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ON A GENERALIZED OPIAL TYPE INEQUALITY IN TWO INDEPENDENT VARIABLES

BY

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1. Introduction. In 1982 G. S. Yang [5] proved the following inequality of Opial type (see [1, pp. 154—162]) in two independent variables.

If $f(s,t)$, $f_1(s,t)$, $f_{12}(s,t)$ are continuous functions on $[a,b]\times[c,d]$ and if $f(a,t)=f(b,t)=f_1(s,c)=f_1(s,d)=0$ for $a\le s\le b$, $c\le t\le d$, then

$$\int_a^b\Big(\int_c^d |f(s,t)|\,|f_{12}(s,t)|\,dt\Big)ds \le M\int_a^b\Big(\int_c^d |f_{12}(s,t)|^2\,dt\Big)ds, \tag{1}$$

where $M=(b-a)(d-c)/8$ and $f_1(s,t)=\frac{\partial}{\partial s}f(s,t)$, $f_{12}(s,t)=\frac{\partial^2}{\partial s\partial t}f(s,t)$.

Recently, in a series of papers [2—4], the present author has established several interesting and new integral inequalities of Opial type involving functions and their partial derivatives in two independent variables. The aim of the present paper is to establish a new integral inequality involving functions of two independent variables and their partial derivatives which claim their origin to Yang's inequality given in (1). Our result in the special cases yield the Yang's inequality (1) and its generalization given by the present author in [2, Theorem 1].

2. Statement of results. Throughout we use the notation $Dh(r)=dh(r)/dr$,

$$D_1h(s,t)=\frac{\partial}{\partial s}h(s,t),\quad D_2h(s,t)=\frac{\partial}{\partial t}h(s,t),\quad D_2D_1h(s,t)=\frac{\partial^2}{\partial t\partial s}h(s,t)$$

and $\Delta=[a,b]\times[c,d]$, $\Delta_1=[a,\alpha]\times[c,\beta]$, $\Delta_2=[a,\alpha]\times[\beta,d]$,

$\Delta_3=[\alpha,b]\times[c,\beta]$, $\Delta_4=[\alpha,b]\times[\beta,d]$ for $a\le\alpha\le b$, $c\le\beta\le d$.

Our main result is established in the following

Theorem 1. *Let $f(s,t)$, $D_1f(s,t)$, $D_2D_1f(s,t)$ and $g(s,t)$, $D_1g(s,t)$, $D_2D_1g(s,t)$ be real-valued continuous functions on Δ and $f(a,t)=f(b,t)=D_1f(s,c)=D_1f(s,d)=0$, $g(a,t)=g(b,t)=D_1g(s,c)=D_1g(s,d)=0$ for $a\le s\le b$, $c\le t\le d$. Let $\Phi(r)$ and $\Psi(r)$ be nonnegative, continuous, nondecreasing functions for $r\ge 0$ and $\Phi(0)=\Psi(0)=0$ such that $D\Phi(r)$ and $D\Psi(r)$ exist, nonnegative, continuous and nondecreasing for $r\ge 0$. Then*

$$(2)\qquad \int_a^b\int_c^d [\Phi(|f(s,t)|)\, D\Psi(|g(s,t)|)\, |D_2D_1g(s,t)|$$

$$+\Psi(|g(s,t)|)\, D\Phi(|f(s,t)|)\, |D_2D_1f(s,t)|]\, dtds$$

$$\leq \Phi\Big(\int_a^\alpha\int_c^\beta |D_2D_1f(s,t)|\, dtds\Big)\Psi\Big(\int_a^\alpha\int_c^\beta |D_2D_1g(s,t)|\, dtds\Big)$$

$$+\Phi\Big(\int_a^\alpha\int_\beta^d |D_2D_1f(s,t)|\, dtds\Big)\Psi\Big(\int_a^\alpha\int_\beta^d |D_2D_1g(s,t)|\, dtds\Big)$$

$$+\Phi\Big(\int_\alpha^b\int_c^\beta |D_2D_1f(s,t)|\, dtds\Big)\Psi\Big(\int_\alpha^b\int_c^\beta |D_2D_1g(s,t)|\, dtds\Big)$$

$$+\Phi\Big(\int_\alpha^b\int_\beta^d |D_2D_1f(s,t)|\, dtds\Big)\Psi\Big(\int_\alpha^b\int_\beta^d |D_2D_1g(s,t)|\, dtds\Big),$$

for $a\leqslant\alpha\leqslant b$, $c\leqslant\beta\leqslant d$.

The particular cases of Theorem 1 are contained in the following corollaries.

Corollary 2. *Assume that in the hypotheses of Theorem 1 we have $g=f$ and $\Psi=\Phi$. Then*

$$(3)\qquad \int_a^b\int_c^d \Phi(|f(s,t)|)\, D\Phi(|f(s,t)|)\, |D_2D_1f(s,t)|\, dtds$$

$$\leqslant \frac{1}{2}\Big[\Big\{\Phi\Big(\int_a^\alpha\int_c^\beta |D_2D_1f(s,t)|\, dtds\Big)\Big\}^2+\Big\{\Phi\Big(\int_a^\alpha\int_\beta^d |D_2D_1f(s,t)|\, dtds\Big)\Big\}^2+$$

$$+\Big\{\Phi\Big(\int_\alpha^b\int_c^\beta |D_2D_1f(s,t)|\, dtds\Big)\Big\}^2+\Big\{\Phi\Big(\int_\alpha^b\int_\beta^d |D_2D_1f(s,t)|\, dtds\Big)\Big\}^2\Big],$$

for $a\leqslant\alpha\leqslant b$, $c\leqslant\beta\leqslant d$.

Corollary 3. *Assume that in the hypotheses of Theorem 1 we have $\Phi(r)=r^{m+1}$, $\Psi(r)=r^{m+1}$, where $m\geqslant 0$ is a constant. Then*

$$(4)\qquad \int_a^b\int_c^d |f(s,t)|^m\, |g(s,t)|^m\, [|f(s,t)|\, |D_2D_1g(s,t)|+$$

$$+|g(s,t)|\, |D_2D_1f(s,t)|]\, dtds\leqslant \frac{1}{2(m+1)}\Big[\frac{(b-a)(d-c)}{4}\Big]^{2m+1}\times$$

$$\times\int_a^b\int_c^d [|D_2D_1f(s,t)|^{2(m+1)}+|D_2D_1g(s,t)|^{2(m+1)}]\, dtds.$$

We note that the inequality (3) is analogous to Godunova and Levin's inequality (see [1, Theorem 13, p. 159]) and we believe that it is new to the literature. Putting $\Phi(r)=r$ and hence $D\Phi(r)=1$ in (3) and applying Schwarz inequality twice to each resulting integral on the right side and then taking $\alpha=(a+b)/2$ and $\beta=(c+d)/2$, we get the Yang's inequality given in (1). We also note that the inequality (4) is established by the present author in [2, Theorem 1] which in turn contains as a special case the inequality established by Yang in (1) (see [2, p. 265]).

3. Proofs of Theorem 1 and Corollaries 2—3. In order to prove Theorem 1 we obtain the estimates in the following four cases.

Case 1. Let $(s,t)\in\Delta_1$ and define

$$(5)\qquad z(s,t)=\int_a^s\int_c^t |D_2D_1f(u,v)|\, dvdu,$$

$$(6)\qquad w(s,t)=\int_a^s\int_c^t |D_2D_1g(u,v)|\, dvdu.$$

Then

$$(7)\qquad D_1z(s,t)=\int_c^t |D_2D_1f(s,v)|\, dv,\quad D_2z(s,t)=\int_a^s |D_2D_1f(u,t)|\, du,$$

and

$$(8)\qquad D_1w(s,t)=\int_c^t |D_2D_1g(s,v)|\, dv,\quad D_2w(s,t)=\int_a^s |D_2D_1g(u,t)|\, du.$$

From this, we observe that for each fixed s, $z(s,t)$ and $w(s,t)$ are nondecreasing for t on $[c,\beta]$. Since $f(a,t)=g(a,t)=0$ and $D_1f(s,c)=D_1g(s,c)=0$ for $(s,t)\in\Delta_1$, we have

$$(9)\qquad |f(s,t)|\leqslant\int_a^s |D_1f(u,t)|\, du,\quad |g(s,t)|\leqslant\int_a^s |D_1g(u,t)|\, du,$$

$$(10)\qquad |D_1f(s,t)|\leqslant\int_c^t |D_2D_1f(s,v)|\, dv=D_1z(s,t),$$

$$(11)\qquad |D_1g(s,t)|\leqslant\int_c^t |D_2D_1g(s,v)|\, dv=D_1w(s,t).$$

From (9)—(11) we observe that

$$(12)\qquad |f(s,t)|\leqslant\int_a^s D_1z(u,t)\, du=z(s,t),$$

$$(13)\qquad |g(s,t)|\leqslant\int_a^s D_1w(u,t)\, du=w(s,t).$$

From (12), (13), (7), (8), (5), (6) and the hypotheses of Theorem 1 we observe that

$$(14)\qquad \int_a^\alpha\int_c^\beta [\Phi(|f(s,t)|)\, D\Psi(|g(s,t)|)\, |D_2D_1g(s,t)|$$

$$+\Psi(|g(s,t)|)\, D\Phi(|f(s,t)|)\, |D_2D_1f(s,t)|]\, dtds$$

$$\leqslant\int_a^\alpha\int_c^\beta [\Phi(z(s,t))\, D\Psi(w(s,t))\, |D_2D_1g(s,t)|$$

$$+\Psi(w(s,t))\, D\Phi(z(s,t))\, |D_2D_1f(s,t)|]\, dtds$$

$$\leqslant\int_a^\alpha [\Phi(z(s,\beta))\, D\Psi(w(s,\beta))\int_c^\beta |D_2D_1g(s,t)|$$

$$+\Psi(w(s,\beta))\,D\Phi(z(s,\beta))\int_c^\beta |D_2D_1f(s,t)|]\,dtds$$

$$=\int_a^\alpha [\Phi(z(s,\beta))\,D\Psi(w(s,\beta))\,D_1w(s,\beta)+\Psi(w(s,\beta))\,D\Phi(z(s,\beta))\,D_1z(s,\beta)]\,ds$$

$$=\int_a^\alpha \frac{d}{ds}[\Phi(z(s,\beta))\Psi(w(s,\beta))]\,ds=\Phi(z(\alpha,\beta))\Psi(w(\alpha,\beta))$$

$$=\Phi\Big(\int_a^\alpha\int_c^\beta |D_2D_1f(s,t)|\,dtds\Big)\,\Psi\Big(\int_a^\alpha\int_c^\beta |D_2D_1g(s,t)|\,dtds\Big).$$

Case II. Let $(s,t)\in\Delta_2$ and define

$$z(s,t)=\int_a^s\int_t^d |D_2D_1f(u,v)|\,dvdu,\quad w(s,t)=\int_a^s\int_t^d |D_2D_1g(u,v)|\,dvdu.$$

Case III. Let $(s,t)\in\Delta_3$ and define

$$z(s,t)=\int_s^b\int_c^t |D_2D_1f(u,v)|\,dvdu,\quad w(s,t)=\int_s^b\int_c^t |D_2D_1g(u,v)|\,dvdu.$$

Case IV. Let $(s,t)\in\Delta_4$ and define

$$z(s,t)=\int_s^b\int_t^d |D_2D_1f(u,v)|\,dvdu,\quad w(s,t)=\int_s^b\int_t^d |D_2D_1g(u,v)|\,dvdu.$$

Now by following similar arguments to those in the proof of Case I, but with suitable modifications, we obtain the following estimates in cases II, III and IV:

$$\int_a^\alpha\int_\beta^d [\Phi(|f(s,t)|)\,D\Psi(|g(s,t)|)\,|D_2D_1g(d,t)| \tag{15}$$
$$+\Psi(|g(s,t)|)\,D\Phi(|f(s,t)|)\,|D_2D_1f(s,t)|]\,dtds$$
$$\leqslant\Phi\Big(\int_a^\alpha\int_\beta^d |D_2D_1f(s,t)|\,dtds\Big)\Psi\Big(\int_a^\alpha\int_\beta^d |D_2D_1g(s,t)|\,dtds\Big),$$

$$\int_\alpha^b\int_c^\beta [\Phi(|f(s,t)|)\,D\Psi(|g(s,t)|)\,|D_2D_1g(s,t)| \tag{16}$$
$$+\Psi(|g(s,t)|)\,D\Phi(|f(s,t)|)\,|D_2D_1f(s,t)|]\,dtds$$
$$\leqslant\Phi\Big(\int_\alpha^b\int_c^\beta |D_2D_1f(s,t)|\,dtds\Big)\Psi\Big(\int_\alpha^b\int_c^\beta |D_2D_1g(s,t)|\,dtds\Big),$$

and

$$\int_\alpha^b\int_\beta^d [\Phi(|f(s,t)|)\,D\Psi(|g(s,t)|)\,|D_2D_1g(s,t)| \tag{17}$$
$$+\Psi(|g(s,t)|)\,D\Phi(|f(s,t)|)\,|D_2D_1f(s,t)|]\,dtds$$
$$\leqslant\Phi\Big(\int_\alpha^b\int_\beta^d |D_2D_1f(s,t)|\,dtds\Big)\Psi\Big(\int_\alpha^b\int_\beta^d (|D_2D_1g(s,t)|\,dtds\Big),$$

respectively. Adding (14)–(17) the desired inequality in (2) follows. This completes the proof of Theorem 1.

The proof of Corollary 2 is immediate if we substitute $g=f$ and $\Psi=\Phi$ in (2).

By putting $\Phi(r)=r^{m+1}$, $\Psi(r)=r^{m+1}$ and hence $D\Phi(r)=(m+1)\,r^m$, $D\Psi(r)=(m+1)\,r^m$ in (2) and using the elementary inequality $c_1c_2\leqslant(c_1^2+c_2^2)/2$ for $c_1, c_2\geqslant 0$ reals and Hölder's inequality twice with indices $2(m+1)$, $2(m+1)/(2m+1)$ to each resulting integral on the right side and then taking $\alpha=(a+b)/2$ and $\beta=(c+d)/2$, we get the required inequality in (1). This completes the proof of Corollary 3.

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