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SOME REMARKS ON EXISTENCE THEOREMS FOR PARETO PROBLEMS

BY

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Introduction. Existence theorems for Pareto minima have been largely investigated: we limit ourselves to quote the ones proved in [1–5].

Here we wish to compare existence results obtained by the authors in [1] with the interesting one proved by C o r l e y in [2]. To this end let us briefly recall the notations and the general setting of the problem which are used in both the works, because they may be different, we also make some preliminar simplifications which allow us an easy comparison.

First of all we wish to remark that we constantly refer to the terminology about cones used in [3]: particularly we use the concept of acuteness and of pointedness of a cone as it is stated there and we wish to emphasize that this does not agree with the terminology used in [2] where the same name stands for a different concept.

Secondly we recall that Corley uses a convex cone which is not necessarily closed, although he assumes the pointedness of its closure. For sake of simplicity we shall limit ourselves to consider only closed convex cones.

As soon as we agree on previous assumptions, we may summarize Corley's results as follows: *let V be a Banach space and let $K \subset V$ be a pointed closed convex cone; then every K-lower semicontinuous function assumes at least a Pareto minimum on a compact set.*

The setting in [1] is slight different. We define a Pareto infimum, which we prove to be not-empty, and we show that: *if $K \subset V$ is an acute closed convex cone, every function which satisfies a different kind of K-lower semicontinuity, assumes all its Pareto infima on a compact set.* We see at once that, although our result is sharper (it assures that all Pareto infima are assumed), it needs more assumptions, first af all because we suppose our cone acute and secondly because a comparison between the two kinds of K-lower semicontinuity shows that they are equivalent in finite dimensional spaces, but generally speaking, Corley's semicontinuity is weaker that our one.

Here we intend to show first of all that pointedness and acuteness of cones are equivalent in a very large class of Banach spaces and secondly we define a weaker kind of K-lower semicontinuity which is still sufficient to prove an existence result of the type proved in [1]. In this way we obtain a sharper result of the type proved in [1] under the weaker assumptions of [2].

Some properties of cones. Let V be a Banach space and let V' be its strong dual space. Let $K\subset V$ be a closed convex cone and let K^0 be its polar cone.

It is well known that K^0 and K^{00} are themselves closed convex cones defined as

$$K^0=\{p\in V':\langle p,\ v\rangle\le 0\ \ \forall v\in K\}\,.$$

$$K^{00}=\{p\in V'':\langle p,\ v'\rangle\le 0\ \ \forall p\in K^0\}\,.$$

Generally $K\subset K^{00}$ and, when V is reflexive, $K=K^{00}$.

Following [3] we say that

Definition 1. *The cone K is pointed when*

$$K\cap(-K)=\{0\}\,.$$

The cone K is acute when

$$\exists p\in K^0:\langle p,\ x\rangle<0\ \ \forall x\in K\setminus\{0\}.$$

We also say that a cone K is bidually pointed when

$$K^{00}\cap(-K^{00})=\{0\}.$$

It's a simple matter to prove that

Lemma 2. *A closed convex cone $K\subset V$ is pointed if and only if*

$$w^*-\mathrm{cl}\ \mathrm{aff}\,(K^0)=w^*-\mathrm{cl}\,(K^0-K^0)=V'$$

(where we use $\mathrm{aff}(K^0)$ to indicate the affine hull of K^0 and $w^*-\mathrm{cl}$ to indicate the weak closure).*

Moreover it is bidually pointed if and only if

$$\mathrm{cl}\ \mathrm{aff}(K^0)=\mathrm{cl}(K^0-K^0)=V'.$$

Obviously when V is a reflexive space every pointed cone is also bidually pointed.

Proof. The cone K is pointed if and only if

$$K\cap(-K)=\{0\}.$$

Now $K\cap(-K)\neq\{0\}$ if and only if there is $v\in V$, $v\neq 0$ such that

$$R=\{\lambda v:\lambda\in\mathbf{R}\}\subset K$$

but the latter is true if and only if

$$R^0=\{p\in V':\langle p,\ v\rangle=0\}\supset K^0$$

that is if and only if there is a closed hyperplane $H=R^0$ in V' such that $H\supset K^0$ i.e. (by corollary 5.3.4 in [7]) if and only if $w^*-\mathrm{cl}\ \mathrm{aff}\ (K^0)\neq V'$.

The second assertion can be proven analogously.

Y u also proved in [5] that

Theorem 3. *Let V be a finite dimensional Banach space and let $K\subset V$ be a closed convex cone. Then K is acute if and only if it is pointed.*

We are able to prove the same result also for a large class of infinite dimensional Banach spaces which we are going to introduce.

Definition 4. [6]-*Let V be a Banach space: $T\subset V'$ is a total subset for V' when*

$$v\in V,\ \langle t,\ v\rangle=0\ \ \forall t\in T\Rightarrow v=0.$$

It is well known that [6, 7]

Theorem 5. *A subset $T\subset V'$ is a total subset for V' if and only if*

$$w^*-\mathrm{cl}\ \mathrm{aff}(T)=V'$$

i.e. when the linear hull of T is w^*-dense in V'.

In the following pages we shall need to deal with a space V whose dual space V' has a countable total subset.

To simplify our notations we set the following

Definition 6. *A Banach space V is said to be countably separated if its dual space has a countable total subset $T=\{p_n:n\in\mathbf{N}\}$.*

Theorem 5 assures that every Banach spaces V whose dual space V' is w^*-separable is countably separated: hence every space V with strongly separable dual space V' is countably separated too. Moreover we can easily verify that most of commonly used Banach spaces such as $L^1(0,1)$ or l^1 which have not strongly separable dual spaces, are countably separated.

Indeed let $V=l^1$ the space of all real sequences a_k such that

$$\|a_k\|_1=\sum_{k=1}^{+\infty}|a_k|<+\infty$$

We have that $V'=l^\infty$ is the space of all real bounded sequences a_k such that

$$\|a_k\|_\infty=\sup\{|a_k|:k\in\mathbf{N}\}$$

Clearly l^∞ is not a strongly separable space, but if we choose

$$T=\{(p_k)_n:n\in\mathbf{N}\}$$

defined by

$$(p_k)_n=\begin{cases}1, & \text{if } k=n\,;\\ 0, & \text{if } k\neq n.\end{cases}$$

it is easily verified that T is a total subset of V'; indeed if we take $(a_k)\in l^1$ such that

$$0=\langle a_k,\ (p_k)_n\rangle=\sum_{k=1}^{+\infty}a_k(p_k)_n=a_n\ \ \forall n\in\mathbf{N}$$

we have $a_k=0$.

Similarly if we take $V=L^1(0,1)$, the usual Lebesgue space of real summable functions on $[0,1]$, we have $V'=L^\infty(0,1)$, the space of all Lebesgue measurable real functions on $[0,1]$ which are essentially bounded and we can consider the set $T\subset V'$

$$T=\{p_\tau:\tau\in\mathbf{Q}\cap[0,1]\}$$

defined by

$$p_\tau(t)=\begin{cases} 1, & \text{if } t\in[0,\ \tau], \\ 0, & \text{if } t\notin[0,\ \tau]; \end{cases}$$

T is a total subset in V' because is we take $v\in L^1(0,\ 1)$, we have

$$0=\langle p_\tau,\ v\rangle=\int_0^1 p_\tau(t)v(t)dt=\int_0^\tau v(t)dt \qquad \forall \tau\in Q\cap[0,1].$$

So if we set

$$w(t)=\int_0^t v(s)ds$$

we have

$$w(\tau)=0 \quad \forall\tau\in Q\cap[0,\ 1]$$

and since w is absolutely continuous it is the null function so that

$$\dot{w}=v=0.$$

Let us now prove that, when V is countably separated, acuteness and pointedness are equivalent.

Theorem 7. *Let V be a countably separated Banach space and let $K\subset V$ be a convex closed cone. Then K is bidually pointed if and only if it is acute.*

Proof. First of all we observe that, obviously, acuteness implies pointedness.

Conversely let us suppose that K is a pointed cone and let us verify that it is acute.

Since V is countably separated, there is a countable total subset

$$T=\{p_n : n\in\mathbf{N}\}$$

in V'; moreover, since $\mathrm{cl}\,(K^0-K^0)=\mathrm{cl\,aff}(K^0)=V'$, for every $p_n\in T$ we can find a sequence $p_n^k\in K^0-K^0$ strongly convergent to p_n.

Let us denote

$$T_1=\{p_n^k : n\in\mathbf{N},\ k\in\mathbf{N}\}:$$

then T_1 is a countable set, we can set

$$T_1=\{\xi_n : n\in\mathbf{N}\}\ \text{ where }\ \xi_n\in K^0-K^0,$$

and we have

$$\xi_n=\alpha_n-\beta_n \quad \alpha_n,\ \beta_n\in K^0.$$

Let's define

$$p_0=\sum_{n=1}^{\infty}\left(\frac{\alpha_n}{2^n\|\alpha_n\|}+\frac{\beta_n}{2^n\|\beta_n\|}\right)\in K^0$$

and let us prove that

$$\langle p_0,\ v\rangle<0 \quad \forall v\in K\setminus\{0\}.$$

Indeed we have

$$\langle p_0,\ v\rangle=\sum_{n=1}^{\infty}\frac{\langle\alpha_n,\ v\rangle}{2^n\|\alpha_n\|}+\sum_{n=1}^{\infty}\frac{\langle\beta_n,\ v\rangle}{2^n\|\beta_n\|}\leq 0 \qquad \forall v\in K$$

because $\langle\alpha_n,\ v\rangle,\ \langle\beta_n,\ v\rangle\leq 0\ \ \forall v\in K$.

So, if $\langle p_0,\ v\rangle=0$, we should have

$$\langle\alpha_n,\ v\rangle,\ \langle\beta_n,\ v\rangle=0 \quad \forall n\in\mathbf{N}$$

and

$$\langle\xi_n,\ v\rangle=0 \quad \forall n\in\mathbf{N}.$$

Therefore we can assert that

$$\langle p_n^k,\ v\rangle=0 \qquad \forall n,\ k\in\mathbf{N}$$

and

$$\langle p_n,\ v\rangle=0 \quad \forall n\in\mathbf{N}$$

so that $v=0$.

The previous theorem proves that, when V is a countably separated reflexive space, our assumptions on cones are equivalent to the conditions used by Corley.

Moreover it allows us to precise the diagram of implications on cones which is proved in [3].

An existence result. Let us now examine the existence results for Pareto minima in Banach spaces. In this section we shall suppose that V is a countably separated Banach space and that $K\subset V$ is a closed convex bidually pointed cone.

Let's now introduce some notations which we usually adopt [1—4]:

- $\ll$ indicate the order induced in V by the cone K;
- $C(v)$ indicate the subset of all elements in V which are comparable with $v\in V$, i.e.

$$C(v)=(v-K)\cup(v+K);$$

- $G\subset K$ is a subset of generators of K i.e.

$$K=\mathrm{cl}\left\{\sum_{i=1}^{m}\lambda_i g_i,\ \lambda_i\geq 0,\quad g_i\in G,\quad m\in\mathbf{N}\right\};$$

- $G^0\subset K^0$ will be a subset of generators of K^0.

We can always suppose that both G and G^0 are norm bounded, e.g. by 1.

We extend V using two elements $+\infty$ and $-\infty$ which satisfy obvious relations, see [1—4]. We use the symbol V^E to indicate $V\cup\{\pm\infty\}$.

Following [1] we also set

Definition 8. *Let $x_n\in V^E$ be a sequence. We define*

$$x\in K-\lim x_n$$

when one of the following conditions holds

1) $x\in V$ *and* $x_n\to x$ *weakly in* V;
2) $x=+\infty$ *and* $\lim_n(\sup\{\langle x_n,\ p\rangle : p\in -G^0\})=+\infty$
3) $x=-\infty$ *and* $\lim_n$ $(\inf\{\langle x_n,\ p\rangle : p\in -G^0\})=-\infty$.

We remark that these conditions do not exclude each other so that we are led to define

$$\mathrm{LIM}\ x_n = \begin{cases} K-\lim x_n & \text{when it is a singleton} \\ +\infty & \text{otherwise.} \end{cases}$$

We also remark that $K - \lim x_n$ is not a singleton only when it contains both $+\infty$ and $-\infty$.

Now we are able to introduce a concept of Pareto infimum.

Definition 9. *Let $A \subset V^E$, $A \neq \emptyset$; we say that*

$$x_0 \in \mathrm{INF}\ A$$

when

1) $x_0 \ll x \quad \forall x \in A \cap C(x_0)$

2) $\exists x_n \in A \cap C(x_0),\ x_{n+1} \ll x_n,\ \mathrm{LIM}\ x_n = x_0.$

A simple extension of lemma A.1 in [1], which can be found in the appendix, shows that

Lemma 10. *Let $A \subset V^E$, $A \neq \emptyset$, then*

$$\mathrm{INF}\ A \neq \emptyset$$

and

$$\forall x \in A \quad \exists \lambda \in \mathrm{INF}\ A \quad \textit{such that } \lambda \ll x.$$

Definition 9 allows us to consider the following problem:

let

$$\lambda \in \mathrm{INF}\ \{f(x) : x \in X\},$$

is there any $x_0 \in X$ such that

$$f(x_0) = \lambda\ ?$$

Obviously the answer depends upon lower semicontinuity and compactness arguments.

In [1] we used the following kind of lower semicontinuity with respect to the cone K.

Definition 11. *Let us suppose that X is a topological space and let $f : X \to V^E$. We say that f is sequentially K — lsc when*

$$\forall x_k \in X, \quad x_k \to x \ \textit{ such that } \ K - \lim_k f(x_k) \neq \emptyset$$

we have

$$\mathrm{LIM}\ f(x_k) \gg f(x).$$

In [4] we proved that, when $V = \mathbf{R}^n$, and int $K \neq \emptyset$ definition 11 may be stated in the following form: for every $M \in \mathbf{R}^n$

$$\{x \in X : f(x) \ll M\} = \{x \in X : f(x) \in M - K\} = f^{-1}(M - K)$$

is sequentially closed.

The latter condition is the one adopted by Corley to obtain his existence theorem. An easy example shows that Corley's semicontinuity can be strictly weaker than our one. It can be stated in the following form.

Let $X = \{1/n : n \in \mathbf{N}\} \cup \{0\}$ with the topology induced by the euclidean topology of $\mathbf{R}$; let $V = l^2$ be the space of all real sequences a_k such that

$$\| a_i \|_2^2 = \sum_{k=1}^{+\infty} a_k^2 < +\infty;$$

let $e_n \in l^2$ be the sequence which is identically 0 except for the n-th term which is set to 1; let $K \subset V$ be the non-negative orthant.

Let's also define $f : X \to V$ by

$$f(t) = \begin{cases} e_n & \text{if } t = 1/n \\ e_1 & \text{if } t = 0. \end{cases}$$

We see at once that

$$\mathrm{LIM}\ f\left(\frac{1}{n}\right) = 0 \ll e_1 = f(0)$$

so that f is not K-lower semicontinuous in the sense of definition 11.

On the contrary f has closed levels because

$$t \in f^{-1}(\alpha - K) \quad \alpha \in l^2$$

iff

$$\langle f(t),\ e_i \rangle \leq (\alpha,\ e_i \rangle \quad \forall_i \in \mathbf{N}.$$

Since

$$\alpha = \sum_{i=1}^{+\infty} e_i(\alpha,\ e_i\rangle = \sum_{i=1}^{+\infty} \alpha_i e_i \ \textit{ with } \ \sum_{i=1}^{+\infty} \alpha_i^2 < +\infty$$

we have:

$$\frac{1}{k} \in f^{-1}(\alpha - K) \Leftrightarrow 1 \leq \alpha_k, \quad 0 \leq \alpha_i, \quad i \neq k$$

while

$$0 \in f^{-1}(\alpha) \Leftrightarrow 1 \leq \alpha_1, \quad 0 \leq \alpha_i, \quad i \neq 1.$$

We can easily deduce that, as $\alpha_k \to 0$, only a finite number of elements is contained in $f^{-1}(\alpha - K)$, which is closed.

We wish to remark that the preceeding example can be modified defining $f(t) = +\infty \ \ \forall t \in (0, 1] \setminus \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$, without any change in its semicontinuity properties.

We can also assert that the implication $(4.V) \Rightarrow (1.1.V)$ of [4] is no longer true when we deal with infinite dimensional spaces and with empty interior cones.

To obtain a kind of lower semicontinuity as weaker as possible we can set the following

Definition 12. *Let X be a topological space and let us consider $f : X \to V^E$, we say that f is K-lower semicontinuous if*

$$\forall x_k \in X,\ x_k \to x,\ f(x_{k+1}) \ll f(x_k),$$

$$K - \lim_k f(x_k) \neq \emptyset \ \textit{ we have } \ \mathrm{LIM}\ f(x_k) \gg f(x).$$

Clearly the only difference is that in definition 12 we consider only sequences x_k such that $f(x_k)$ is decreasing.

As to compactness conditions we use the following one.

Let

$$f_p(x)=\langle p, f(x)\rangle, \quad \forall p\in -K^0$$

and let's define

$$\Phi(x)=\sup\{f_p(x) : \|p\|\leq 1, \ p\in -K^0\}$$

and let us consider the following condition:

(*) $\{x:\Phi(x)\leq M\}$ *is relatively sequentially compact in* X.

Exactly as in the Appendix of [1] we can prove that:

Theorem 13. *Let* $f: X\to V^E$ *a K-lower semicontinuous function which satisfies condition* (*)*: then*

$$\forall \lambda\in \mathrm{INF}\ A, \quad \exists \bar{x}\in X, \quad f(\bar{x})=\lambda.$$

Theorem 13, in this formulation, covers cases which are out of range of Corley's result also when finite valued functions are involved.

Indeed if we consider $X=(0,1]$, $f: X\to \mathbf{R}^2$ defined by

$$f(t)=(t, -t)$$

we can easily see that theorem 13 is still applicable but it is not possible to apply Corley's result.

We conclude with a remark: condition (*) is, e.g., true when we can find $\bar{p}\in -K^0 : \|\bar{p}\|\leq 1$ such that

$$\{x : f_{\bar{p}}(x)\leq M\}$$

is a sequentially compact set; indeed

$$\{x : \Phi(x)\leq M\}\subset\{x : f_{\bar{p}}(x)\leq M\}.$$

Appendix.

Lemma A. 1 — *Let* $A\subset V^E$, $A\neq 0$, *then* $\mathrm{INF}\ A\neq 0$ *and for every* $u\in A$ *there is* $v\in\mathrm{INF}\ A$ *such that* $v\ll u$.

Proof. Let

$$A_0=A\cap(u-K).$$

Two cases are possible: A_0 is a bounded or a unbounded set.

When A_0 is unbounded, let us show that $-\infty\in\mathrm{INF}\ A$. Indeed we can choose $u_n\in A_0$ such that $\|u_n\|\to+\infty$ and we also have

$$\langle p, u_n\rangle\ll\langle p, u\rangle \quad \forall p\in(-K^0).$$

If $\inf\{\langle p, u_n\rangle : p\in(-G^0)\}$ were a bounded sequence, since we have $\mathrm{claff}\ K^0 = V'$, by the uniform boundedness theorem we should have $\|u_n\|\leq M$ and this is not possible.

So we may choose a subsequence $u_{n'}$ such that

$$\inf\{\langle p, u_{n'}\rangle : p\in(-K^0)\}\to -\infty$$

and $-\infty\in\mathrm{INF}\ A$.

Let us now consider the case when A_0 is a bounded set. Let $p\in V'$ as in definition 1 and let $p_0=-p$: let

$$u_1\in A_0 \text{ such that } \langle p_0, u_1\rangle\leq\inf\{\langle p_0, u\rangle : u\in A_0\}+1;$$

let

$$A_n=A\cap(u_n-K)$$

and let us choose

$$u_{n+1}\in A_n$$

such that

$$\langle p_0, u_{n+1}\rangle\leq\inf\{\langle p_0, u\rangle : u\in A_n\}+1/(n+1).$$

Clearly we have

$$u_{n+1}\ll u_n\ll\ldots\ll u$$

and, since A_0 is a bounded set and $\langle u_n, p\rangle$ is a monotone sequence $\forall p\in K^0$, we have

$$\langle u_n, p\rangle\to\alpha_p \quad \forall p\in K^0-K^0.$$

Since K^0-K^0 is a norm dense subset of V', by [7] theorem 9.3.4, we have that $u_n\to v$ weakly in V.

Since $u_n-u_k\in K$, for $k>n$, taking $k\to+\infty$ we obtain

$$u_n-v\in K \text{ and } v\ll u_n \ \forall n.$$

So condition 2) of definition 9 is satisfied and we only must prove that there is no $w\in A$, $w\ll v$, $w\neq v$.

Let us suppose that there is such an element w. We should have $v-w\in K\setminus\{0\}$ and

$$\langle p_0, v-w\rangle>0$$

whence

$$\langle p_0, w\rangle=\langle p_0, v\rangle-a \text{ with } a>0.$$

On the other hand $w\ll v\ll u_n$ $\forall n$, $w\in A$, and so $w\in A_n$ $\forall n$; so that

$$\langle p_0, w\rangle=\langle p_0, v\rangle-a\ll\langle p_0, u_{n+1}\rangle-a\leq$$
$$\leq\inf\{\langle p_0, u\rangle : u\in A_n\}+1/(n+1)-a<$$
$$<\inf\{\langle p_0, u\rangle : u\in A_n\}\leq\langle p_0, w\rangle$$

for n sufficiently large. And this is false.

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ON SOME TYPES OF SECOND ORDER CONVEXITY

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Introduction. Recently, some classes of generalized convex functions were introduced in order to state and prove duality results of higher order in non linear programming. Let $D\subset \mathbf{R}^n$ be a nonempty open set. The class of continuous functions $f: D\to\mathbf{R}$ whose p^{th} order partial derivatives $(p\geqslant 1)$ exist and are continuous is denoted by C^p; $d^k f(x)$ denotes the differential of order k of f at x.

Let $f\in C^2$ and $x\in D$. Using the terminology of [1], we say that

(i) f is *bonvex* at x if for all $u\in D$ and $p\in\mathbf{R}^n$

$$f(u)-f(x)\geqslant d^1f(x)(u-x)+d^2f(x)(u-x,p)-{}^1/_2 d^2f(x)(p,p); \tag{1}$$

(ii) f is *strictly bonvex* at x if for all $u\in D\setminus\{x\}$, $p\in\mathbf{R}^n$

$$f(u)-f(x)>d^1f(x)(u-x)+d^2f(x)(u-x,p)-{}^1/_2 d^2f(x)(p,p); \tag{2}$$

(iii) f is *pseudo-bonvex* at x if for all $u\in D$, $p\in\mathbf{R}^n$

$$d^1f(x)(u-x)+d^2f(x)(u-x,p)\geqslant 0\Rightarrow f(u)\geqslant f(x)-{}^1/_2 d^2f(x)(p,p); \tag{3}$$

(iv) f is *strictly pseudo-bonvex* at x if for all $u\in D\setminus\{x\}$, $p\in\mathbf{R}^n$

$$d^1f(x)(u-x)+d^2f(x)(u-x,p)\geqslant 0\Rightarrow f(u)>f(x)-{}^1/_2 d^2f(x)(p,p); \tag{4}$$

v) f is *quasi-bonvex* at x if for all $u\in D$, $p\in\mathbf{R}^n$

$$f(u)\leqslant f(x)-{}^1/_2 d^2f(x)(p,p)\Rightarrow d^1f(x)(u-x)+d^2f(x)(u-x,p)\leqslant 0. \tag{5}$$

f is said to be *bonvex* (*strictly bonvex*, ...) if f is bonvex (strictly bonvex, ...) at every $x\in D$.

The notions defined above were introduced by Bector and Bector in [1]. The class of functions satisfying (1) for all x, $u\in D=\mathbf{R}^n$, $p\in\mathbf{R}^n$ was introduced by Mond [6]. Mond showed that every convex quadratic function is bonvex (second order convex in the terminology used in [3], [6]). The class of functions satisfying (3) and (5) for all x, u, p where introduced in [3] and [6] and called second order pseudo-convex and second order quasi-convex, respectively.

In this note we characterize these classes of functions and show that every C^3-bonvex function is convex quadratic. It follows that there are not C^3-strictly bonvex functions.