$$d^1 f(x)(h) = \varphi'(\omega(x))\,\omega(h) \quad \forall x,\ h \in \mathbf{R}^n.$$

As $\varphi'(\omega(x)) \neq 0$, if $d^1 f(x)(u-x)=0$ then $\omega(u)=\omega(x)$, and so, by (20), $f(u)=f(x)$. Therefore f is pseudo-linear.

Conversely, let us take f a pseudo-linear function. The conclusion is obvious if f is constant or $n=1$. Assume that f is not constant and $n \geq 2$. Let us first note that $d^1 f(x) \neq 0$ for all $x \in \mathbf{R}^n$ (otherwise f is constant). Let $\tilde{x} \in \mathbf{R}^n$ such that $f(\tilde{x}) \neq f(0)$ and $X_0 = \ker d^1 f(0) = \{h \in \mathbf{R}^n : d^1 f(0)(h)=0\}$, $\tilde{X} = \ker d^1 f(\tilde{x})$. We have that $\tilde{X} = X_0$. If this is not the case, as $\dim \tilde{X} = \dim X_0 = n-1$, there exists $\bar{x} \in X_0 \cap (\tilde{x} + \tilde{X})$, and so, by (19), $f(0) = f(\bar{x}) = f(\tilde{x})$, a contradiction. Let now $x \in \mathbf{R}^n$ be arbitrarily chosen: then $f(x) \neq f(0)$ or $f(x) \neq f(\tilde{x})$. As above we obtain $\ker d^1 f(x) = X_0$ or $\ker d^1 f(x) = \tilde{X}$. Hence $\ker d^1 f(x) = X_0$ for all $x \in \mathbf{R}^n$. Take $\omega = d^1 f(0)$ and $\bar{x} \in \mathbf{R}^n$ such that $\omega(\bar{x}) = 1$. Then every $x \in \mathbf{R}^n$ can be written as $x = (x - \omega(x)\,\bar{x}) + \omega(x)\,\bar{x}$, and so, by (19), $f(x) = f(\omega(x)\,\bar{x})$. Taking $\varphi : \mathbf{R} \to \mathbf{R}$, $\varphi(t) = f(t\bar{x})$, (20) holds. Moreover $\varphi'(t) = d^1 f(t\bar{x})(\bar{x}) \neq 0$. Therefore φ is pseudo-linear. The proof is complete.

Remark. The results concerning (strictly) bonvex, (strictly) pseudo-bonvex and quasi-bonvex functions stated in this note are also valid in infinite dimensional normed spaces, while Theorem 12 is valid in every topological linear space, taking the Gateaux differential of f.

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SUFFICIENT CONDITIONS FOR THE EXISTENCE OF NONOSCILLATORY SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

BY

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The present paper deals with finding sufficient conditions for the existence of at least one non-oscillatory solution of the equation

$$[(Lx)(t)]^{(n)} + p(t)\, f(x(t),\ x(\tau(t))) = 0, \tag{1}$$

where $(Lx)(t) := x(t) + a(t)\, x(\tau(t))$, $t \geq t_0$, $t_0 \in \mathbf{R}$.

Analogous results concerning ordinary differential equations and equations with a retarding argument have been obtained in a number of contributions. A detailed bibliography in this respect is contained in [1], [2].

Denote by $C(I)$ the space of all continuous functions $\varphi : I \to \mathbf{R}$ where $I = [t_0, \infty)$, $\infty = +\infty$, by $AC^k(I)$ denote the linear space of all functions possessing locally absolutely continuous derivatives to order k inclusively, and by $AC^{-1}(I)$ denote the space of locally summable functions.

We shall say that conditions (A) are fulfilled if the following conditions hold:

A1. The function $a \in AC^{n-1}(I)$, and is non-negative.
A2. The function $p \in AC^{-1}(I)$ and is non-negative.
A3. The function $f \in C(\mathbf{R}^2)$ and $f(u, v) > 0$ for $u, v > 0$.
A4. The function $\tau \in AC^{n-1}(E_{t_0} \cup I)$, $E_{t_0} = \{t \mid \tau(t) \leq t_0,\ t \in I\}$, $\tau(t) \leq t$, for $t \in I$ and $\lim_{t \to \infty} \tau(t) = \infty$.

As in [3], define the function

$$\gamma(t) := \inf \{s \mid \tau(s) > t,\ s \in I\}$$

(obviously $\gamma(t)$ is defined for all $t \in I$ since $\lim_{t \to \infty} \tau(t) = \infty$). We shall mention some of its properties [4]

1. The inequality $t < \gamma(t)$ holds for all $t \in I$.
2. If $t_1 < t_2$, then $\gamma(t_1) < \gamma(t_2)$.
3. The function $\gamma(t)$ is semicontinuous from the right and is continuous if and only if $\tau(t)$ is monotonely increasing and $\lim_{t \to \infty} \tau(t) = \infty$.
4. For any $t \in I$ the equality $\tau(\gamma(t)) = t$ holds.

Introduce the notations

$$\gamma_0(t)=t,\ \gamma_1(t)=\gamma(t),\ \tau_1(t)=\tau(t),\ a_1(t)=a(t),$$

$$(2)\qquad \gamma_m(t)=\gamma(\gamma_{m-1}(t)),\ \tau_m(t)=\tau_{m-1}(\tau(t)),$$

$$a_m(t)=a(t)\,a(\tau(t))\ldots a(\tau_{m-1}(t)),\ m=2,3,\ldots$$

It can be immediately seen that

$$(3)\qquad \tau_r(\gamma_m(t))=\tau_{r-1}(\gamma_{m-1}(t))\ \text{for}\ r\geq 2,\ m\geq 1,\ t\in I.$$

Consider the initial value problem

$$(4)\qquad x(t)+a(t)x(\tau(t))=y(t),\ t\in I$$

$$(5)\qquad x(t)=\varphi(t),\ t\in E_{t_0}.$$

Definition 1. *We shall say that the function $x\in C(E_{t_0}\cup I)$ is a solution of problem (4), (5) if x satisfies equation (1) almost everywhere on I and if $x(t)=\varphi(t)$ for $t\in E_{t_0}$.*

Lemma 1. *Let the following conditions hold:*

1. The functions $a: I\to[0,\infty)$, $\tau: E_{t_0}\cup I\to\mathbf{R}$ are continuous.

2. $\lim_{m\to\infty}\gamma_m(t_0)=\infty$ where γ_m is the function defined by (2).

3. The function $\varphi\in C(E_{t_0})$ and $\varphi(\tau(t_0))=0$.

Then problem (4), (5) has a unique continuous solution for any function $y\in C(I)$ such that $y(t_0)=\varphi(t_0)$.

Proof. In view of (2) and condition 2 of Lemma 1, the method of steps can be applied, then

$$(6)\qquad x(t)=y(t)+\sum_{k=1}^{m-1}(-1)^k a_k(t)\,y(\tau_k(t))+(-1)^m a_m(t)\,\varphi(\tau_m(t)),$$

for $t\in[\gamma_{m-1}(t_0),\gamma_m(t_0)]$ and $m\geq 1$, and in the case when $m=1$ the sum in the right-hand side of equality (6) is considered to be identically equal to zero.

The continuity of $x(t)$ at the points $t\in I$, $t\neq\gamma_m(t_0)$ for $m\geq 1$ is obvious, while for $t=\gamma_m(t_0)$ it is immediately implied by condition 3 of the lemma and equalities (3) and (6).

Equality (6) yields that when $y(t)\equiv 0$ for $t\in I$ and $\varphi(t)\equiv 0$ for $t\in E_{t_0}$, then $x(t)\equiv 0$ for $t\in E_{t_0}\cup I$, i.e. the solution of problem (4), (5) is unique.

Remark 1. It can be immediately seen from equality (6) that if the functions φ, y, τ and a are continuous, then the condition $\varphi(\tau(t_0))=0$ is necessary and sufficient for the solution $x(t)$ to be continuous.

Lemma 2. *Let the following conditions be satisfied*

1. The conditions of Lemma 1 hold and the function $y\in C(I)$.

2. The functions φ and y are monotonely increasing non-negative and $\varphi(t_0)=y(t_0)\neq 0$.

3. $a(t)<1$ for $t\in I$.

Then problem (4), (5) has a unique positive solution $x(t)$ which will be bounded if y is a bounded function.

Proof. For $t\in[\gamma_{m-1}(t_0),\gamma_m(t_0)]$ and $m\geq 1$ the following inequalities hold

$$a_m(t)\,\varphi(\tau_m(t))\geq 0,$$

$$(7)\qquad a_m(t)\,y(\tau_m(t))-a_{m+1}(t)\,y(\tau_{m+1}(t))\geq 0.$$

Then the assertion of Lemma 2 follows immediately from equality (6) and inequalities (7).

Lemma 3. *Let the following conditions be fulfilled:*

1. Conditions A1, A4 and conditions 2 and 3 of Lemma 1 hold.

2. The function $y\in AC^{n-1}(I)$ and the function $\varphi\in AC^{n-1}(E_{t_0})$.

Then the conditions $\varphi^{(i)}(\tau(t_0))=0$ and $\varphi_-^{(i)}(t_0)=y_+^{(i)}(t_0)$, $i=1,\ldots,n-1$ are necessary and sufficient for the problem (4), (5) to have a unique solution $x\in AC^{n-1}(E_{t_0}\cup I)$.

(Here $\varphi_-^{(i)}(t_0)$ and $y_+^{(i)}(t_0)$ denote the left and right derivatives of the functions φ and y, respectively).

Proof. 1. Sufficiency. Let $m\geq 1$, $\varphi^{(i)}(\tau(t_0))=0$ and $\varphi_-^{(i)}(t_0)=y_+^{(i)}(t_0)$. Lemma 1 and (6) imply that problem (4), (5) has a continuous solution $x\in AC^{n-1}((\gamma_{m-1}(t_0),\gamma_m(t_0)))$. We shall show that the derivatives $x^{(i)}(t)$ exist at the points $t=\gamma_m(t_0)$. Indeed, (3) implies that for $i=0,1,\ldots,n-1$ the equality

$$\varphi^{(i)}(\tau_s(\gamma_s(t_0)))=y^{(i)}(\tau_s(\gamma_s(t_0))),\quad s=1,2,\ldots,m$$

is equivalent to the equality

$$\varphi_-^{(i)}(t_0)=y_+^{(i)}(t_0)$$

and then

$$\lim_{t\to\gamma_m(t_0)-0}x^{(i)}(t)=\lim_{t\to\gamma_m(t_0)+0}x^{(i)}(t).$$

2. The necessity is immediately implied by the above equality and by (6).

Denote by $\mathbf{B}$ the Banach space of bounded functions $y\in C(I)$ with a norm $\|y\|=\sup\{|y(t)|\mid t\in I\}$, and by K^+ denote the cone of monotonely increasing and non-negative functions.

Definition 2 [5]. *We shall say that the relation $y_1\leqslant y_2$ holds for $y_1, y_2\in\mathbf{B}$ if $y_2-y_1\in K^+$. The ordering introduced in $\mathbf{B}$ is called regular if every bounded monotone sequence has a limit. The cone used to introduce the ordering is also called regular.*

It is immediately verified that the ordering introduced by means of K^+ is regular.

Lemma 4. *Let $y_1, y_2\in\mathbf{B}$, $y_1\leqslant y_2$, $\varphi_1, \varphi_2\in C(E_{t_0})$ and let the difference $\varphi_2(t)-\varphi_1(t)$ be a non-negative monotonely increasing function for $t\in E_{t_0}$. Moreover, let $\varphi_j(t)$ and $y_j(t)$, $j=1,2$ satisfy the conditions of Lemma 2. Then $x_1(t)\leqslant x_2(t)$ for any $t\in E_{t_0}\cup I$ where x_j is a solution of problem (4), (5) with an initial function φ_j and right-hand side y_j.*

The proof of Lemma 4 follows immediately from equality (6).

Definition 3. *The function $x\in C(I)$ is called a weak (strong) solution of equation (1) if $Lx\in AC^{n-1}(I)$ ($x\in AC^{n-1}(I)$) and $x(t)$ satisfies (1) almost everywhere for $t\in I$.*

It is obvious that every strong solution of (1) is a weak solution at the same time.

Consider the integral equations

$$(8)\quad (Lx)(t)=C^{*}+\frac{1}{(n-2)!}\int_{t_0}^{t}(s-t_0)^{n-1}\,p(s)\,f(x(s),\ x(\tau(s)))\,ds+\frac{t-t_0}{(n-2)!}\int_{t}^{\infty}(s-t_0)^{n-2}\,p(s)\,f(x(s),\ x(\tau(s)))\,ds.$$

for n even and

$$(9)\quad (Lx)(t)=C^{*}+\frac{1}{(n-3)!}\int_{t_0}^{t}\Big[\int_{t_0}^{\alpha}(s-t_0)^{n-2}\,p(s)f(x(s),\ x(\tau(s)))\,ds+(\alpha-t_0)\int_{\alpha}^{\infty}(s-t_0)^{n-3}\,p(s)\,f(x(s),x(\tau(s)))\,ds\Big]d\alpha.$$

for n odd, $t\geqslant t_0$, $c^{*}>0$, $n\geqslant 2$.

By means of immediate differentiation it is verified that each continuous solution $x(t)$ of equations (8) and (9) is a weak solution of equation (1). (Lx obviously belongs to $AC^{n-1}(I)$).

Let $y\in \mathbf{B}\cap AC^{n-1}(I)$ and let $\varphi\in AC^{n-1}(E_{t_0})$ be an arbitrary fixed function satisfying the following conditions

1. $\varphi(t)$ is monotonely increasing and non-negative for $t\in E_{t_0}$.
2. $\varphi(t)=0$ for $t\in E_{t_0}$, $t\leqslant\tau(t_0)<t_0$ and $\varphi(t_0)=C^{*}$.

Consider the function

$$(10)\qquad \Phi_y^{\varphi}(t)=y(t)+\sum_{k=1}^{m-1}(-1)^{k}\,a_k(t)\,y(\tau_k(t))+(-1)^{m}\,a_m(t)\,\varphi(\tau_m(t)),$$

for $t\in[\gamma_{m-1}(t_0),\ \gamma_m(t_0)]$ and $m\geqslant 1$. It is obvious that if $\varphi(t_0)=y(t_0)$, then Φ_y^{φ} is a continuous solution of problem (4), (5) with a right-hand side $y(t)$ and an initial function $\varphi(t)$ and if $\varphi^{(i)}(t_0)=y^{(i)}(t_0)$ for $i=0,\ldots,n-1$, then $\Phi_y^{\varphi}\in AC^{n-1}(E_{t_0}\cup I)$ is bounded as well.

Definition 4 [5]. *For any two functions $y_1, y_2\in\mathbf{B}$ such that $y_1\leqslant y_2$ the set $<y_1,\ y_2>=\{y\,/\,y\in\mathbf{B},\ y_1\leqslant y\leqslant y_2\}$ will be called a cone segment.*

Definition 5 [5]. *The operator $A:\mathbf{B}\to\mathbf{B}$ is called monotone on $T\subseteq\mathbf{B}$ if $Ay_1-Ay_2\in K^{+}$ for any two elements $y_1,\ y_2\in T$ such that $y_2-y_1\in K^{+}$* (i.e. if $y_1\leqslant y_2$, *then* $Ay_1\leqslant Ay_2$).

Consider the operators

$$(11)\quad (\mathcal{F}y)(t)=\begin{cases}\varphi(t_0)+\dfrac{1}{(n-2)!}\displaystyle\int_{t_0}^{t}(s-t_0)^{n-1}\,(p(s)f(\Phi_y^{\varphi}(s),\ \Phi_y^{\varphi}(\tau(s)))\,ds+\dfrac{t-t_0}{(n-2)!}\displaystyle\int_{t}^{\infty}(s-t_0)^{n-2}\,p(s)\,f(\Phi_y^{\varphi}(s),\Phi_y^{\varphi}(\tau(s)))\,ds, & t\geqslant t_0\\ \varphi(t), & t\in E_{t_0}.\end{cases}$$

when n is an even number and

$$(12)\quad (Sy)(t)=\begin{cases}\varphi(t_0)+\dfrac{1}{(n-3)!}\displaystyle\int_{t_0}^{t}\Big[\int_{t_0}^{\alpha}(s-t_0)^{n-2}\,p(s)\,f(\Phi_y^{\varphi}(s),\ \Phi_y^{\varphi}(\tau(s)))\,ds+(\alpha-t_0)\int_{\alpha}^{\infty}(s-t_0)^{n-3}p(s)\,f(\Phi_y^{\varphi}(s),\Phi_y^{\varphi}(\tau(s)))\,ds\Big]d\alpha, & t\geqslant t_0\\ \varphi(t), & t\in E_{t_0}\end{cases}$$

when n is an odd number.

Theorem 1. *Let the following conditions be fulfilled along with conditions (A)*

1. *The function $f(u,v)$ is bounded, i.e. for every point $(u,v)\in\mathbf{R}^2$ we have $|f(u,v)|\leqslant M$ and if $u_1\leqslant u_2$, $v_1\leqslant v_2$, then $f(u_1,\ v_1)\leqslant f(u_2,\ v_2)$.*
2. *The function $p(t)$ satisfies the inequality*

$$(13)\qquad I_0=\int_{t_0}^{\infty}t^{n-1}\,p(t)\,dt<\infty.$$

3. $\lim\limits_{m\to\infty}\gamma_m(t_0)=\infty$ *and* $\tau(t_0)<t_0$.

Then the operator $\mathcal{F}$ has at least one fixed point $y_0\in K^{+}$.

Proof. Introduce the notation $T=\{y\mid y\in\mathbf{B},\ y(t_0)=\varphi(t_0)\}$. If $y\in T$, then in view of Lemma 1 $\Phi_y^{\varphi}(t)$ is a continuous function and (11) implies that $\mathcal{F}y\in\mathbf{B}$. Moreover, $(\mathcal{F}y)(t_0)=\varphi(t_0)$ and therefore $\mathcal{F}(T)\subseteq T$.

Let $y_1,\ y_2\in T$ be arbitrary and such that $y_1\leqslant y_2$. Then condition 1 of the theorem yields

$$(\mathcal{F}y_2)(t)-(\mathcal{F}y_1)(t)=\frac{1}{(n-2)!}\int_{t_0}^{t}(s-t_0)^{n-1}\,p(s)\,[f(\Phi_{y_2}^{\varphi}(s),\ \Phi_{y_2}^{\varphi}(\tau(s)))-f(\Phi_{y_1}^{\varphi}(s),\ \Phi_{y_1}^{\varphi}(\tau(s)))]\,ds+\frac{t-t_0}{(n-2)!}\int_{t}^{\infty}(s-t_0)^{n-2}p(s)[f(\Phi_{y_2}^{\varphi}(s),\ \Phi_{y_2}^{\varphi}(\tau(s)))-f(\Phi_{y_1}^{\varphi}(s),\ \Phi_{y_1}^{\varphi}(\tau(s)))]\,ds.$$

for $t\in I$.

Since for any function $y\in T$

$$(14)\qquad [(\mathcal{F}y)(t)]'=\frac{1}{(n-2)!}\int_{t}^{\infty}(s-t_0)^{n-2}\,p(s)f(\Phi_y^{\varphi}(s),\ \Phi_y^{\varphi}(\tau(s)))\,ds.$$

then condition 1 of the theorem implies that if $y_1,\ y_2\in T$ and $y_1\leqslant y_2$, then the inequality $[(\mathcal{F}y_2)(t)-(\mathcal{F}y_1)(t)]\geqslant 0$ holds for $t\in E_{t_0}\cup I$, i.e. $\mathcal{F}y_2-\mathcal{F}y_1\in K^{+}$, hence the operator $\mathcal{F}$ is monotone on T.

Introduce the notation $T^{+}=\{y\mid y\in T,\ y(t)\geqslant 0,\ t\in I\}$. Obviously, if $y\in T^{+}$, the $\mathcal{F}y\in K^{+}$, i.e. the operator is positive on T^{+}.

The operator $\mathcal{F}$ is continuous on T, i.e. if $\lim\limits_{n\to\infty}\|y_n-y_0\|=0$, then $\lim\limits_{n\to\infty}\|\mathcal{F}y_n-\mathcal{F}y_0\|=0$ as well. (T is a closed set).

Denote by $I_n(t)$ the difference

$$I_n(t)=f(\Phi^{\varphi}_{y_n}(t),\ \Phi^{\varphi}_{y_n}(\tau(t)))-f(\Phi^{\varphi}_{y_0}(t),\ \Phi^{\varphi}_{y_0}(\tau(t))) \tag{15}$$

and then

$$\|\mathcal{F}y_n-\mathcal{F}y_0\|=\sup_{t\in I}\Big|\int\limits_{t_0}^{t}\frac{(s-t_0)^{n-1}}{(n-2)!}\,p(s)\,I_n(s)\,ds+ \tag{16}$$
$$+\frac{t-t_0}{(n-2)!}\int\limits_{t}^{\infty}(s-t_0)\,p(s)\,I_n(s)\,ds\Big|\leqslant\frac{I_0}{(n-2)!}\sup_{t\in I}|I_n(t)|.$$

Since all functions participating in (15) are continuous, then for any $\varepsilon>0$ there exist numbers $\nu(\varepsilon)$, $\delta(\varepsilon)>0$ such that for $n>\nu$ and $\sup\limits_{t\in I}|y_n(t)-y_0(t)|<\delta$ the inequality $\sup\limits_{t\in I}|I_n(t)|<\dfrac{\varepsilon(n-2)!}{I_0}$ holds. Then (16) implies the inequality $\|\mathcal{F}y_n-\mathcal{F}y_0\|<\varepsilon$ for $n>\nu$.

Let $\bar{y}(t)=\varphi(t_0)$ for all $t\in I$. Then $\bar{y}(t)\in K^+\cap T^+$ and $(\mathcal{F}\bar{y})(t)\geqslant\bar{y}(t)$ for $t\in I$. Since $\bar{y}(t)>0$ for $t\in I$, then $\Phi^{\varphi}_{\bar{y}}(t)>0$ for all $t\in I$. In view of (14) we obtain that

$$[(\mathcal{F}\bar{y})(t)-\bar{y}(t)]'=\frac{1}{(n-2)!}\int\limits_{t}^{\infty}(s-t_0)^{n-2}\,p(s)\,f(\Phi^{\varphi}_{\bar{y}}(s),\ \Phi^{\varphi}_{\bar{y}}(\tau(s)))\,ds\geqslant 0$$

for $t\in I$, whence we conclude that $F\bar{y}\in K^+\cap T^+$ and $\bar{y}\leqslant F\bar{y}$.

On the other hand, for $y\in\mathbf{B}$

$$\|\mathcal{F}y\|=\sup_{t\in I}|(\mathcal{F}y)(t)|\leqslant\varphi(t_0)+\frac{MI_0}{(n-2)!}. \tag{17}$$

Introduce the notation

$$y^*(t)=\varphi(t_0)+\frac{M}{(n-2)!}\int\limits_{t_0}^{t}(s-t_0)^{n-1}\,p(s)\,ds+\frac{M(t-t_0)}{(n-2)!}\int\limits_{t}^{\infty}(s-t_0)^{n-2}p(s)ds+$$
$$+\frac{M}{(n-2)!}\int\limits_{t_0}^{t}\Big[\int\limits_{\alpha}^{\infty}(s-t_0)^{n-2}p(s)\,ds\Big]d\alpha.$$

It is easily verified that $y^*(t)\geqslant\varphi(t_0)$ for $t\in I$ and

$$y^{*\prime}(t)=\frac{2M}{(n-2)!}\int\limits_{t}^{\infty}(s-t_0)^{n-2}\,p(s)\,ds\geqslant 0\qquad\text{for}\qquad t\in I.$$

Therefore, $y^*\in K^+\cap T^+$ and $y^*-y\in K^+\cap T^+$.

Consider the difference

$$y^*(t)-(\mathcal{F}y^*)(t)=\frac{1}{(n-2)!}\Big[\int\limits_{t_0}^{t}(s-t_0)^{n-1}\,p(s)\,(M-f(\Phi^{\varphi}_{y^*}(s),\ \Phi^{\varphi}_{y^*}(\tau(s))))ds+$$
$$+(t-t_0)\int\limits_{t}^{\infty}(s-t_0)^{n-2}\,p(s)(M-f(\Phi^{\varphi}_{y^*}(s),\ \Phi^{\varphi}_{y^*}(\tau(s))))\,ds+ \tag{18}$$
$$+M\int\limits_{t_0}^{t}\Big(\int\limits_{\alpha}^{\infty}(s-t_0)^{n-2}p(s)\,ds\Big)d\alpha\Big].$$

Condition 1 of Theorem 1 yields that the first two addends on the right-hand side of equality (18) are non-negative, hence $y^*(t)-(\mathcal{F}y^*)(t)\geqslant 0$ for $t\in I$. On the other hand, from (14) and condition 1 of the theorem we obtain that

$$[y^*(t)-(\mathcal{F}y^*)(t)]'\geqslant 0,$$

for $t\in I$ and hence $\mathcal{F}y^*\leqslant y^*$, i.e. $y^*-\mathcal{F}y^*\in K^+\cap T^+$. Whence it follows that the monotone operator, positive on $K^+\cap T^+$, transforms the cone segment $<\bar{y},\ y^*>$ onto itself and in view of Theorem 4.1 [5] there exists a function $y_0\in\ <\bar{y},\ y^*>\subset K^+\cap T^+$ such that $\mathcal{F}y_0=y_0$.

Thus Theorem 1 is proved.

Remark 3. The definition of the operator $\mathcal{F}$ implies that $y_0\in AC^{n-1}(I)$.

Corollary 1. *Let the conditions of Theorem 1 be fulfilled and let y_0 be the fixed point of the operator $\mathcal{F}$ belonging to $K^+\cap T^+$. Then the function $x_0(t)=\Phi^{\varphi}_{y_0}(t)$ is a positive weak solution of equation* (1).

Proof. In view of Lemmas 1 and 2, the function $x_0(t)=\Phi^{\varphi}_{y_0}(t)$ is the unique continuous, bounded and positive solution of problem (4), (5) with a right-hand side $y_0(t)$ and an initial function $\varphi(t)$ and then, according to Theorem 1, $x_0(t)$ is a continuous positive solution of the integral equation (8) for $C^*=\varphi(t_0)$. Therefore, $Lx_0\in AC^{n-1}(I)$ and $x_0(t)$ satisfies (1) almost everywhere for $t\in I$.

This completes the proof of Corollary 3.

Theorem 2. *Let the conditions of Theorem 1 be fulfilled. Then the operator S has at least one fixed point $y_0\in K^+$.*

Proof. Let $y\in\mathbf{B}$. It is easily seen that for $y(t_0)\neq\varphi(t_0)$ the function $\Phi^{\varphi}_{y}(t)$ can have only bounded jumps. Therefore, the function

$$\Psi_y(t)=\int\limits_{t_0}^{t}(s-t_0)^{n-2}\,p(s)f(\Phi^{\varphi}_{y}(s),\ \Phi^{\varphi}_{y}(\tau(s)))\,ds+ \tag{19}$$
$$+(t-t_0)\int\limits_{t}^{\infty}(s-t_0)^{n-3}\,p(s)f(\Phi^{\varphi}_{y}(s),\ \Phi^{\varphi}_{y}(\tau(s)))\,ds,$$

is continuous for $t\in I$ and $S(\mathbf{B})\subseteq\mathbf{B}$.

From condition 1 of Theorem 1 we obtain that if $y_1, y_2 \in \mathbf{B}$ and $y_1 \leqslant y_2$ i.e. $y_2 - y_1 \in K^+$, then

$$\Psi_{y_1}(t) \leqslant \Psi_{y_2}(t) \quad \text{for} \quad t \in I. \tag{20}$$

Moreover, for $y \in \mathbf{B}$

$$[(Sy)(t)]' = ((n-3)!)^{-1}\Psi_y(t) \quad \text{for} \quad t \in I. \tag{21}$$

Let $y_1, y_2 \in \mathbf{B}$ and $y_2 - y_1 \in K^+$. Then (20) and (21) imply that $Sy_2 - Sy_1 \in K^+$, i.e. the operator S is monotone in $\mathbf{B}$.

The continuity of the operator S is proved analogously to that of $\mathcal{F}$ with the difference that S is continuous in $\mathbf{B}$ and not only in T.

Introduce the notation $\mathbf{B}^+ = \{y \mid y \in \mathbf{B},\ y(t) \geqslant 0,\ t \in I\}$. If $y \in \mathbf{B}^+$, then (21) implies $Sy \in K^+$, i.e. $S(\mathbf{B}^+) \subset K^+$ and hence the operator S is positive in $\mathbf{B}^+$.

If we set $\bar{y}(t) = \varphi(t_0)$ for $t \in I$, then $\bar{y} \in K^+$ and hence $S\bar{y} \in K^+$. Then (12) implies that $(S\bar{y})(t) - \bar{y}(t) \geqslant 0$ for $t \in I$, while (21) yields $[(S\bar{y})(t) - \bar{y}(t)]' = ((n-3)!)^{-1}\Psi_{\bar{y}}(t)$ for $t \in I$, hence $S\bar{y} - \bar{y} \in K^+$.

If $y \in \mathbf{B}$, then

$$\| Sy \| \leqslant \varphi(t_0) + MI_0/(n-3)!\,.$$

Introduce the notation

$$y^* = \varphi(t_0) + \frac{M}{(n-3)!}\int_{t_0}^{t}\left(\int_{t_0}^{\alpha}(s-t_0)^{n-2}\,p(s)ds + (\alpha - t_0)\int_{\alpha}^{\infty}(s-t_0)\,p(s)ds\right)d\alpha.$$

and obviously, $y^* \in K^+$, $y^*(t) - \bar{y}(t) \geqslant 0$ for $t \in I$,

$$[y^*(t) - \bar{y}(t)]' > 0,$$

for $t \in I$, i.e. $y^* - \bar{y} \in K^+$. On the other hand, condition 1 of Theorem 1 implies that for $t \in I$, $y^*(t) - (Sy^*)(t) \geqslant 0$ and

$$[y^*(t) - (Sy^*)(t)]' \geqslant 0,$$

i.e. $y^* - Sy^* \in K^+$ and the monotone continuous and positive operator S transforms the cone segment $<\bar{y}, y^*>$ onto itself. In view of Theorem 4.1 [5] there is at least one fixed point $y_0 \in <\bar{y}, y^*> \subset K^+$.

Theorem 2 is thus proved.

Remark 4. The definition of the operator S implies that $\varphi(t_0) = y(t_0)$ and, moreover, equality (12) yields that $y_0 \in AC^{n-1}(I)$.

Corollary 2. *Let the conditions of Theorem 2 be fulfilled and let y_0 be the fixed point of the operator S, $y_0 \in K^+$. Then the function $x_0(t) = \Phi^{\mp}_{y_0}(t)$ is a positive weak solution of equation* (1).

The proof of Corollary 2 is analogous to that of Corollary 1 with the difference that $x_0(t)$ is a solution of (9).

Corollary 3. *Let the conditions of Theorem 1 (Theorem 2) be fulfilled and $a^{(i)}(t_0) = 0$, $i = 0, \ldots, n$. Then the function $x_0(t) = \Phi^{\mp}_{y_0}(t)$ is a positive strong solution of equation* (1) *where y_0 is the fixed point of the operator $\mathcal{F}(S)$ in the cone K^+.*

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