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Received 26. IV. 1988
Revised 20. X. 1988

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Matematica, 35 (1989)

ON SOME TYPES OF SECOND ORDER CONVEXITY

BY

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Introduction. Recently, some classes of generalized convex functions were introduced in order to state and prove duality results of higher order in non linear programming. Let $D\subset\mathbf{R}^n$ be a nonempty open set. The class of continuous functions $f: D\to\mathbf{R}$ whose p^{th} order partial derivatives $(p\geqslant 1)$ exist and are continuous is denoted by C^p ; $d^k f(x)$ denotes the differential of order k of f at x.

Let $f\in C^2$ and $x\in D$. Using the terminology of [1], we say that

(i) f is *bonvex* at x if for all $u\in D$ and $p\in\mathbf{R}^n$

$$f(u)-f(x)\geqslant d^1f(x)\,(u-x)+d^2f(x)\,(u-x,\,p)-{}^1\!/_2 d^2f(x)\,(p,\,p)\,; \tag{1}$$

(ii) f is *strictly bonvex* at x if for all $u\in D\setminus\{x\}$, $p\in\mathbf{R}^n$

$$f(u)-f(x)>d^1f(x)\,(u-x)+d^2f(x)\,(u-x,\,p)-{}^1\!/_2 d^2f(x)\,(p,\,p)\,; \tag{2}$$

(iii) f is *pseudo-bonvex* at x if for all $u\in D$, $p\in\mathbf{R}^n$

$$d^1f(x)\,(u-x)+d^2f(x)\,(u-x,\,p)\geqslant 0\Rightarrow f(u)\geqslant f(x)-{}^1\!/_2 d^2f(x)\,(p,\,p)\,; \tag{3}$$

(iv) f is *strictly pseudo-bonvex* at x if for all $u\in D\setminus\{x\}$, $p\in\mathbf{R}^n$

$$d^1f(x)\,(u-x)+d^2f(x)\,(u-x,\,p)\geqslant 0\Rightarrow f(u)>f(x)-{}^1\!/_2 d^2f(x)\,(p,\,p)\,; \tag{4}$$

v) f is *quasi-bonvex* at x if for all $u\in D$, $p\in\mathbf{R}^n$

$$f(u)\leqslant f(x)-{}^1\!/_2 d^2f(x)\,(p,\,p)\Rightarrow d^1f(x)\,(u-x)+d^2f(x)\,(u-x,\,p)\leqslant 0. \tag{5}$$

f is said to be *bonvex* (*strictly bonvex*, ...) if f is bonvex (strictly bonvex, ...) at every $x\in D$.

The notions defined above were introduced by Bector and Bector in [1]. The class of functions satisfying (1) for all x, $u\in D=\mathbf{R}^n$, $p\in\mathbf{R}^n$ was introduced by Mond [6]. Mond showed that every convex quadratic function is bonvex (second order convex in the terminology used in [3], [6]). The class of functions satisfying (3) and (5) for all x, u, p where introduced in [3] and [6] and called second order pseudo-convex and second order quasi-convex, respectively.

In this note we characterize these classes of functions and show that every C^2-bonvex function is convex quadratic. It follows that there are not C^2-strictly bonvex functions.

Results. Let $f\in C^2$ and $x\in D$. By $D-x$ we mean $\{u-x : u\in D\}$.

Theorem 1. *f is bonvex at x if and only if $d^2f(x)$ is positive semi-definite and*

$$f(x+h)\geq f(x)+d^1f(x)(h)+{}^1/_2d^2f(x)(h,h)\quad \forall h\in D-x. \tag{6}$$

Proof. Let f be bonvex at x. Taking $u=x$ in (1) we obtain ${}^1/_2d^2f(x)(p,p)\geq 0$ for every $p\in\mathbf{R}^n$. Therefore $d^2f(x)$ is positive semi-definite. Taking now $h\in D-x$, $u=x+h$ and $p=h$ in (1) we obtain (6).

Conversely, suppose that $d^2f(x)$ is positive semi-definite and (6) holds. Fix $h\in D-x$ and consider $g:\mathbf{R}^n\to\mathbf{R}$, $g(p)={}^1/_2d^2f(x)(p,p)-d^2f(x)(h,p)$. g is a convex function and $d^1g(p)=d^2f(x)(p,\cdot)-d^2f(x)(h,\cdot)$. Thus $d^1g(h)=0$, which shows that h minimizes g on $\mathbf{R}^n$. From (6) we have

$$f(x+h)-f(x)\geq d^1f(x)(h)-g(h)\geq d^1f(x)(h)-g(p)$$
$$=d^1f(x)(h)+d^2f(x)(h,p)-{}^1/_2d^2f(x)(p,p)\quad \forall p\in\mathbf{R}^n.$$

Therefore (1) is satisfied.

Corollary 2. *f is strictly bonvex at x if and only if $d^2f(x)$ is positive semi-definite and*

$$f(x+h)>f(x)+d^1f(x)(h)+{}^1/_2d^2f(x)(h,h)\quad \forall h\in D-x,\ h\neq 0. \tag{7}$$

Proof. Let f be strictly bonvex at x. Letting $u\to x$ in (2) we obtain that f is bonvex at x, too. By Theorem 1 we obtain that $d^2f(x)$ is positive semi-definite. Taking $h\in D-x$, $h\neq 0$ and $p=h$ in (2) we get (7). The converse implication follows exactly as in Theorem 1.

In the sequel we shall need the following result.

Lemma 3. *Let $H:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$ be a symmetric positive semi-definite bilinear form, $a\in\mathbf{R}^n$ such that $H(a,a)\neq 0$ and $\alpha\geq 0$. Denote $C=\{p : H(a,p)\geq\alpha\}$ and $E=\{p : H(a,p)>\alpha\}$. Then there exists $p_0=(\alpha/H(a,a))\,a\in C$ such that*

$$p\in C\Rightarrow H(p,p)\geq H(p_0,p_0)=\frac{\alpha^2}{H(a,a)}\ \textit{and}\ p\in E\Rightarrow H(p,p)>\frac{\alpha^2}{H(a,a)}.$$

Proof. Use Schwarz' inequality $(H(x,y))^2\leq H(x,x)\,H(y,y)$.

Theorem 4. *f is pseudo-bonvex at x if and only if $d^2f(x)$ is positive semi-definite and for every $h\in D-x$ we have*

$$d^1f(x)(h)\geq 0\Rightarrow f(x+h)\geq f(x)\ \textit{and} \tag{8}$$

$$d^1f(x)(h)<0,\ d^2f(x)(h,h)\neq 0\Rightarrow f(x+h)\geq f(x)-\frac{1}{2}\frac{(d^1f(x)(h))^2}{d^2f(x)(h,h)}. \tag{9}$$

Proof. Suppose that f is pseudo-bonvex at x. Taking $u=x$ in (3) we obtain ${}^1/_2d^2f(x)(p,p)\geq 0$ for all $p\in\mathbf{R}^n$, i.e. $d^2f(x)$ is positive semi-definite. Fix $h\in D-x$. If $d^1f(x)(h)\geq 0$ then $d^1f(x)(h)+d^2f(x)(h,0)\geq 0$ and so, by (3), $f(x+h)\geq f(x)$. Suppose now that $d^1f(x)(h)<0$ and $d^2f(x)(h,h)\neq 0$. Taking $H=d^2f(x)$, $a=h$ and $\alpha=-d^1f(x)(h)\geq 0$ in Lemma 3, there exists p_0 such that $d^1f(x)(h)+d^2f(x)(h,p_0)=0$ and ${}^1/_2d^2f(x)(p_0,p_0)=\frac{1}{2}\frac{(d^1f(x)(h))^2}{d^2f(x)(h,h)}$.

Now (9) follows taking $p=p_0$ in (3).

Conversely, suppose that $d^2f(x)$ is positive semi-definite and (8) and (9) hold. Let $u\in D$ and $h=u-x$ be fixed and $p\in\mathbf{R}^n$ such that

$$d^1f(x)(h)+d^2f(x)(h,p)\geq 0. \tag{10}$$

If $d^1f(x)(h)\geq 0$ then, by (8), $f(u)\geq f(x)\geq f(x)-{}^1/_2d^2f(x)(p,p)$ (since $d^2f(x)$ is positive semi-definite). If $d^1f(x)(h)<0$ then $d^2f(x)(h,h)\neq 0$. Otherwise $d^2f(x)(h,\cdot)=0$ and (10) can not take place. Once again we apply Lemma 3 with H, a and α as above. As $p\in C$, by (10), we have

$$f(u)=f(x+h)\geq f(x)-\alpha(P)\geq f(x)-{}^1/_2\,d^2f(x)(p,p).$$

Hence (3) holds in this case, too.

Theorem 5. *f is strictly pseudo-bonvex at x if and only if $d^2f(x)$ is positive semi-definite and for every $h\in D-x$, $h\neq 0$ we have*

$$d^1f(x)(h)\geq 0\Rightarrow f(x+h)>f(x),\ \textit{and} \tag{11}$$

$$d^1f(x)(h)<0,\ d^2f(x)(h,h)\neq 0\Rightarrow f(x+h)>f(x)-\frac{1}{2}\frac{(d^1f(x)(h))^2}{d^2f(x)(h,h)}. \tag{12}$$

Proof. Suppose that f is strictly pseudo-bonvex at x. Let us show first that $d^2f(x)$ is positive semi-definite. We may suppose that $d^2f(x)\neq 0$. As $D-x$ is a neighborhood of the origin in $\mathbf{R}^n$, there exists $h\in D-x$ such that $d^2f(x)(h,\cdot)\neq 0$. Then, by (4),

$$p\in\tilde{C}=\{p : d^1f(x)(h)+d^2f(x)(h,p)\geq 0\}\Rightarrow{}^1/_2\,d^2f(x)(p,p)>f(x)-f(x+h)=:\beta.$$

Of course, $\tilde{C}\neq 0$. Let $\bar{p}\in\tilde{C}$ be fixed and $p\in\mathbf{R}^n$ such that $d^2f(x)(h,p)\geq 0$. Then $\bar{p}+tp\in\tilde{C}$ for every $t\geq 0$ and so

$$d^2f(x)(\bar{p}+tp,\bar{p}+tp)=d^2f(x)(\bar{p},\bar{p})+2td^2f(x)(\bar{p},p)+t^2d^2f(x)(p,p)\geq 2\beta\ \forall t\geq 0.$$

Taking the limit as $t\to\infty$ we get $d^2f(x)(p,p)\geq 0$. If $d^2f(x)(h,p)<0$ then $d^2f(x)(h,-p)>0$ and so $d^2f(x)(p,p)=d^2f(x)(-p,-p)\geq 0$. Therefore $d^2f(x)$ is positive semi-definite. The rest of the proof is the same as that of Theorem 4.

Theorem 6. *f is quasi-bonvex at x if and only if $d^2f(x)$ is positive semi-definite and for every $h\in D-x$ we have*

$$d^1f(x)(h)>0\Rightarrow f(x+h)>f(x)\ \textit{and} \tag{13}$$

$$d^1f(x)(h)\leq 0,\ d^2f(x)(h,h)\neq 0\Rightarrow f(x+h)\geq f(x)-\frac{1}{2}\frac{(d^1f(x)(h))^2}{d^2f(x)(h,h)}. \tag{14}$$

Proof. Note that (5) is equivalent to: for every $h\in D-x$, $p\in\mathbf{R}^n$

$$d^1f(x)(h)+d^2f(x)(h,p)>0\Rightarrow f(x+h)>f(x)-{}^1/_2\,d^2f(x)(p,p). \tag{15}$$

Suppose that f is quasi-bonvex at x. Let us first show that $d^2f(x)$ is positive semi-definite. We may assume that $d^2f(x)\neq 0$. As in the proof of Theorem 5, there exists $h\in D-x$ such that $d^2f(x)(h,\cdot)\neq 0$. Let

$$\tilde{E}=\{p\in R^n : d^1f(x)(h)+d^2f(x)(h,p)>0\}.$$

It is obvious that $\tilde{E}\neq\emptyset$ and $\mathrm{cl}\,\tilde{E}=\tilde{C}$ (where $\tilde{C}$ is defined in the proof of Theorem 5). The fact that $d^2f(x)\,(p,p)\geq 0$ for $p\in\mathbf{R}^n$ follows now as in the proof of Theorem 5, taking $\bar{p}\in\tilde{E}$ instead of $\bar{p}\in\tilde{C}$.

Let now $h\in D-x$ be fixed. If $d^1f(x)\,(h)>0$ then $0\in E$ and so, by (15), $f(x+h)\geq f(x)$, i.e. (13) holds. Assume now that $d^1f(x)\,(h)\leq 0$ and $d^2f(x)\,(h,h)\neq 0$. From (18) we have $d^2f(x)\,(p,p)>2(f(x)-f(x+h))$ for $p\in\tilde{E}$, and so

$$p\in\tilde{C}\Rightarrow {}^1\!/_2\, d^2f(x)\,(p,p)\geq f(x)-f(x+h).$$

Applying Lemma 3 for $H=d^2f(x)$, $a=h$, $\alpha=-d^1f(x)\,(h)$, we get

$$\frac{1}{2}\frac{(d^1f(x)\,(h))^2}{d^2f(x)\,(h,h)}\geq f(x)-f(x+h),$$

whence (14) follows.

Conversely, suppose that $d^2f(x)$ is positive semi-definite and (13), (14) hold. Let $h\in D-x$ be fixed and take $p\in\tilde{E}$. If $d^1f(x)\,(h)>0$ then, by (13),

$$f(x+h)>f(x)\geq f(x)-{}^1\!/_2\, d^2f(x)\,(p,p),$$

i.e. (15) holds in this case. If $d^1f(x)\,(h)\leq 0$ then $d^2f(x)\,(h,h)\neq 0$ (otherwise $\tilde{E}=\emptyset$). Applying once again Lemma 3 we have that for every $q\in\tilde{E}$,

$$\frac{1}{2}\, d^2f(x)\,(q,q)>v(P)=\frac{1}{2}\frac{(d^1f(x)\,(h))^2}{d^2f(x)\,(h,h)},$$

whence, by (14),

$$f(x+h)\geq f(x)-\frac{1}{2}\frac{(d^1f(x)\,(h))^2}{d^2f(x)\,(h,h)}>f(x)-{}^1\!/_2\, d^2f(x)\,(p,p).$$

Thus (15) is verified in this case, too.

The characterizations given above show that if f is bonvex (pseudo-bonvex, quasi-bonvex) at x then f is locally convex (pseudo-convex, quasi-convex) at x (see [5] for these definitions).

Corollary 7. (i) *If f is strictly pseudo-bonvex at x then f is quasi-bonvex at x.*

(ii) *If f is quasi-bonvex at x and $d^1f(x)\neq 0$ or $d^2f(x)\neq 0$ then f is pseudo-bonvex at x.*

(iii) *If f is strictly pseudo-bonvex at x then f is pseudo-bonvex at x.*

Proof. (i) Let f be strictly pseudo-bonvex at x and $h\in D-x$, $h\neq 0$. If $d^1f(x)\,(h)>0$ then, by (11), $f(x+h)>f(x)$, and (13) holds. Assume now that $d^1f(x)\,(h)\leq 0$ and $d^2f(x)\,(h,h)\neq 0$. If $d^1f(x)\,(h)=0$, we have, again by (11),

$$f(x+h)>f(x)=f(x)-\frac{1}{2}\frac{(d^1f(x)\,(h))^2}{d^2f(x)\,(h,h)}.$$

If $d^1f(x)\,(h)<0$, by (12), we have

$$f(x+h)>f(x)-\frac{1}{2}\frac{(d^1f(x)\,(h))^2}{d^2f(x)\,(h,h)}.$$

Therefore (14) holds (even with strict inequality).

(ii) Let f be quasi-bonvex at x. Of course (9) holds. Assume first that $d^1f(x)\neq 0$. Then

$$\{h\in D-x : d^1f(x)\,(h)\geq 0\}\subset\mathrm{cl}\,\{h\in D-x : d^1f(x)\,(h)>0\}.$$

Let $h\in D-x$ be such that $d^1f(x)\,(h)\geq 0$. By the above relation, there exists $(h_n)\subset D-x$ such that $h_n\to h$ and $d^1f(x)\,(h_n)>0$ for all $n\in\mathbf{N}$. By (13) we have $f(x+h_n)\geq f(x)$. Taking the limit we get $f(x+h)\geq f(x)$. Assume now that $d^1f(x)=0$ but $d^2f(x)\neq 0$. Then

$$X=\{p\in\mathbf{R}^n : d^2f(x)\,(p,p)=0\}$$

is a linear subspace of $\mathbf{R}^n$, $X\neq\mathbf{R}^n$. It follows that $\mathrm{cl}\,(\mathbf{R}^n\setminus X)=\mathbf{R}^n$. Take $h\in D-x$; there exists $(h_n)\subset(D-x)\setminus X$ such that $h_n\to h$. By (14) we have that $f(x+h_n)\geq f(x)$. Passing to the limit we get $f(x+h)\geq f(x)$, i.e. (8) holds.

(iii) follows immediately from Theorems 4 and 5.

Note that if $d^1f(x)=0$ and $d^2f(x)=0$ then f is quasi-bonvex, while f (in this case) is pseudo-bonvex if and only if x is a global minimum of f.

It was remarked in [1] that f is pseudo-bonvex and quasi-bonvex (strictly pseudo-bonvex) at x when f is bonvex (strictly bonvex) at x.

Corollary 8. *Let D be convex and $f\in C^2$. Then the following assertions are equivalent:*

(i) *f is pseudo-bonvex.*

(ii) *f is quasi-bonvex.*

(iii) *f is convex and (9) holds for all $x\in D$, $h\in D-x$.*

Proof. (i)$\Rightarrow$(iii). By Theorem 4 $d^2f(x)$ is positive semi-definite for every $x\in D$. By a well known characterization of convex functions (see [8, Th. 4.5]) f is convex. Of course (9) holds for all $x\in D$, $h\in D-x$.

(iii)$\Rightarrow$(i). If f is convex then (by the same characterization of convex functions) $d^2f(x)$ is positive semi-definite for $x\in D$. Another characterization of convex functions shows that f is convex if and only if

$$d^1f(x)\,(u-x)\leq f(u)-f(x)\quad\forall u,x\in D. \tag{16}$$

Therefore (8) holds, and so f is pseudo-bonvex.

(ii)$\Rightarrow$(iii). By Theorem 6 $d^2f(x)$ is positive semi-definite for every $x\in D$, and so f is convex. From (14) we have that (9) holds for all $x\in D$, $h\in D-x$.

(iii)$\Rightarrow$(ii). f being convex, $d^2f(x)$ is positive semi-definite and (16) holds for all $x,u\in D$. (13) is a consequence of (16). Let $d^1f(x)\,(h)\leq 0$ and $d^2f(x)\,(h,h)\neq 0$, where $x\in D$ and $h\in D-x$. If $d^1f(x)\,(h)=0$, (14) follows from (16), while for $d^1f(x)\,(h)<0$ (14) follows from (9). Hence f is quasi-bonvex.

When $n=1$ the characterizations given above become simpler and two cases must be distinguished: $f''(x)=0$ and $f''(x)\neq 0$.

If $h\in\mathbf{R}^n\setminus\{0\}$ we can consider the function $\varphi_h : I_h\to\mathbf{R}$, $\varphi_h(t)=f(x+th)$, where $x\in D$ is fixed and $I_h=\{t\in\mathbf{R} : x+th\in D\}$. We say that f is bonvex (strictly bonvex, ...) at x in direction h if φ_h is bonvex (strictly bonvex, ...) at 0. From the characterizations given above and the relations $\varphi_h'(0)=d^1f(x)\,(h)$, $\varphi_h''(0)=d^2f(x)\,(h,h)$ we see that f is bonvex (strictly bonvex, ...) at x if and only if f is bonvex (strictly bonvex, ...) at x in every direction.

Example 1. Let $D=\mathbf{R}^n$, $a\in\mathbf{R}^n\setminus\{0\}$, $f:\mathbf{R}^n\to\mathbf{R}$, $f(x)=\langle a, x\rangle$. Then, by Corollary 8, f is pseudo-bonvex and quasi-bonvex, but it is obvious that f is not strictly pseudo-bonvex.

Example 2. Let $f:\mathbf{R}\to\mathbf{R}$ be defined by

$$f(x)=\begin{cases}\sin x \ \text{ for } x\leqslant\pi/2,\\ \dfrac{1}{6}\left(\dfrac{1}{3}x^3-\dfrac{a+b}{2}x^2+abx+\dfrac{1}{6}b^3-\dfrac{1}{2}ab^2\right)\text{ for } x>\pi/2,\end{cases}$$

where $a=\pi/2$, $b=a+\sqrt{6}$. f is pseudo-bonvex at 0, but it is not quasi-bonvex at 0. Indeed, $f'(0)=1$, $f''(0)=0$ and $h>0\Rightarrow f(h)\geqslant f(0)$, but $f(b)=0=f(0)$.

These examples show that the converse implications are, generally, false in Corollary 7.

Example 3. Let $f:\mathbf{R}\to\mathbf{R}$, $f(x)=e^x$. Of course f is convex, but condition (9) does not hold. Therefore f is not pseudo-bonvex or quasi-bonvex on $\mathbf{R}$.

Indeed, $f'(x)=f''(x)=e^x$. Condition (9) becomes

$$e^{x+h}\geqslant e^x-{}^1/_2\,e^x \ \ \forall\, h<0\Leftrightarrow e^h\geqslant {}^1/_2 \ \ \forall h<0\Leftrightarrow 0\geqslant {}^1/_2,$$

a contradiction.

Theorem 9. *Let D be convex and $f\in C^2$. Then f is bonvex if and only if f is convex quadratic.*

For the proof of Theorem 9 we need the following lemma.

Lemma 10. *Let $I\subset R$ be an open interval and $f\in C^2(I)$. Then*

(17) $$f(y)\geqslant f(x)+f'(x)\,(y-x)+{}^1/_2\,f''(x)\,(y-x)^2 \ \ \forall x,\ y\in I$$

if and only if f'' is constant. Moreover, in such a case equality holds in (17).

Proof. Let us note that for all x, $y\in I$ we have

$$f(y)=f(x)+\int_x^y f'(t)\,dt=f(x)+f'(x)\,(y-x)-\int_x^y (t-y)\,f''(t)\,dt.$$

So, from (17) we get

(18) $$0\geqslant \int_x^y (t-y)\,(f''(t)-f''(x))\,dt \ \ \forall x,\ y\in I.$$

Let us show first that f'' is not decreasing. Suppose that there exist a, $b\in I$, $a<b$ such that $f''(a)>f''(b)$. As f'' is continuous, there exists $\bar{x}\in[a, b]$ such that $f''(\bar{x})\geqslant f''(t)$ for all $t\in[a, b]$. Of course, $\bar{x}<b$. Then, by (18),

$$0\geqslant \int_{\bar{x}}^b (t-b)\,(f''(t)-f''(\bar{x}))\,dt.$$

As $\varphi(t)=(t-b)\,(f''(t)-f''(\bar{x}))\geqslant 0$ for all $t\in[\bar{x}, b]$ and φ is continuous, we obtain that $f''(t)=f''(\bar{x})$ for all $t\in[\bar{x}, b]$, and so $f(b)=f(\bar{x})\geqslant f(a)$, a contradiction.

Let now x, $y\in I$, $y<x$. Then $f''(t)\leqslant f''(x)$ for all $t\in[y, x]$.

On the other hand, by (18), we have

$$0\leqslant \int_y^x (t-y)\,(f''(t)-f''(x))\,dt.$$

Because f'' is nondecreasing we have that $(t-y)\,(f''(t)-f''(x))\leqslant 0$ for $t\in[y, x]$, and so, as above, we obtain that $f''(t)=f''(x)$ for $t\in[y, x]$. The conclusion follows. If f'' is constant it is easy to see that equality holds in (17).

Proof of Theorem 9. As mentioned in the Introduction, the sufficiency is proved by M o n d [6]. Assume that f is bonvex. Then, by Theorem 1, we have that

$$f(y)\geqslant f(x)+d^1f(x)\,(y-x)+{}^1/_2\,d^2f(x)\,(y-x,\ y-x) \ \ \forall x,\ y\in D.$$

Let us fix $x_0\in D$ and take $\varphi: I\to R$, $\varphi(t)=f(x_0+td)$, where $I=\{t\in\mathbf{R}:\ x_0+td\in D\}$. Thus, for $x=x_0+\tau d$, $y=x_0+td$, with t, $\tau\in I$, we have that

$$\varphi(t)\geqslant\varphi(\tau)+\varphi'(\tau)\,(t-\tau)+{}^1/_2\,\varphi''(\tau)\,(t-\tau)^2 \ \ \forall\, t,\ \tau\in I.$$

Using Lemma 10 we obtain that

$$\varphi(t)=\varphi(\tau)+\varphi'(\tau)\,(t-\tau)+{}^1/_2\,\varphi''(\tau)\,(t-\tau)^2 \ \ \forall t,\ \tau\in I.$$

Taking $y\in D$, $d=y-x$, $\tau=0$ and $t=1$ we get

$$f(y)=f(x_0)+d^1(x_0)\,(y-x_0)+{}^1/_2\,d^2f(x_0)\,(y-x_0,\ y-x_0) \ \ \forall y\in D,$$

which shows that f is quadratic. As, by Theorem 6, $d^2f(x_0)$ is positive semidefinite, it follows that f is convex quadratic.

Remark. The result stated in Theorem 9 is valid if we suppose that D is open and connected.

Corollary 11. *There are not bonvex functions $f\in C^2$, with D convex, that are strictly bonvex at some point.*

Proof. Let $f\in C^2$ be bonvex. Then f is quadratic, by Theorem 9. Therefore

$$f(x+h)=f(x)+d^1f(x)\,(h)+{}^1/_2\,d^2f(x)\,(h,\ h) \ \ \forall x\in D,\ h\in D-x.$$

This relation shows, using Corollary 2, that f can not be strictly bonvex at some point $x\in D$.

B e c t o r and C h a n d r a [2] say that the function $f:\mathbf{R}^n\to\mathbf{R}$ is *pseudo-linear* if f is differentiable and for all x, $u\in\mathbf{R}^n$

(19) $$d^1f(x)\,(u-x)=0\Rightarrow f(u)=f(x).$$

Note that $f:\mathbf{R}\to\mathbf{R}$ is pseudo-linear if and only if f is constant or $f'(x)\neq 0$ for all $x\in\mathbf{R}$.

The next result characterizes the pseudo-linear functions. This characterization is of the same type as those of quasiaffine functions [4, Th. 2.31] and evenly quasiaffine functions [4, Th. 2.36].

Theorem 12. *Let $f:\mathbf{R}^n\to\mathbf{R}$. Then f is pseudo-linear if and only if there exist $\omega:\mathbf{R}^n\to\mathbf{R}$ linear and $\varphi:\mathbf{R}\to\mathbf{R}$ a pseudo-linear function such that*

(20) $$f=\varphi\circ\omega.$$

Proof. Suppose that there exist a linear function $\omega:\mathbf{R}^n\to\mathbf{R}$ and a pseudo-linear function $\varphi:\mathbf{R}\to\mathbf{R}$ such that $f=\varphi\circ\omega$. If φ is constant or $\omega=0$ then f is constant, and so pseudo-linear. If $\omega\neq 0$ and φ is not constant then $\varphi'(t)\neq$ $\neq 0$ for all $t\in\mathbf{R}$, and

$$d^1 f(x)(h) = \varphi'(\omega(x))\,\omega(h) \quad \forall x,\ h \in \mathbf{R}^n.$$

As $\varphi'(\omega(x)) \neq 0$, if $d^1 f(x)(u-x)=0$ then $\omega(u)=\omega(x)$, and so, by (20), $f(u)=f(x)$. Therefore f is pseudo-linear.

Conversely, let us take f a pseudo-linear function. The conclusion is obvious if f is constant or $n=1$. Assume that f is not constant and $n \geq 2$. Let us first note that $d^1 f(x) \neq 0$ for all $x \in \mathbf{R}^n$ (otherwise f is constant). Let $\tilde{x} \in \mathbf{R}^n$ such that $f(\tilde{x}) \neq f(0)$ and $X_0 = \ker d^1 f(0) = \{h \in \mathbf{R}^n : d^1 f(0)(h) = 0\}$, $\tilde{X} = \ker d^1 f(\tilde{x})$. We have that $\tilde{X} = X_0$. If this is not the case, as $\dim \tilde{X} = \dim X_0 = n-1$, there exists $\bar{x} \in X_0 \cap (\tilde{x} + \tilde{X})$, and so, by (19), $f(0) = f(\bar{x}) = f(\tilde{x})$, a contradiction. Let now $x \in \mathbf{R}^n$ be arbitrarily chosen: then $f(x) \neq f(0)$ or $f(x) \neq f(\tilde{x})$. As above we obtain $\ker d^1 f(x) = X_0$ or $\ker d^1 f(x) = \tilde{X}$. Hence $\ker d^1 f(x) = X_0$ for all $x \in \mathbf{R}^n$. Take $\omega = d^1 f(0)$ and $\bar{x} \in \mathbf{R}^n$ such that $\omega(\bar{x}) = 1$. Then every $x \in \mathbf{R}^n$ can be written as $x = (x - \omega(x)\bar{x}) + \omega(x)\bar{x}$, and so, by (19), $f(x) = f(\omega(x)\bar{x})$. Taking $\varphi : \mathbf{R} \to \mathbf{R}$, $\varphi(t) = f(t\bar{x})$, (20) holds. Moreover $\varphi'(t) = d^1 f(t\bar{x})(\bar{x}) \neq 0$. Therefore φ is pseudo-linear. The proof is complete.

Remark. The results concerning (strictly) bonvex, (strictly) pseudo-bonvex and quasi-bonvex functions stated in this note are also valid in infinite dimensional normed spaces, while Theorem 12 is valid in every topological linear space, taking the Gateaux differential of f.

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Received 15. I. 1988

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Matematica, 35 (1989)

SUFFICIENT CONDITIONS FOR THE EXISTENCE OF NONOSCILLATORY SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

BY

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The present paper deals with finding sufficient conditions for the existence of at least one non-oscillatory solution of the equation

$$[(Lx)(t)]^{(n)} + p(t)\, f(x(t),\ x(\tau(t))) = 0, \tag{1}$$

where $(Lx)(t) := x(t) + a(t)\, x(\tau(t))$, $t \geq t_0$, $t_0 \in \mathbf{R}$.

Analogous results concerning ordinary differential equations and equations with a retarding argument have been obtained in a number of contributions. A detailed bibliography in this respect is contained in [1], [2].

Denote by $C(I)$ the space of all continuous functions $\varphi : I \to \mathbf{R}$ where $I = [t_0, \infty)$, $\infty = +\infty$, by $AC^k(I)$ denote the linear space of all functions possessing locally absolutely continuous derivatives to order k inclusively, and by $AC^{-1}(I)$ denote the space of locally summable functions.

We shall say that conditions (A) are fulfilled if the following conditions hold :

A1. The function $a \in AC^{n-1}(I)$, and is non-negative.

A2. The function $p \in AC^{-1}(I)$ and is non-negative.

A3. The function $f \in C(\mathbf{R}^2)$ and $f(u, v) > 0$ for $u, v > 0$.

A4. The function $\tau \in AC^{n-1}(E_{t_0} \cup I)$, $E_{t_0} = \{t \mid \tau(t) \leq t_0,\ t \in I\}$, $\tau(t) \leq t$, for $t \in I$ and $\lim_{t\to\infty} \tau(t) = \infty$.

As in [3], define the function

$$\gamma(t) := \inf\ \{s \mid \tau(s) > t,\ s \in I\}$$

(obviously $\gamma(t)$ is defined for all $t \in I$ since $\lim_{t\to\infty} \tau(t) = \infty$). We shall mention some of its properties [4]

1. The inequality $t < \gamma(t)$ holds for all $t \in I$.
2. If $t_1 < t_2$, then $\gamma(t_1) < \gamma(t_2)$.
3. The function $\gamma(t)$ is semicontinuous from the right and is continuous if and only if $\tau(t)$ is monotonely increasing and $\lim_{t\to\infty} \tau(t) = \infty$.
4. For any $t \in I$ the equality $\tau(\gamma(t)) = t$ holds.

Introduce the notations