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VARIATIONAL SOLUTIONS FOR ELLIPTIC BOUNDARY VALUE PROBLEMS

BY

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1. Introduction. The purpose of the present paper is to investigate the following nonlinear boundary value problem:

$$T_{2m}u := \sum_{k=0}^{m} (-1)^k [A_k(x, u, u', \ldots, u^{(m)})]^{(k)} = f(x), \tag{E}$$

for $0 < x < 1$ $(m \geq 1)$.

$$\begin{bmatrix} (L_{2m-1}\, u)(0) \\ -(L_{2m-1}\, u)(1) \\ (L_{2m-2}\, u)(0) \\ -(L_{2m-2}\, u)(1) \\ \cdots\cdots\cdots \\ (L_m\, u)(0) \\ -(L_m\, u)(1) \end{bmatrix} \in \beta \left(\begin{bmatrix} u(0) \\ u(1) \\ u'(0) \\ u'(1) \\ \cdots\cdots \\ u^{(m-1)}(0) \\ u^{(m-1)}(1) \end{bmatrix} \right) \tag{B.C.}$$

where

$$L_{2m-j}\, u := \sum_{k=j}^{m} (-1)^{k-j} [A_k(x, u, u', \ldots, u^{(m)})]^{(k-j)} \qquad (j = 1, 2, \ldots, m).$$

Basic assumptions. Assume that the functions A_k satisfy the following standard hypotheses:

(H_1) *All the functions* $A_k = A_k(x, \xi)$: $]0,1[\times R^{m+1} \to R$ $(k = 0, 1, \ldots, m;\ R =]-\infty, +\infty[)$ *are of Caratheodory type, i.e.*

(a) $x \to A_k(x, \xi)$ *are measurable, for every* $\xi \in R^{m+1}$

and

(b) $\xi \to A_k(x, \xi)$ *are continuous, for a.e.* $x \in]0, 1[$.

In addition, there exist $p > 1$, $g \in L^q(0,1)$ $(1/p + 1/q = 1)$ *and* $C \geq 0$ *such that*

$$|A_k(x, \xi)| \leq C \|\xi\|_{R^{m+1}}^{p-1} + g(x),\ (\forall)\ \xi \in R^{m+1}, \tag{c}$$

a.e. $x \in]0, 1[$ $(k = 0, 1, \ldots, m)$.

Moreover, functions A_k satisfy the ellipticity condition :

$$\text{(H}_2\text{)} \qquad \sum_{k=0}^{m} (A_k(x, \xi) - A_k(x, \eta))\,(\xi_k - \eta_k) \geq 0, \text{ a.e. } x \in]0,1[$$

$$(\forall)\ \xi = (\xi_0, \xi_1, \ldots, \xi_m),\ \eta = (\eta_0, \ldots, \eta_m) \in R^{m+1}.$$

As regards β in the boundary condition (B.C.) we assume :
(H$_3$) $\beta : D(\beta) \subset R^{2m} \to R^{2m}$ *is a maximal monotone (possibly multivalued) mapping.*

Note that equation (E) is just the one-dimensional case ($0 < x < 1$) of the well-known nonlinear divergence equation, which has been studied by many authors (see, e.g., Yu. A. Dubinskii [3], J.L. Lions [4], D. Pascali and S. Sburlan [6], M.I. Višik [7] and the references therein).

Instead, we associate to equation (E) the boundary condition (B.C.) (where β satisfies (H$_3$)) which, to our knowledge, was never considered until now in the literature in this general form. This is the novelty of this paper.

Notice that many classical boundary conditions can be derived from (B.C.) by making suitable choices of β and of A_k (see Section 5).

2. Some notation and terminology. Let $W^{k,r}(0,1)$ (k natural, $1 \leq r \leq \infty$) be the usual Sobolev space endowed with the usual norm. For convenience, we shall denote it by $W^{k,r}$. Also we shall denote the space $L^r(0,1)$ by L^r.

The space of all continuous functions : $[0,1] \to R$ with continuous derivatives up to the order k will be denoted by $C^k[0,1]$. This is a Banach space with respect to its usual norm. Finally, we introduce the notation

$$W_\beta^{k,r} := \{u \in W^{k,r};\ \mathrm{col}\,[u(0),\ u(1),\ u'(0),\ u'(1), \ldots, u^{(k-1)}(0),\ u^{(k-1)}(1)]$$
$$\in D(\beta)\}.$$

We assume the familiarity of the reader with the monotone operator theory. For terminology and background material in this direction see, e.g., V. Barbu [1], H. Brezis [2].

3. The main result. Let $a : W^{m,p} \times W^{m,p} \to R$ be the form defined by

$$a(u, v) := \sum_{k=0}^{m} \int_0^1 A_k(x, u, u', \ldots, u^{(m)})\, v^{(k)}\, dx.$$

Define also the (possibly multivalued) mapping

$$a_\beta : W_\beta^{m,p} \times W^{m,p} \to R \text{ by}$$

$$a_\beta(u, v) := \langle \beta\,(\mathrm{col}\ [u(0), u(1), \ldots, u^{(m-1)}(0),\ u^{(m-1)}(1)]),$$
$$\mathrm{col}\ [v(0),\ v(1), \ldots, v^{(m-1)}(0),\ v^{(m-1)}(1)] \rangle_{R^{2m}}.$$

where $\langle \cdot, \cdot \rangle_{R^{2m}}$ represents the scalar product of R^{2m}. We have denoted by $\langle \beta(w_1),\ w_2 \rangle_{R^{2m}}$ the set $\{\langle y, w_2 \rangle_{R^{2m}} :\ y \in \beta(w_1)\}$.

Let us associate to problem (E), (B.C.) the form $b : W_\beta^{m,p} \times W^{m,p} \to R$ defined by

$$b(u, v) = a(u, v) + a_\beta(u, v).$$

Definition 1. *Let f be an element of $(W^{m,p})'$ (the dual of $W^{m,p}$). The function u is said to be a variational solution of problem (E), (B.C.) if :*

$$u \in W_\beta^{m,p} \ \textit{and}\ \ f(v) \in b(u, v),\ (\forall)\ v \in W^{m,p}.$$

Remark 1. Let us assume that $A_k (k = 0, 1, \ldots, m)$ and f are sufficiently good functions and u is a classical solution of problem (E), (B.C). Then a straightforward computation shows that u is also a variational solution for this problem in the sense of Definition 1.

It is well-known (see, e.g., V. Barbu [1, pp. 48—50]) that, under hypotheses (H$_1$), (H$_2$), the mapping a is well defined on $W^{m,p} \times W^{m,p}$ and, for every $u \in W^{m,p}$, $v \to a(u, v)$ is a linear continuous functional on $W^{m,p}$. Therefore $a(u, v)$ may be written as

$$a(u, v) = (Au)\,(v),\ (\forall)\ u, v \in W^{m,p}.$$

where A is an operator from $W^{m,p}$ into its dual. Moreover, operator $A : W^{m,p} \to (W^{m,p})'$ is monotone and continuous. Therefore, A *is maximal monotone.*

It is also easy to see that for every $y \in R^{2m}$ the mapping $v \to \langle y, \mathrm{col}\ [v(0),\ v(1), \ldots, v^{(m-1)}(0),\ v^{(m-1)}(1)] \rangle_{R^{2m}}$ is a linear continuous functional on $W^{m,p}$. The continuity of this functional follows from the fact that $W^{m,p}$ is continuously embedded in $C^{m-1}[0,1]$. Therefore a_β can be written as

$$a_\beta(u, v) = (A^\beta u)\,(v),\ (\forall)\ u \in W_\beta^{m,p},\ v \in W^{m,p},$$

where A^β is a multivalued operator from $W^{m,p}$ into its dual.

We are now able to state the main result of this paper :

Theorem 1. *Operator $A^\beta : W_\beta^{m,p} \to (W^{m,p})'$ defined above, by means of a_β is maximal monotone.*

Proof. First of all, as β is monotone, we deduce that A^β is *monotone* too. Let us prove now that A^β is *maximal* monotone. In the case in which β is everywhere defined on R^{2m}, single-valued and continuous the maximality of A^β follows easily. Indeed, since $W^{m,p}$ is continuously embedded in $C^{m-1}[0,1]$, it follows that A^β is demicontinuous (i.e. $u_j \to u$ strongly in $W^{m,p}$ implies $A^\beta u_j \to A^\beta u$, weakly in $(W^{m,p})'$). Therefore in this special case A^β is maximal monotone. So, in particular, for every $\lambda > 0$, the operator $A^{\beta_\lambda} : W^{m,p} \to (W^{m,p})'$ is maximal monotone (β_λ denotes the Yosida approximation of β). Equivalently, for every $h \in (W^{m,p})'$ and for every $\lambda > 0$, there exists $u_\lambda \in W^{m,p}$ such that

$$\text{(1)} \qquad F(u_\lambda) + A^{\beta_\lambda} u_\lambda = h.$$

where $F : W^{m,p} \to (W^{m,p})'$ is the duality mapping of $W^{m,p}$.

Let $u_0 \in W_\beta^{m,p}$ be fixed and denote

$$\text{(2)} \qquad h_0 = F(u_0) + A^\beta u_0.$$

(For convenience consider for the time being that β is single-valued, i.e. A^β is single-valued too).

From (1) and (2) we have

$$(F(u_\lambda)-F(u_0))\,(u-u_0)+(A^{\beta_\lambda}u-A^\beta u_0)\,(u_\lambda-u_0)=(h-h_0)\,(u_\lambda-u_0),$$

which yields

$$\text{(3)}\qquad \|u_\lambda\|^2_{W^{m,p}}+\langle\beta_\lambda(\operatorname{col}[u_\lambda(0),\ u_\lambda(1),\ \dots,\ u_\lambda^{(m-1)}(0),\ u_\lambda^{(m-1)}(1)])-$$
$$-\beta(\operatorname{col}[u_0(0),\ u_0(1),\ \dots,\ u_0^{(m-1)}(0),\ u_0^{(m-1)}(1)]),$$
$$\operatorname{col}[u_\lambda(0)-u_0(0),\ u_\lambda(1)-u_0(1),\ \dots]\rangle_{R^{2m}}\leqslant C_1\|u_\lambda\|_{W^{m,p}}+C_2.$$

From (3) we obtain by monotonicity of β_λ

$$\text{(4)}\qquad \|u_\lambda\|^2_{W^{m,p}}+\langle\beta_\lambda(\operatorname{col}[u_0(0),\ u_0(1),\ \dots])-$$
$$-\beta(\operatorname{col}[u_0(0),\ u_0(1),\ \dots]),\quad \operatorname{col}\ [u_\lambda(0)-$$
$$-u_0(0),\ u_\lambda(1)-u_0(1),\ \dots]\rangle_{R^{2m}}\leqslant C_1\|u_\lambda\|_{W^{m,p}}+C_2$$

On the other hand

$$\text{(5)}\qquad \lim_{\lambda\to 0}\beta_\lambda(\operatorname{col}[u_0(0),\ u_0(1),\ \dots])=\beta^0(\operatorname{col}[u_0(0),\ u_0(1),\ \dots]),$$

where β^0 is the minimal section of β. Also, as $W^{m,p}$ is continuously embedded in $C^{m-1}[0,1]$, we have

$$\text{(6)}\qquad \|\operatorname{col}[u_\lambda(0),\ u_\lambda(1),\ \dots]\|_{R^{2m}}\leqslant C_3\|u_\lambda\|_{W^{m,p}}.$$

Now, by (4), (5), (6) we obtain that the set

$$\text{(7)}\qquad \{u_\lambda;\ \lambda>0\}\ \text{is bounded in}\ W^{m,p}.$$

Taking into account eq. (1) we also have

$$\text{(8)}\qquad \{A^{\beta_\lambda}u_\lambda;\ \lambda>0\}\ \text{is bounded in}\ (W^{m,p})'.$$

Since $W^{m,p}$ is compactly embedded in $C^{m-1}[0,1]$ it follows by (7) that there is $u\in W^{m,p}$ such that (eventually on a subsequence)

$$\text{(9)}\qquad u_\lambda^{(j)}\to u^{(j)},\ \text{as}\ \lambda\to 0,\ \text{in}\ C[0,1]\ (j=0,1,\dots,m-1).$$

In particular (on the same subsequence)

$$\text{(10)}\qquad u_\lambda^{(j)}(0)\to u^{(j)}(0),\quad u_\lambda^{(j)}(1)\to u^{(j)}(1),\quad \text{as}\ \lambda\to 0\quad (j=0,1,\dots,m-1).$$

According to (8), the set

$$\{\langle\beta_\lambda(\operatorname{col}[u_\lambda(0),\ u_\lambda(1),\ \dots]),\ \operatorname{col}[v(0),\ v(1),\ \dots]\rangle_{R^{2m}},\ \lambda>0\}$$

is bounded, for every $v\in W^{m,p}$. Therefore

$$\text{(11)}\qquad \{\beta_\lambda(\operatorname{col}[u_\lambda(0),\ u_\lambda(1),\ \dots]);\ \lambda>0\}\ \text{is bounded in}\ R^{2m}$$

(because the set $\{\operatorname{col}[v(0),\ v(1),\ \dots];\ v\in W^{m,p}\}$ coincides to R^{2m}). Using (10) and (11) it follows by a standard argument that $u\in W^{m,p}_\beta$ and there exists a sequence $\lambda_n\to 0$ (a subsequence of that extracted above) such that

$$\text{(12)}\qquad \beta_{\lambda_n}(\operatorname{col}[u_{\lambda_n}(0),\ u_{\lambda_n}(1),\ \dots])\to w_0\in\beta(\operatorname{col}[u(0),\ u(1),\ \dots]).$$

In other words

$$A^{\beta_{\lambda_n}}u_{\lambda_n}\to v^\beta\in A^\beta(u),\ \text{weakly in}\ (W^{m,p})'.$$

Therefore, taking into account eq. (1), we have

$$\text{(13)}\qquad F(u_{\lambda_n})\to h-v^\beta,\ \text{weakly in}\ (W^{m,p})',\ \text{with}\ v^\beta\in A^\beta u.$$

Now using (10), (12) and (1) we deduce that

$$\text{(14)}\qquad \lim_{\substack{n\to\infty\\ m\to\infty}}(F(u_{\lambda_n})-F(u_{\lambda_m}))\,(u_{\lambda_n}-u_{\lambda_m})=0.$$

Remember that F is a maximal monotone operator from $W^{m,p}$ into $(W^{m,p})'$. Then, by (13) and (14) and the fact that $u_{\lambda_n}\to u$ weakly in $W^{m,p}$ we deduce, by virtue of Lemma 1.3 in V. Barbu [1, p. 42] that

$$h-v^\beta=F(u),\ \text{where}\ v^\beta\in A^\beta u.$$

Therefore A^β is maximal monotone. Q.E.D.

4. Existence and regularity of variational solutions. Assume that assumptions (H_1), (H_2) and (H_3) are fulfilled. Then, as seen above both operators A and A^β are maximal monotone. In addition A is everywhere defined on $W^{m,p}$. Therefore $A+A^\beta$ *is maximal monotone* too, by the well-known perturbation result due to R. T. Rockafellar (see, e.g., V. Barbu [1, pp. 46–48]). So, a sufficient condition for the surjectivity of $A+A^\beta$ (i.e. for the existence of at least one variational solution for problem (E), (B.C.)) is the *coerciveness* of $A+A^\beta$. Specifically, let us assume (in addition to (H_1), (H_2), (H_3)) that

$$(H_4)\qquad (\exists)\ u_0\in W^{m,p}_\beta\ \text{such that}\ \lim_{\|u\|_{W^{m,p}}\to\infty}a(u,\ u-u_0)/\|u\|_{W^{m,p}}=+\infty$$

(i.e. operator A is coercive with respect to u_0). Then, it is easy to see that $A+A^\beta$ is also coercive with respect to u_0. Indeed, by the monotonicity of A^β, we have

$$(Au+A^\beta u)\,(u-u_0)\geqslant(Au)\,(u-u_0)+(A^\beta u_0)\,(u-u_0),$$

which by (H_4) implies that $A+A^\beta$ is coercive with respect to u_0. Then it is easy to see that $(A+A^\beta)^{-1}$ is bounded on bounded subsets of its domain (which coincides to the range of $A+A^\beta$). Then by a standard result in the monotone operator theory (see V. Barbu [1, p. 45]) it follows that the range of $A+A^\beta$ is the whole $(W^{m,p})'$. Hence we have proved the following

Theorem 2. *Assume $(H_1)\sim(H_4)$ hold. Then, for any $f\in(W^{m,p})'$ problem (E), (B.C.) has at least one variational solution.*

Remark 2. Hypotheses (H_4) is trivially satisfied if A_k verify, for example, the following condition:

$$\text{(i)}\qquad \text{there exist}\ r^0=(r^0_0,\ r^0_1,\ \dots,\ r^0_m)\in D(\beta),\ C_0>0$$

and $g_1\in L^1(0,1)$ such that

$$\sum_{k=0}^{m} A_k(x, \xi)(\xi_k - \eta_k^0) \geq C_0 \|\xi\|_{R^{m+1}}^p + g_1(x), \text{ a.e. } x \in]0,1[$$
$$(\forall)\ \xi \in R^{m+1}.$$

It is also obvious that the existence of variational solutions for $f \in (W^{m,p})'$ is assured by (H_1), (H_3) together with the strong ellipticity condition:

$$(H_2)' \qquad \sum_{k=0}^{m} (A_k(x, \xi) - A_k(x, \eta))(\xi_k - \eta_k) \geq \omega \|\xi - \eta\|_{R^{m+1}}^p \ (\omega > 0),$$

for a.e. $x \in]0,1[$, $(\forall)\ \xi, \eta \in R^{m+1}$.
Indeed, we have the following implications

$$(H_2)' \Rightarrow (H_2) \text{ and } (H_1) + (H_2)' \Rightarrow (I).$$

Moreover, in this case we also have the uniqueness of the variational solutions.

Remark 3. It should be noticed that in certain situations the variational solutions (if they exist) are classical solutions of equation (E) and verify (B.C.) in usual sense, provided that f belongs to some function space.

Let us consider here only the simplest case in which (E) is a linear equation. Namely, let us assume that

$$(15) \qquad A_k(x, \xi) = \sum_{k=0}^{m} a_{ik}(x)\, \xi_i \quad (k = 0, 1, \ldots, m),$$

where the functions a_{ik} satisfy the following hypotheses:

$$(I_1) \qquad a_{ik} \in W^{k,\infty}(0,1) \quad (i, k = 0, 1, \ldots, m),$$

$$(I_2) \qquad \sum_{i,k=0}^{m} a_{ik}(x)\, \xi_i \xi_k \geq \omega \|\xi\|_{R^{m+1}}^2, \quad (\omega > 0), \quad \text{a.e. } x \in]0,1[,$$
$$(\forall)\ \xi \in R^{m+1}.$$

We are able to prove the following existence and regularity result.

Proposition 1. *Assume that* (I_1), (I_2) *and* (H_3) *hold and* $f \in L^r(0,1)$, *for some* $r \in [1, +\infty]$. *Then problem* (E), (B.C.) *(where* A_k *are defined by* (15)) *has a unique classical solution* $u \in W^{2m,r}$ *(more precisely,* u *satisfies* (E) *for a.e.* $x \in]0,1[$ *and* u *satisfies* (B.C.) *for every* $x \in [0,1]$).

Proof. First of all, we remark that $(I_1) \Rightarrow (H_1)$ (with $p = 2$, and $g(x) \equiv 0$) and $(I_2) \Rightarrow (H_2)'$. Therefore, according to the Theorem 2 and Remark 2, there exists a unique variational solution $u \in W_\beta^{m,2}$ of problem (E), (B.C.), i.e.

$$(16) \qquad \int_0^1 fv\,dx \in a(u, v) + a_\beta(u, v), \quad (\forall)\ v \in W^{m,2}.$$

In particular (16) holds for every test function $v \in \mathcal{D}(0,1)$, i.e.

$$(17) \qquad \sum_{i,k=0}^{m} (-1)^k [a_{ik} u^{(i)}]^{(k)} = f, \text{ in the sense of } \mathcal{D}'(0,1).$$

Now, define the function

$$(18) \qquad w(x) := \sum_{i=0}^{m} (-1)^m a_{im}(x)\, u^{(i)}(x) + \sum_{i=0}^{m} (-1)^{m-1} \int_0^x a_{i,m-1}(s)\, u^{(i)}(s)\, ds +$$
$$+ \sum_{i=0}^{m} \sum_{k=0}^{m} (-1)^k \int_0^x ds_{m-k-1} \int_0^{s_{m-k-1}} ds_{m-k-2} \cdots \int_0^{s_1} a_{ik}(s)\, u^{(i)}(s)\, ds.$$

Obviously $w \in L^2(0,1)$ and its distributional derivative of order m coincides to f. Therefore $w \in W^{m,r}$. So by performing successively in (18) the derivatives w', w'', ..., $w^{(m)}$ we deduce by virtue of hypotheses (I_1) and (I_2) (which implies in particular that $a_{mm}(x) \geq \omega$, for $0 \leq x \leq 1$) that $u \in W^{2m,r}$. Then, obviously u satisfies (17) (i.e. u satisfies (E)) for a.e. $x \in]0,1[$.

Now, using again (16) we deduce easily that u satisfies (B.C.) for every $x \in [0,1]$, (where the derivatives are understood in usual sense). Q.E.D.

5. Examples. Let us first point out two important particular cases for which the variational solutions in the sense of Definition 1 (see Section 3 above) are just variational solutions in usual sense (see, e.g., [6, pp. 272–274]), provided that f is in $(W^{m,p})'$.

Example 1 (Dirichlet). Let β be the subdifferential of Ψ, where $\Psi : R^{2m} \to]-\infty, +\infty]$ is defined by

$$\Psi(z) = \begin{cases} 0, & \text{for } z = 0 \in R^{2m} \\ +\infty, & \text{otherwise.} \end{cases}$$

Then $W_\beta^{m,p}$ coincides to $W_0^{m,p}$ and problem (E), (B.C.) becomes

$$\begin{cases} T_{2m} u = f \\ u^{(j)}(0) = u^{(j)}(1) = 0 \quad (j = 0, 1, \ldots, m-1). \end{cases}$$

Example 2 (Neumann). Let $\beta \equiv 0$. Then $W_\beta^{m,p}$ coincides to $W^{m,p}$ and problem (E), (B.C.) becomes

$$\begin{cases} T_{2m}\, u = f \\ L_{2m-j}\, u = 0, \text{ for } x = 0 \text{ and } x = 1 \quad (j = 1, 2, .., m). \end{cases}$$

Example 3. Let β be a $2m \times 2m$-positive semi-definite matrix (not necessarly symmetric, i.e. β is not necessarly a subdifferential). In this case (B.C.) (where „$\in$" must be replaced by „$=$") represents some *linear boundary conditions*. Notice that even in this linear case we effectively go beyond the case of subdifferential in (B.C.).

Example 4. Assume that A_k are defined by (15), where a_{ik} satisfy, for instance, (I_1), (I_2) and, besides,

$$a_{ik}^{(j)}(0) = a_{ik}^{(j)}(1) \qquad (i = 0, 1, \ldots, m;\ j = 0, 1, \ldots, k-1;\ k = 1, \ldots, m).$$

Let β be the subdifferential of the function
$\Psi_1 : R^{2m} \to]-\infty, +\infty]$ defined by

$$\Psi_1(\text{col}\,[r_1, s_1, \ldots, r_m, s_m]) = \begin{cases} 0, & \text{if } r_k = s_k (k = 1, 2, \ldots, m) \\ +\infty, & \text{otherwise.} \end{cases}$$

Then it is easy to see that (E), (B.C.) becomes

$$\begin{cases} \sum_{i,k=0}^{m} (-1)^k [a_{ik} u^{(i)}]^{(k)} = f \\ u^{(j)}(0) = u^{(j)}(1) \quad (j = 0, 1, \ldots 2m-1) \text{ (periodic BC's).} \end{cases}$$

For other examples of boundary conditions which can be put in the form (B.C.) we refer the reader to the paper [5] of the authors.

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A NOTE ON INTEGRODIFFERENTIAL INEQUALITIES FOR DIFFERENTIAL EQUATIONS

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§1. Introduction. A result of fundamental and far-reaching importance in the study of existence, uniqueness, boundedness and stability properties of ordinary differential equations is the Gronwall inequality [1, p. 134]. Recently, the author has established some new integral inequalities of this type [9], [10]. In this paper, we wish to establish some new integral inequalities and integrodifferential inequalities which may be used as a tool in applications (see [1], [4] — [6]). The results obtained are extensions to those in [2], [3], [8] — [11].

§2. Main results. In this section we wish to establish some interesting and useful integrodifferential inequalities which can be used in the study of many problems concerning the behaviour of solutions of a class of differential and integrodifferential equations. A useful general version of Gronwall's inequality may be stated as follows:

Theorem 2.1. *Let $x(t)$, $f(t)$, $g(t)$ and $h(t)$ be real-valued nonnegative continuous functions defined on I, for which the inequality*

$$x(t) \leq x_0 + \int_0^t f(s)\, x(s)\, ds + \int_0^t f(s) \left(\int_0^s g(\tau) x(\tau)\, d\tau\right) ds + \int_0^t f(s) \left(\int_0^s g(\tau) \left(\int_0^t h(k)\, x^{\alpha}(k)\, dk\right) d\tau\right) ds \tag{2.1}$$

holds for all $t \in I$, where x_0 is a nonnegative constant and $0 \leq \alpha < 1$, then

$$x(t) \leq x_0 + \int_0^t f(s) \exp\left(\int_0^s f(\tau)\, d\tau\right) \Big[x_0 + \int_0^s g(\tau) \exp\left(\int_0^\tau g(k)\, dk\right) \left[x_0^{1-\alpha} + (1-\alpha) \int_0^\tau h(k) \exp\left((-1+\alpha) \int_0^k [f(n)+g(n)]\, dn\right) dk\right]^{(1-\alpha)^{-1}} d\tau\Big]\, ds \tag{2.2}$$

for all $t \in I$.

Proof. Define a function $m(t)$ by the right member of (2.1). Then

$$m'(t) = f(t)x(t) + f(t)\int_0^t g(\tau)\, x(\tau)\, d\tau + f(t)\int_0^t g(\tau) \left(\int_0^\tau h(k)\, x^{\alpha}(k)\, dk\right) d\tau, \quad m(0) = x_0$$

which, in view of (2.1), implies