl}=A_{klij},\ \ B_{ij}=B_{ji}\ \text{in}\ \Omega_T.$$

3. Energy equation and uniqueness theorem. We now aim to lay down an energy balance equation that will be basic for uniqueness. Let then $(\mathbf{v},\ \psi)$ and $(\mathbf{v}+\mathbf{u},\ \psi+\varphi)$ be two regular solutions of problem (1), (2), (4) with the same data. Write simply $\mathbf{T}$ for $\hat{\mathbf{T}}(\nabla \mathbf{v}, \psi, \nabla\psi)$ and $\mathbf{T}'$ for $\hat{\mathbf{T}}(\nabla\mathbf{v}+\nabla\mathbf{u}, \psi+\varphi, \nabla\psi+\nabla\varphi)$, and likewise for $\mathbf{h}$ and g; also, k for $\hat{K}(\psi)$ and k' for $\hat{K}(\psi+\varphi)$. By difference, from (1) we get:

$$\mathrm{Div}\,(\mathbf{T}'-\mathbf{T})=\rho\ddot{\mathbf{u}},$$

$$\mathrm{Div}\,(\mathbf{h}'-\mathbf{h})+(g'-g)=\rho[(\ddot{k}'-\ddot{k})\dot\psi+(k'-k)\ddot\psi+(k'\dot\varphi)^{\cdot}],$$

with $(\mathbf{T}'-\mathbf{T})\,\mathbf{n}=\mathbf{0}$ on $\partial_1\Omega\times(0, T]$, $(\mathbf{h}'-\mathbf{h})\cdot\mathbf{n}=0$ on $\partial_2\Omega\times(0, T]$, and $\mathbf{u}=\dot{\mathbf{u}}=\varphi=\dot\varphi=0$ on $\Omega\times\{0\}$; further, $(\mathbf{u}, \varphi)$ belongs to $\mathscr{H}$.

Now, multiplication of the foregoing equations for $\dot{\mathbf{u}}$ and $\dot\varphi$, respectively, and integration over $\Omega\times(0, t)$, $t\in(0, T)$, give (via divergence theorem):

$$1/2\int_\Omega(\rho\dot{\mathbf{u}}^2+\rho k'\dot\varphi^2)\,(t)=-\int\int_{\Omega\times(0,t)}\{1/2\,\rho\dot{k}'\dot\varphi^2(\tau)\,d\tau+\rho[(\dot{k}'-\dot{k})\,\dot\psi\dot\varphi+$$

$$+(k'-k)\ddot\psi\dot\varphi]\}\,(\tau)\,d\tau-\int\int_{\Omega\times(0,t)}[(\mathbf{T}'-\mathbf{T}):\nabla\dot{\mathbf{u}}+ \tag{5}$$

$$+(\mathbf{h}'-\mathbf{h})\cdot\nabla\dot\varphi-(g'-g)\dot\varphi](\tau)d\tau.$$

For simplicity, we explicitly indicate only the time-dependence of the fields.

This equation is the starting point for the following:

Theorem of uniqueness. *Provided* $\rho, k\geqslant c_0>0$, *the problem* (1), (2), (4) *has at most one regular solution belonging to* $F(\mathscr{B})$.

Proof. Let $(\mathbf{v}, \psi)$ and $(\mathbf{v}+\mathbf{u}, \psi+\varphi)$ be as above and belong to $F(\mathscr{B})$: consider equation (5). By a second-order Taylor expansion (cf. [10]):

$$(\mathbf{T}'-\mathbf{T}):\nabla\dot{\mathbf{u}}=\mathbf{A}\nabla\mathbf{u}:\nabla\dot{\mathbf{u}}+\mathbf{B}'\varphi:\nabla\dot{\mathbf{u}}+\mathbf{A}'\nabla\varphi:\nabla\dot{\mathbf{u}}+\mathbf{u}:\nabla\dot{\mathbf{u}},$$

$$(\mathbf{h}'-\mathbf{h})\cdot\nabla\dot\varphi=\mathbf{A}'\nabla\dot\varphi:\nabla\mathbf{u}+\mathbf{B}\nabla\varphi\cdot\nabla\dot\varphi+\mathbf{e}'\varphi\cdot\nabla\dot\varphi+\mathbf{v}\cdot\nabla\dot\varphi,$$

$$(g'-g)\,\dot\varphi=-\mathbf{B}':\nabla\mathbf{u}\dot\varphi-\mathbf{e}'\cdot\nabla\varphi\,\dot\varphi-c\varphi\,\dot\varphi+z\dot\varphi,$$

where (*) $|\mathbf{u}|, |\mathbf{v}|, |z|\ll|\nabla\mathbf{u}|^2+|\varphi|^2+|\nabla\varphi|^2$ in Ω_T.
A similar argument can be used to yield $|k'-k|\ll|\varphi|+|\varphi|^2$ and $|\dot{k}'-\dot{k}|\ll|\varphi|+|\dot\varphi|+|\varphi|^2$
We thus may write:

$$1/2\int_\Omega(\rho\dot{\mathbf{u}}^2+\rho k'\dot\varphi^2)\,(t)+1/2\int_\Omega[\mathbf{A}\nabla\mathbf{u}:\nabla\mathbf{u}+\mathbf{B}\nabla\varphi\cdot\nabla\varphi+c\varphi^2+2\varphi\mathbf{B}':\nabla\mathbf{u}+$$

$$+2\mathbf{A}'\nabla\varphi:\nabla\mathbf{u}+2\mathbf{e}'\varphi\cdot\nabla\varphi]\,(t)=-\int\int_{\Omega\times(0,t)}\{1/2\,\rho\dot{k}'\dot\varphi^2(\tau)\,d\tau+\rho[(\dot{k}'-\dot{k})\,\dot\psi\dot\varphi+$$

$$+(k'-k)\,\ddot\psi\dot\varphi]\}\,(\tau)\,d\tau+1/2\int\int_{\Omega\times(0,t)}[\dot{\mathbf{A}}\nabla\mathbf{u}:\nabla\mathbf{u}+\dot{\mathbf{B}}\nabla\varphi\cdot\nabla\varphi+\dot{c}\varphi^2+2\varphi\dot{\mathbf{B}}':\nabla\mathbf{u}+$$

(*) In what follows by „$a\ll b$“ we mean that a positive constant s can be computed such that $a\leqslant sb$.

$$+2\dot{\mathbf{A}}'\nabla\varphi:\nabla\mathbf{u}+2\dot{\mathbf{e}}'\varphi\cdot\nabla\varphi]\,(\tau)\,d\tau-\int\int_{\Omega\times(0,t)}(\mathbf{u}:\nabla\dot{\mathbf{u}}+\mathbf{v}\cdot\nabla\dot\varphi+z\dot\varphi)\,(\tau)\,d\tau\ll$$

$$\int\int_{\Omega\times(0,t)}\ll(|\varphi|^2+|\dot\varphi|^2+|\nabla\mathbf{u}|^2+|\nabla\varphi|^2)\,(\tau)\,d\tau.$$

In the last step, Cauchy inequality and the maxima on $\bar\Omega_T$ of various fields have been taken into account.

Recall now the definition of $F(\mathscr{B})$, so that, since $(\mathbf{u}, \varphi)\in\mathscr{H}$,

$$\int_\Omega[(\dot{\mathbf{u}}^2+\dot\varphi^2)+(|\nabla\mathbf{u}|^2+|\varphi|^2+|\nabla\varphi|^2)]\,(t)$$

$$\ll\int\int_{\Omega\times(0,t)}(|\dot\varphi|^2+|\nabla\mathbf{u}|^2+|\varphi|^2+|\nabla\varphi|^2)\,(\tau)\,d\tau,$$

where c_0 is also included in the understood positive constant.

The thesis is then achieved by noting that well-known Gronwall lemma [11, p. 286] implies the vanishing of the left side above for arbitrary $t\in(0, T)$, and thus we conclude that $\mathbf{u}\equiv\mathbf{0}$, $\varphi\equiv0$ in Ω_T. Q.E.D.

Achnowledgment. This paper has been performed under the auspices of G.N.F.M. of the Italian Research Council (C.N.R.).

REFERENCES

1. Nunziato J. W. and Cowin S. C. — *A non-linear theory of elastic materials with voids*, Arch. Rat. Mech. Anal. 72 (1979), 175.
2. Goodman M. A. and Cowin S. C. — *A continuum theory for granular materials*, Arch. Rat. Mech. Anal. 44 (1972), 249.
3. Cowin S. C. and Nunziato J. W. — *Linear elastic materials with voids*, Journal of Elasticity, 13 (1983), 125.
4. Ieşan D. — *Some results in the theory of elastic materials with voids*, Journal of Elasticity, 15 (1985), 215.
5. Chandrasekharaiah D. S. — *A uniqueness theorem in the theory of elastic materials with voids*, Journal of Elasticity, 18 (1987), 173.
6. Ieşan D. — *Continuous dependence, uniqueness and existence theorems in nonlinear theory of elastic-materials with voids*, Rev. Roum. Sci. Techn. Mec. Appl. 32 n. 4. (1987) 351.
7. Dafermos C. M. — *The second law of thermodynamics and stability*. Arch. Rat. Mech. Anal. 70 (1979), 167.
8. Band W. — *Local heating in shocked granular materials*, Stanford Research Institute, Internal Report 002-67 (1967).
9. Ieşan D. — *A theory of thermoelastic materials with voids*, Acta Mechanica, 60 (1986), 67.
10. Wheeler L. and Nachlinger R. R. — *Uniqueness theorems for finite elastodynamics*, Journal of Elasticity, 4 (1974) 27.
11. Dieudonné J. — *Foundations of Modern Analysis*, New York: Academic Press (1969).

Received 29. VIII. 1988

Istituto di Fisica Matematica ed Informatica
Università di Salerno, 84100 Salerno, Italia