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POLYNOMIAL SOLUTIONS IN THE AXIAL COORDINATE FOR ELASTIC BEAMS LOADED ON THE LATERAL SURFACE. UNIQUENESS OF SOLUTION FOR THE GENERAL PROBLEM OF THE BEAMS

BY

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We consider an orthotropic beam limited by two planes $x_3=0$, $x_3=h$ $(h>0)$ and by a cylindrical surface $\mathcal{F}$ having the generators parallel to x_3 axis. We will denote by $\mathcal{V}$ the domain occupied by the beam and also the volume of this domain; by S we denote the domain and also the area of a cross-section of the beam. The boundary of S will be denoted by ∂S.

1. Formulation of the problem. The material of the beam is homogeneus with the following constitutive equations

$$(1.1)\qquad \begin{gathered} t_{ii}=c_{1i}\varepsilon_{11}+c_{2i}\varepsilon_{22}+c_{3i}\varepsilon_{33} \quad (\text{n.s.})^{*}), \\ t_{23}=c_{44}\varepsilon_{23},\quad t_{31}=c_{55}\varepsilon_{31},\quad t_{12}=c_{66}\varepsilon_{12}, \end{gathered}$$

where t_{ij} are the stress components, $c_{ij}=c_{ji}$ are constants which characterize the elastic properties of the material of the beam and ε_{ij} are the strain components given by

$$\varepsilon_{ii}=u_{i,i}\ (\text{n. s.}),\quad \varepsilon_{ij}=u_{i,j}+u_{j,i}\ (i\neq j).$$

The inverse relations of (1.1) are taken in the form

$$(1.2)\qquad \begin{gathered} E\varepsilon_{\alpha\alpha}=\nu_{1\alpha}t_{11}+\nu_{2\alpha}t_{22}-\nu_{\alpha}t_{33}\ ^{**}), \\ E\varepsilon_{33}=-\nu_1 t_{11}-\nu_2 t_{22}+t_{33}, \\ c_{44}\varepsilon_{23}=t_{23},\quad c_{55}\varepsilon_{31}=t_{31},\quad c_{66}\varepsilon_{12}=t_{12}, \end{gathered}$$

where E, $\nu_{\alpha\beta}$, ν_α may be expressed in terms of moduli of elasticity [2, 9].

We suppose that there are no body forces so that the equilibrium equations are given by

$$(1.3)\qquad t_{ij,j}=0 \text{ in } \mathcal{V}.$$

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* We use the summation convention over repeated indices (excepting the indication n. s.). The index j after comma indicates partial differentiation with respect to x_j. All Latin indices take the values 1, 2, 3 and all Greek indices take the values 1, 2.

** It is obvious that here and in any analogous case we have not summation over α.

The tractions applied on the lateral surface are

$$t_{i3}n_3=\sum_{k=0}^{n}\tau_i^{(k)}(x_1, x_2)x_3^k \text{ on } \mathcal{F}, \tag{1.4}$$

where the functions τ_i (...) are given and $\mathbf{n}(n_1, n_2, 0)$ is the exterior normal to the surface $\mathcal{F}$. On the ends the beam is loaded in such a way that it is in equilibrium.

The problem of loaded beams on the lateral surface was considered, for the first time, by Michell [11] who has solved the problem (1.1)–(1.4) for the orthotropic beams when $n=0$ and Almansi [1] who has solved the problem (1.1)–(1.4) for the isotropic beams with arbitrary n. The problem (1.1)–(1.4) when $n=0$ is known as Almansi-Michell problem and when the lateral surface tractions have polynomial forms with respect to the axial coordinate the problem is known as Almansi problem.

Till now a lot of papers were published concerning the beams loaded on the lateral surface, in the isotropic case and for various types of anisotropies. Among them, we should like to point out the paper [5]–[11].

In the mentioned paper, generally, by a semi-inverse method, are finds solutions which satisfy the conditions on the lateral surface. The end conditions are satisfied by superposing the solution of a problem of Saint-Venant.

Our purpose, for such a beam, is to look for the most general solution of the equations (1.1)–(1.4) which is polynomial in x_3, then

$$u_i=\sum_{k=0}^{N}U_i^{(k)}(x_1, x_2)x_3^k=\sum_{s=0}^{N}U_i^{(N-s)}x_3^{N-s}, \tag{1.5}$$

where $U_i^{(k)}(...)$ and the integer N must be determined.

2. General equations. If we denote by $\varepsilon_{\alpha3}^{(k)}$ the strains of the displacements $(U_1^{(k)}, U_2^{(k)}, 0)$ and by $t_{\alpha3}^{(k)}$ the corresponding stresses, then we have

$$\varepsilon_{\alpha3}=\sum\varepsilon_{\alpha3}^{(k)}x_3^k,\quad \varepsilon_{i3}=\sum kU_i^{(k)}x_3^{k-1}+\sum U_{3,i}^{(k)}\ x_3^k$$

and

$$\begin{gathered} t_{ii}=\sum t_{ii}^{(k)}x_3^k+c_{i3}\sum kU_3^{(k)}x_3^{k-1},\quad t_{12}=\sum t_{12}^{(k)}x_3{}^k, \\ t_{\alpha3}=c_{\omega+3,\,\omega+3}[\sum kU_\alpha^{(k)}\ x_3^{k-1}+\sum U_{3,\alpha}^{(k)}\ x_3^k] \quad (\omega\neq\alpha). \end{gathered} \tag{2.1}$$

The equilibrium equations (1.3) and the boundary conditions (1.4) will be satisfied if and only if, for arbitrary k, we have

$$t_{\alpha\beta,\beta}^{(k)}+(k+1)\{c_{\alpha3}U_{3,\alpha}^{(k+1)}+c_{\omega+3,\,\omega+3}[U_{3,\alpha}^{(k+1)}+(k+2)\ U_\alpha^{(k+2)}]\}=0 \text{ in } S, \tag{2.2}$$

$$t_{\alpha3}^{(k)}n_3=\tau_\alpha^{(k)}-c_{\alpha3}\,(k+1)\ U_3^{(k+1)}n_\alpha \quad \text{on } \partial S\ (\omega\neq\alpha) \tag{2.3}$$

and

$$\Delta U_3^{(k)}+(k+1)[c_{55}U_{1,1}^{(k+1)}+c_{44}U_{2,2}^{(k+1)}+t_{33}^{(k+1)}+c_{33}(k+2)U_3^{(k+2)}]=0 \text{ in } S, \tag{2.4}$$

$$DU_3^{(k)}=\tau_3^{(k)}-(k+1)[c_{55}U_1^{(k+1)}n_1+c_{44}U_2^{(k+1)}n_2] \text{ on } \partial S, \tag{2.5}$$

where

$$\Delta=c_{55}\frac{\partial^2}{\partial x_1^2}+c_{44}\frac{\partial^2}{\partial x_2^2},\quad D=c_{55}n_1\frac{\partial}{\partial x_1}+c_{44}n_2\frac{\partial}{\partial x_2}, \tag{2.6}$$

$$\tau_i^{(n+p)}=U_i^{(N+p)}=0 \quad (p=1, 2, ...).$$

The problem (2.2), (2.3) is a plane problem with body forces of the type

$$t_{\alpha\beta,\beta}+X_\alpha=0 \text{ in } S,\quad t_{\alpha\beta}n_\beta=p_\alpha \text{ on } \partial S. \tag{2.7}$$

The necessary and sufficient conditions for the existence of a solution for (2.7) are given by

$$\begin{gathered} \int_{\partial S}p_\alpha ds+\int_S X_\alpha d\sigma=0, \\ \int_{\partial S}(x_1p_2-x_2p_1)\,ds+\int_S(x_1X_2-x_2X_1)\,d\sigma=0. \end{gathered} \tag{2.8}$$

For the problem (2.2), (2.3) the conditions (2.8) become

$$\begin{gathered} \int_{\partial S}\tau_\alpha^{(k)}ds+(k+1)\int_S c_{\omega+3,\,\omega+3}[U_{3,\alpha}^{(k+1)}+(k+2)\,U_\alpha^{(k+2)}]\,d\sigma=0, \\ \int_{\partial S}(x_1\tau_2^{(k)}-x_2\tau_1^{(k)})ds+(k+1)\int_S[c_{44}x_1U_{3,2}^{(k+1)}-c_{55}x_2U_{3,1}^{(k+1)}]\,d\sigma+ \\ +(k+1)\,(k+2)\int_S[c_{44}x_1U_2^{(k+2)}-c_{55}x_2U_1^{(k+2)}]\,d\sigma=0. \end{gathered} \tag{2.9}$$

The problem (2.4), (2.5) is of the Neumann type. The necessary and sufficient condition for the existence of a solution riquires that

$$\int_{\partial S}\tau_3^{(k)}ds+(k+1)\int_S[t_{33}^{(k+1)}+c_{33}(k+2)\ U_3^{(k+2)}]\,d\sigma=0. \tag{2.10}$$

In our problem the function φ of torsion defined by

$$\Delta\varphi=0 \text{ in } S,\ D\varphi=c_{55}x_2n_1-c_{44}x_1n_2 \text{ on } \partial S \tag{2.11}$$

has a special role.

We can prove that [2, 9]

$$\begin{gathered} \int_S c_{55}(\varphi_{,1}-x_2)\,d\sigma=0,\quad \int_S c_{44}(\varphi_{,2}+x_1)\,d\sigma=0, \\ D_t=\int_S[c_{44}x_1(x_1+\varphi_{,2})+c_{55}x_2(x_2-\varphi_{,1})]\,d\sigma>0 \text{ (torsional rigidity).} \end{gathered} \tag{2.12}$$

For any function ψ, which satisfies the ecuation $\Delta\psi=0$, we have

$$\int_{\partial S}U_3^{(k)}D\psi ds=\int_{\partial S}\psi DU_3^{(k)}ds-\int_S\psi\Delta U_3^{(k)}\,d\sigma.$$

Taking by turn $\psi=x_1$, x_2, φ we can deduce the following useful formulae

$$\begin{gathered} \int_S c_{\omega+3,\,\omega+3}[U_{3,\alpha}^{(k)}+(k+1)\ U_\alpha^{(k+1)}]\,d\sigma=\int_{\partial S}x_\alpha\tau_3^{(k)}ds+(k+1)\int_S[t_{33}^{(k+1)}+ \\ +(k+2)\ U_3^{(k+2)}]\,x_\alpha d\sigma, \end{gathered} \tag{2.13}$$

$$\begin{gathered} \int_S[c_{55}x_2U_{3,1}^{(k)}-c_{44}x_1U_{3,2}^{(k)}]\,d\sigma=\int_{\partial S}\varphi\tau_3^{(k)}ds+ \\ +(k+1)\int_S[\varphi t_{33}^{(k+1)}-c_{55}\varphi_{,1}U_1^{(k+1)}-c_{44}\varphi_{,2}U_2^{(k+1)}]\,d\sigma+ \\ +c_{33}(k+1)\,(k+2)\int_S\varphi U_3^{(k+2)}\,d\sigma. \end{gathered} \tag{2.14}$$

3. Auxiliary plane problems. The solution of our problem is clossely connected with three plane strain problems. Denoting by $\tau_{\alpha\beta}^{(i)}$ $(i=1,\ 2,\ 3)$ the stresses, the mentioned problems are defined by

$$\tau_{\alpha\beta,\beta}^{(i)}-\delta_{i\alpha}c_{\alpha s}=0 \text{ in } S,\quad \tau_{\alpha\beta}^{(i)}n_\beta=c_{\alpha s}\,x^{(i)}n_\alpha \text{ on } \partial S \text{ (n. s. } \alpha), \tag{3.1}$$

where $x^{(1)}=x_1,\ x^{(2)}=x_2,\ x^{(3)}=1,\ \delta_{i\alpha}$ is Kronecker's delta.

The conditions (2.8) are satisfied for the problems (3.1).

We have

$$\tau_{\alpha\alpha}^{(i)}=c_{\alpha 3}x^{(i)},\quad \tau_{12}^{(i)}=0,\quad \tau_{33}^{(i)}=(c_{33}-E)\,x^{(i)}\,; \tag{3.2}$$

$$\begin{aligned} u_1^{(1)}&=\frac{1}{2}(\nu_1x_1^2-\nu_2x_2^2),\quad u_1^{(2)}=\nu_1x_1x_2,\quad u_1^{(3)}=\nu_1x_1,\\ u_2^{(1)}&=\nu_2x_1x_2,\quad u_2^{(2)}=-\frac{1}{2}(\nu_1x_1^2-\nu_2x_2^2),\quad u_2^{(3)}=\nu_2x_2. \end{aligned} \tag{3.3}$$

4. The solution of the problem. In order to solve our problem we will use the same method as in [4]. Consequently, we must solve the problems (2.2), (2.3) and (2.4), (2.5) for $k=N,\ N-1,\ N-2,\dots$ $(s=1,\ 2\dots)$. In the following, we will denote by $A_k^{(\alpha)}$, C_k, r_k arbitrary constants.

$k=N$: The equations (2.2)–(2.5), in this case, are

$$t_{\alpha\beta,\beta}^{(N)}=0,\quad \Delta U_3^{(N)}=0 \text{ in } S,$$

$$t_{\alpha\beta}^{(N)}n_\beta=0,\quad DU_3^{(N)}=0 \text{ on } \partial S.$$

It results

$$U_1^{(N)}=A_N^{(1)}-r_Nx_2,\quad U_2^{(N)}=A_N^{(2)}+r_Nx_1,\quad U_3^{(N)}=-C_N. \tag{4.1}$$

$k=N-1$: The corresponding problems are

$$t_{\alpha\beta,\beta}^{(N-1)}=0,\quad \Delta U_3^{(N-1)}=0 \text{ in } S,$$

$$t_{\alpha\beta}^{(N-1)}n_\beta=NC_Nc_{\alpha 3}n_\alpha,$$

$$DU_3^{(N-1)}=-N[c_{55}(A_N^{(1)}-r_Nx_2)\,n_1+c_{44}(A_N^{(2)}+r_Nx_1)\,n_2] \text{ on } \partial S.$$

The solutions of these problems are

$$\begin{aligned} &t_{\alpha\alpha}^{(N-1)}=NC_Nc_{\alpha 3},\quad t_{12}^{(N-1)}=0,\quad t_{33}^{(N-1)}=NC_N(c_{33}-E),\\ &U_\alpha^{(N-1)}=N[C_N\nu_\alpha x_\alpha+A_{N-1}^{(\alpha)}+(-1)^\alpha r_{N-1}\,x_\omega]\ (\omega\neq\alpha),\\ &U_3^{(N-1)}=Nr_N\varphi-N(A_N^{(1)}x_1+A_N^{(2)}x_2+C_{N-1}). \end{aligned} \tag{4.2}$$

$k=N-2$: We have

$$\begin{aligned} &t_{\alpha\beta,\beta}^{(N-2)}+N(N-1)\{c_{\alpha 3}(r_N\varphi_{,\alpha}-A_N^{(\alpha)})+r_Nc_{\omega+3,\ \omega+3}[\varphi_{,\alpha}+(-1)^\alpha x_\omega]\}=0 \text{ in } S,\\ &t_{\alpha\beta}^{(N-2)}n_\beta=-N(N-1)c_{\alpha 3}[r_N\varphi-(A_N^{(1)}x_1+A_N^{(2)}x_2+C_{N-1})]\,n_\alpha \text{ on } \partial S\quad (\omega\neq\alpha) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &\Delta U_3^{(N-2)}+N(N-1)\,(c_{55}\nu_1+c_{44}\nu_2-E)\,C_N=0 \text{ in } S,\\ &DU_3^{(N-2)}=-N(N-1)[c_{55}(C_N\nu_1x_1+A_{N-1}^{(1)}-r_{N-1}x_2)\,n_1+\\ &\quad +c_{44}(C_N\nu_2x_2+A_{N-1}^{(2)}+r_{N-1}x_1)\,n_2] \quad \text{on } \partial S. \end{aligned} \tag{4.4}$$

Taking into account (2.9), for the problem (4.3), we find that $r_N=0$. The existence condition of the function $U_3^{(N-2)}$ requires $C_N=0$.

It results

$$t_{ii}^{(N-2)}=N(N-1)(c_{i3}-\delta_{i3}E)\,(A_N^{(1)}x_1+A_N^{(2)}x_2+C_{N-1}),\ t_{12}^{(N-2)}=0\,; \tag{4.5}$$

$$\begin{aligned} &U_\alpha^{(N-2)}=N(N-1)[A_N^{(1)}u_\alpha^{(1)}+A_N^{(2)}u_\alpha^{(2)}+C_{N-1}\,u_\alpha^{(3)}+A_{N-2}^{(\alpha)}+(-1)^\alpha r_{N-2}\,x_\omega],\\ &U_3^{(N-2)}=N(N-1)[r_{N-1}\,\varphi-(A_{N-1}^{(1)}x_1+A_{N-1}^{(2)}x_2+C_{N-2})]. \end{aligned} \tag{4.6}$$

$k=N-3$: The equations (2.2), (2.3) and (2.4), (2.5) are

$$\begin{aligned} &t_{\alpha 3,\beta}^{(N-3)}+N(N-1)\,(N-2)\{c_{\alpha 3}(r_{N-1}\varphi_{,\alpha}-A_{N-1}^{(\alpha)})+\\ &\quad +r_{N-1}\,c_{\omega+3,\ \omega+3}[\varphi_{,\alpha}+(-1)^\alpha x_\omega]\}=0 \text{ in } S,\\ &t_{\alpha 3}^{(N-3)}n_\beta=-N(N-1)\,(N-2)\,c_{\alpha 3}[r_{N-1}\varphi-(A_{N-1}^{(1)}x_1+\\ &\quad +A_{N-1}^{(2)}x_2+C_{N-2})]\,n_\alpha \text{ on } \partial S \end{aligned} \tag{4.7}$$

and

$$\Delta U_3^{(N-3)}+N(N-1)\,(N-2)\,(c_{55}\nu_1+c_{44}\nu_2-E)\,(A_N^{(1)}x_1+$$
$$+A_N^{(2)}x_2+C_{N-1})=0 \text{ in } S.$$

$$\begin{aligned} DU_3^{(N-3)}=&-N(N-1)(N-2)[c_{55}(A_N^{(1)}\,u_1^{(1)}+A_N^{(2)}u_1^{(2)}+C_{N-1}u_1^{(3)}+\\ &+A_{N-2}^{(1)}-r_{N-2}x_2)\,n_1+c_{44}(A_N^{(1)}u_2^{(1)}+A_N^{(2)}u_2^{(2)}+C_{N-1}u_2^{(3)}+A_{N-2}^{(2)}+\\ &+r_{N-2}\,x_1)\,n_2] \text{ on } \partial S. \end{aligned} \tag{4.8}$$

The problem (4.7) has solution if and only if $r_{N-1}=0$. From the existence condition of the function $U_3^{(N-3)}$ we find

$$\int_S (A_N^{(1)}x_1+A_N^{(2)}x_2+C_{N-1})\,d\sigma=0. \tag{4.9}$$

The solution are given by

$$t_{ii}^{(N-3)}=N(N-1)(N-2)(c_{i3}-E\delta_{i3})(A_{N-1}^{(1)}x_1+A_{N-1}^{(2)}x_2+C_{N-2}),\ t_{12}^{(N-3)}=0\,; \tag{4.10}$$

$$\begin{aligned} U_\alpha^{(N-3)}=&N(N-1)\,(N-2)\,[A_{N-1}^{(3)}u_\alpha^{(3)}+C_{N-2}u_\alpha^{(3)}+A_{N-3}^{(\alpha)}+\\ &+(-1)^\alpha r_{N-3}x_\omega] \qquad (\omega\neq\alpha), \end{aligned} \tag{4.11}$$

$$U_3^{(N-3)}=N(N-1)\,(N-2)\,[V_3^{(N-3)}-(A_{N-2}^{(3)}x_3+C_{N-3})], \tag{4.12}$$

where $V_3^{(N-3)}$ depends on the constants $A_N^{(\alpha)}$, C_{N-1}, r_{N-2} satisfing (4.9)

$k=N-4$: In this case the equations (2.2), (2.3) are

$$\begin{aligned} &t_{\alpha 3,\beta}^{(N-4)}+(N-3)\{c_{\alpha 3}U_{3,\alpha}^{(N-3)}+c_{\omega+3,\ \omega+3}[U_{3,\alpha}^{(N-3)}+(N-2)\,U_\alpha^{(N-2)}]\}=0 \text{ in } S,\\ &t_{\alpha 3}^{(N-4)}n_\beta=\tau_\alpha^{(N-4)}-(N-3)c_{\alpha 3}U_3^{(N-3)}n_\alpha \text{ on } \partial S\ (\omega\neq\alpha). \end{aligned} \tag{4.13}$$

Firstly, we must have $N-4=n$. Contrarily, in addition to $r_N=C_N=0$ we should have also $A_N^{(\alpha)}=0$.

From the existence conditions of the problem (4.13), by using the formulae (2.13), (2.14), we get, at the last

$$E\int_S \left(A^{(1)}_{n+4}x_1 + A^{(2)}_{n+4}x_2 + C_{n+3}\right) x_\alpha d\sigma = \frac{n\,!}{(n+4)\,!}\int_{\partial S} \tau_\alpha^{(n)}\, ds. \tag{4.14}$$

$$\begin{aligned} D_t r_{n+2} = {} & \frac{n\,!}{(n+4)\,!}\int_{\partial S} [x_2\tau_1^{(n)} - x_1\tau_2^{(n)}]\, ds - \int_S \{c_{44}(\varphi_{,2}+x_1)[A^{(3)}_{n+4}u_2^{(3)} + \\ & + C_{n+3}u_2^{(3)}] + c_{55}(\varphi_{,1}-x_2)[A^{(3)}_{n+4}u_1^{(3)} + C_{n+3}u_1^{(3)}]\}\, d\sigma - E\int_S \varphi(A^{(3)}_{n+4}x_3 + C_{n+3})\, d\sigma. \end{aligned} \tag{4.15}$$

The formulae (4.9), (4.14), (4.15) allow us to determine the constants $A_N^{(\alpha)}$, C_{N-1}, r_{N-2} and after its determination the component $U_3^{(N-3)}$ from (4.12) and the solution of problem (4.13) are given by

$$\begin{aligned} U_3^{(n+1)} &= \frac{(n+4)\,!}{(n+1)\,!}[V_3^{(n+1)} - (A^{(3)}_{n+2}x_3 + C_{n+1})], \\ U_\alpha^{(n)} &= \frac{(n+4)\,!}{n\,!}[A^{(3)}_{n+2}u_\alpha^{(3)} + C_{n+1}u_\alpha^{(3)} + A_n^{(\alpha)} + (-1)^\alpha r_n x_\omega + V_\alpha^{(n)}]\ (\omega\neq\alpha). \end{aligned} \tag{4.16}$$

where $V_3^{(n+1)}$, $V_\alpha^{(n)}$ depend on the determined constants.

The structure (4.16) of the solutions for the steps $n-s$ ($s=\overline{1, n}$), remains the same, one changes the arbitrary constants only. We can prove this for, $s=1, 2, 3$. In order to prove that the rule is true for any s let us consider as valid the formulae

$$U_3^{(n-s+3)} = \frac{(n+4)\,!}{(n-s+3)\,!}[V_3^{(n-s+3)} - (A^{(3)}_{n-s+4}x_3 + C_{n-s+3})],$$

$$\begin{aligned} U_\alpha^{(n-s+2)} &= \frac{(n+4)\,!}{(n-s+2)\,!}[A^{(3)}_{n-s+4}u_\alpha^{(3)} + C_{n-s+3}u_\alpha^{(3)} + A^{(\alpha)}_{n-s+2} + (-1)^\alpha r_{n-s+2}x_\omega + V_\alpha^{(n-s+2)}], \\ U_3^{(n-s+2)} &= \frac{(n+4)\,!}{(n-s+2)\,!}[V_3^{(n-s+2)} - (A^{(3)}_{n-s+3}x_3 + C_{n-s+2})], \\ t_{33}^{(n-s+2)} &= \frac{(n+4)\,!}{(n-s+2)\,!}[(c_{33}-E)(A^{(3)}_{n-s+4}x_3 + C_{n-s+3}) + T_{33}^{(n-s+2)}], \\ U_\alpha^{(n-s+1)} &= \frac{(n+4)\,!}{(n-s+1)\,!}[A^{(3)}_{n-s+3}u_\alpha^{(3)} + C_{n-s+3}u_\alpha^{(3)} + A^{(\alpha)}_{n-s+3} + (-1)^\alpha r_{n-s+2}x_\omega + V_\alpha^{(n-s+1)}], \end{aligned} \tag{4.17}$$

where the function $V_i^{(\cdot)}$ and $T_{33}^{(n-s+2)}$ depend only on the constants determined in the previous steps.

In the next step, for the problem (2.4), (2.5) we have

$$\begin{aligned} & \Delta U_3^{(n-s+1)} + (n-s+2)[c_{55}U_{1,1}^{(n-s+2)} + c_{44}U_{2,2}^{(n-s+2)} + t_{33}^{(n-s+2)} + \\ & \qquad + c_{33}(n-s+3)\, U_3^{(n-s+3)}] = 0 \text{ in } S, \\ & DU_3^{(n-s+1)} = \tau_3^{(n-s+1)} - (n-s+2)[c_{55}U_1^{(n-s+2)}n_1 + c_{44}U_2^{(n-s+2)}n_2] \text{ on } \partial S \end{aligned} \tag{4.18}$$

and the problem (2.2), (2.3) is given by

$$\begin{aligned} & t_{\alpha 3,3}^{(n-s)} + (n-s+1)\{c_{\alpha 3}U_{3,\alpha}^{(n-s+1)} + c_{\omega+3,\,\omega+3}[U_{3,\alpha}^{(n-s+1)} + \\ & \qquad + (n-s+2)\, U_\alpha^{(n-s+2)}]\} = 0 \text{ in } S, \\ & t_{\alpha 3}^{(n-s)}n_3 = \tau_\alpha^{(n-s)} - (n-s+1)\, c_{\alpha 3}U_3^{(n-s+1)}n_\alpha \text{ on } \partial S\ (\omega\neq\alpha). \end{aligned} \tag{4.19}$$

By using the formulae (2.13), (2.14), from the condition (2.10) and (2.9) we find that the problems (4.18) and (4.19) have solutions if and only if the constants $A^{(\alpha)}_{n-s+1}$, C_{n-s+3}, r_{n-s+2} are chosen such that

$$E\int_S (A^{(3)}_{n-s+4}x_3 + C_{n-s+3})\, d\sigma = \frac{(n-s+1)\,!}{(n+4)\,!}\int_{\partial S}\tau_3^{(n-s+1)}ds + \int_S [T_{33}^{(n-s+2)} + c_{33}V_3^{(n-s+3)}]\, d\sigma.$$

$$\begin{aligned} E\int_S (A^{(3)}_{n-s+4}x_3 + C_{n-s+3})\, x_\alpha\, d\sigma = {} & \frac{(n-s)\,!}{(n+4)\,!}\int_{\partial S}[\tau_\alpha^{(n-s)} + (n-s+1)x_\alpha\tau_3^{(n-s+1)}]\, ds + \\ & + \int_S [T_{33}^{(n-s+2)} + c_{33}V_3^{(n-s+3)}]\, x_\alpha d\sigma. \end{aligned}$$

$$\begin{aligned} D_t r_{n-s+2} = {} & \frac{(n-s)\,!}{(n+4)\,!}\int_{\partial S}[x_2\tau_1^{(n-s)} - x_1\tau_2^{(n-s)} + (n-s+1)\,\varphi\tau_3^{(n-s+1)}]\, ds - \\ & - \int_S \{c_{55}(\varphi_{,1}-x_2)[A^{(3)}_{n-s+1}u_1^{(3)} + C_{n-s+3}u_1^{(3)} + V_1^{(n-s+2)}] + \\ & + c_{44}(\varphi_{,2}+x_1)[A^{(3)}_{n-s+4}u_2^{(3)} + C_{n-s+3}u_2^{(3)} + V_2^{(n-s+2)}] + \\ & + \varphi[T_{33}^{(n-s+2)} + c_{33}V_3^{(n-s+3)} - E(A^{(3)}_{n-s+4}x_3 + C_{n-s+3})]\}\, d\sigma. \end{aligned}$$

The solutions of the problems (4.18) and (4.19) are given by

$$U_3^{(n-s+1)} = \frac{(n+4)\,!}{(n-s+1)\,!}[V_3^{(n-s+1)} - (A^{(3)}_{n-s+2}x_3 + C_{n-s+1})]. \tag{4.20}$$

$$\begin{aligned} U_\alpha^{(n-s)} &= \frac{(n+4)\,!}{(n-s)\,!}[A^{(3)}_{n-s+2}u_\alpha^{(3)} + C_{n-s+1}u_\alpha^{(3)} + A^{(\alpha)}_{n-s} + (-1)^\alpha r_{n-s}x_\omega + V_\alpha^{(n-s)}], \\ t_{33}^{(n-s)} &= \frac{(n+4)\,!}{(n-s)\,!}[(c_{33}-E)\,(A^{(3)}_{n-s+2}x_3 + C_{n-s+1}) + T_{33}^{(n-s)}]. \end{aligned} \tag{4.21}$$

Thus, the formulae (4.20), (4.21) are valid for any $s=\overline{0, n}$.

Taking into account the above results we can write the solution of our problem. All the functions and all constants $A^{(\alpha)}_{k+1}$, C_{k+3}, r_{k+2} ($k=\overline{0, n}$) are determined. The constants $A_3^{(\alpha)}$, C_2 are connected by the relation

$$E\int_S (A_3^{(1)}x_1 + A_3^{(2)}x_2 + C_2)\, d\sigma = \frac{1}{(n+4)\,!}\int_{\partial S}\tau_3^{(0)}\, ds + \int_S [T_{33}^{(1)} + c_{33}V_3^{(2)}]\, d\sigma$$

and the constants $A_2^{(\alpha)}$, C_1, r_1 are arbitrary; $A_1^{(\alpha)}$, $A_0^{(\alpha)}$, C_0, r_0 generate a rigid displacement and, cosequently, they can by omitted.

Then, the solution of our problem depends on six arbitrary constants only. The arbitrary constants can be uniquely determined if we add the end conditions

$$R_i = -\int_S t_{i3} d\sigma, \quad M_\alpha = (-1)^\alpha \int_S x'_\omega t_{33} d\sigma, \tag{4.22}$$
$$M_3 = \int_S (x_2 t_{13} - x_1 t_{23})\, d\sigma \text{ on } x_3 = 0,$$

where R_i are given components of the resultant tractions and M_i are given components of the resultant moment of the applied tractions on the end $x_3 = 0$.

It results that the problem (1.1)–(1.5), (4.22) has a unique solution.

The problem defined by (1.1)–(1.5), (4.22) is the *general problem of the beams*. Its solution will be called *standard solution* and the corresponding tractions on the ends, *standard repartition*.

On account of the Saint-Venant principle we take this solution as solution for all tractions on the end $x_3 = 0$ which satisfy (4.22).

Making use of (4.1), (4.2), (4.6), (4.20), (4.21) we can write the form of the solution. If we introduce the terms corresponding to constants $A_k^{(\alpha)}$ (which interviene in $U_\alpha^{(k)}$) in $U_\alpha^{(k-2)}$ and the term from $U_3^{(k)}$, corresponding to $-(A_{k+1}^{(1)} x_1 + A_{k+1}^{(2)} x_2 + C_k)$, in $U_3^{(k-1)}$ and make also some modifications of notations, the solution of the problem (1.1)–(1.5), (4.22) may be written in the form

$$u_\alpha = \sum_{k=0}^{n+2} \left[a_k^{(1)} u_\alpha^{(1)} + (-1)^\alpha r_k x_\omega + a_k^{(\alpha)} \frac{x_3^2}{(k+1)\ (k+2)} \right] x_3^k + \sum_{k=0}^{n} V_\alpha^{(k)} x_3^k,$$
$$u_3 = -\sum_{k=0}^{n+2} \frac{1}{k+1} (a_k^{(1)} x_1 + a_k^{(2)} x_2 + a_k^{(3)})\, x_3^{k+1} + \sum_{k=0}^{n+1} V_3^{(k)} x_3^k \quad (i = 1, 2, 3;\ \omega \neq \alpha).$$

which coincides with the form used in [3].

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W. A. COPPEL: *A Chinese-English Mathematics Primer*, Canberra, Australian National University, 1989, Proceedings of the Centre for Mathematical Analysis, Australian National University, Volume 23/1989, VI + 134 p.

Aşa cum afirmă şi autorul, materialul din această lucrare poate face obiectul unui curs de iniţiere în chineza matematicii. Nu avem de-a face cu un substitut de dicţionar — în ciuda listei de 1 800 de cuvinte şi a indexului de 950 de caractere — ci, mai degrabă, cu un supliment de dicţionar. De altfel, autorul recomandă cercetarea în paralel a „abecedarului" său cu "The Pinyin Chinese-English Dictionary". Cartea de faţă nu este nici o lucrare cu ambiţia de a fi exhaustivă, ci doar una care să arate calea pe care se poate ajunge la decodarea unui text matematic scris în limba chineză. Cartea cuprinde o „Introducere în limba chineză" cu: cunoştinţe de scriere în chineză, transliteraţia pinyin şi o schiţă de gramatică chineză, toate avînd ca teren de aplicaţie chineza standard. În partea a doua, se găsesc: cheia listelor de cuvinte, listele de cuvinte (de la 1 la 82) şi un index care are ca principiu de structurare numărul de trăsături de penson necesare pentru desenarea pictogramelor (de la 1 la 20 de trăsături). Este de remarcat faptul că ordonarea cuvintelor în liste nu este făcută în maniera aleatorie a dicţionarelor, ci urmînd o caracteristică comună legată, de pildă, de centrul de interes căruia îi aparţin cuvintele. Aceasta facilitează memorarea termenilor, ei repetîndu-se destul de frecvent într-un text de specialitate.

În concluzie, putem spune că „abecedarul" profesorului Coppel este un instrument foarte util pentru matematicienii care vor să depăşească limitele impuse de scrierea în limba chineză în accesul la informaţia ştiinţifică.

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W. MENZEL, I. NEUMANN, W. ZIMMERMANN (Hrsg.): *Nicht-Monotone Logik*, Seminar an der Universität Karlsruhe 1988, GMD-Studien nr. 156, Gesellschaft fur Mathematik und Datenverarbeitung mbH, St. Augustin, 1989, 51 p.

Volumul cuprinde 4 lucrări prezentate in cadrul seminarului de logică nemonotonă organizat de către Universitatea din Karlsruhe în 1988. Lucrările se bazează pe articole de logică nemonotonă apărute în revista „Artificial Intelligence" 13 (1980). Ele prezintă contribuţii şi extensii ale rezultatelor şi abordărilor prezentate în acele articole. Subiectele tratate în lucrări sînt logica nemonotonă a lui Mc Dermott şi Doyle, logica raţionamentelor cu excepţii şi informaţii implicite (default reasoning) a lui Reiter şi teoria circumscrierilor — o teorie a raţionamentului nemonoton dezvoltată de Mc Carthy.

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A. C. ALBU: *Introducere in fundamentele matematicii, Partea I: Probleme ale fundamentelor logicii. Teorii axiomatice*; Universitatea din Timişoara, Monografii matematice 34—35, Tipografia Universităţii din Timişoara, 1989, XV + 321 p.

Cartea cuprinde: Logica matematică naivă (Logica propoziţiilor, Logica predicatelor, Aplicaţii ale logicii — probleme metodologice), Sisteme axiomatice (Sisteme axiomatice semiformalizate, Sisteme formale), Logica formală (Calculul pro-