$$\begin{cases} \sum_{i,k=0}^{m} (-1)^k [a_{ik} u^{(i)}]^{(k)} = f \\ u^{(j)}(0) = u^{(j)}(1) \quad (j=0,1, \dots, 2m-1) \ (\textit{periodic BC}^*). \end{cases}$$

For other examples of boundary conditions which can be put in the form (B.C.) we refer the reader to the paper [5] of the authors.

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A NOTE ON INTEGRODIFFERENTIAL INEQUALITIES FOR DIFFERENTIAL EQUATIONS

BY

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§1. Introduction. A result of fundamental and far-reaching importance in the study of existence, uniqueness, boundedness and stability properties of ordinary differential equations is the Gronwall inequality [1, p. 134]. Recently, the author has established some new integral inequalities of this type [9], [10]. In this paper, we wish to establish some new integral inequalities and integrodifferential inequalities which may be used as a tool in applications (see [1], [4] — [6]). The results obtained are extensions to those in [2], [3], [8] — [11].

§2. Main results. In this section we wish to establish some interesting and useful integrodifferential inequalities which can be used in the study of many problems concerning the behaviour of solutions of a class of differential and integrodifferential equations. A useful general version of Gronwall's inequality may be stated as follows:

Theorem 2.1. *Let $x(t)$, $f(t)$, $g(t)$ and $h(t)$ be real-valued nonnegative continuous functions defined on I, for which the inequality*

$$x(t) \leqslant x_0 + \int_0^t f(s)\, x(s)\, ds + \int_0^t f(s) \left(\int_0^s g(\tau) x(\tau) d\tau \right) ds \tag{2.1}$$
$$+ \int_0^t f(s) \left(\int_0^s g(\tau) \left(\int_0^\tau h(k)\, x^\alpha(k)\, dk \right) d\tau \right) ds$$

holds for all $t \in I$, where x_0 is a nonnegative constant and $0 \leqslant \alpha < 1$, then

$$x(t) \leqslant x_0 + \int_0^t f(s) \exp \left(\int_0^s f(\tau)\, d\tau \right) [x_0 + \int_0^s g(\tau) \exp \left(\int_0^\tau g(k)\, dk \right) \tag{2.2}$$
$$[x_0^{1-\alpha} + (1-\alpha) \int_0^\tau h(k) \exp \left((-1+\alpha) \int_0^k [f(n)+g(n)]\, dn \right) dk]^{(1-\alpha)^{-1}} d\tau]\, ds$$

for all $t \in I$.

Proof. Define a function $m(t)$ by the right member of (2.1). Then

$$m'(t) = f(t)x(t) + f(t) \int_0^t g(\tau)\, x(\tau)\, d\tau + f(t) \int_0^t g(\tau) \left(\int_0^\tau h(k)\, x^\alpha(k)\, dk \right) d\tau, \quad m(0) = x_0$$

which, in view of (2.1), implies

$$m'(t) \leq f(t)\ \left[m(t) + \int_0^t g(\tau)\, m(\tau)\, d\tau + \int_0^t g(\tau)\left(\int_0^\tau h(k)\, x^\alpha(k)\, dk\right) d\tau\right]. \tag{2.3}$$

Define

$$v(t) = m(t) + \int_0^t g(\tau)\, m(\tau)\, d\tau + \int_0^t g(\tau)\left(\int_0^\tau h(k)\, m^\alpha(k)\, dk\right) d\tau, \tag{2.4}$$

$$v(0) = m(0) = x_0.$$

Then it follows from (2.3) and (2.4) and the fact that $m(t) \leq v(t)$, the inequality

$$v'(t) \leq f(t)\, v(t) + g(t)\ \left[v(t) + \int_0^t h(k)\, v^\alpha(k)\, dk\right]. \tag{2.5}$$

is satisfied. If we define

$$r(t) = v(t) + \int_0^t h(k)\, v^\alpha(k)\, dk,\ r(0) = v(0) = x_0. \tag{2.6}$$

Then, it follows from (2.5) and (2.6) and the fact that $v(t) \leq r(t)$, the inequality

$$r'(t) \leq (f(t) + g(t))\, r(t) + h(t)\, r^\alpha(t)$$

is satisfied which implies the estimation for $r(t)$ such that

$$r(t) \leq \exp\left(\int_0^t (f(n) + g(n))\, dn\right)\ \left[x_0^{(1-\alpha)} + (1-\alpha)\int_0^t h(k)\exp\left((\alpha-1)\int_0^k (f(n) + g(n))\, dn\right) dk\right]^{(1-\alpha)^{-1}}.$$

Then from (2.5) we have

$$v'(t) \leq f(t)v(t) + g(t)r(t).$$

or

$$v'(t) \leq f(t)v(t) + g(t)\ \left\{\exp\left(\int_0^t (f(n)+g(n))dn\right)\ \left[x_0^{1-\alpha} + (1-\alpha)\times \int_0^t h(k)\exp\left((\alpha-1)\int_0^k (f(n)+g(n))\, dn\right) dk\right]^{(1-\alpha)^{-1}}\right\}.$$

which implies the estimate for $v(t)$ such that

$$v(t) \leq \exp\left(\int_0^t f(\tau)\, d\tau\right)\ \left[x_0 + \int_0^t g(\tau)\exp\left(\int_0^\tau g(k)\, dk\right) \times \left[x_0^{1-\alpha} + (1-\alpha)\int_0^\tau h(k)\exp\left((\alpha-1)\int_0^k (f(n)+g(n))\, dn\right) dk\right]^{(1-\alpha)^{-1}} d\tau\right].$$

Substituting this value of $v(t)$ in (2.3) we have

$$m'(t) \leq f(t)\ \left[\exp\int_0^t f(\tau)\, d\tau\right)\ \left[x_0 + \int_0^t g(\tau)\exp\left(\int_0^\tau g(k)\, dk\right) \times \left[x_0^{1-\alpha} + (1-\alpha)\int_0^\tau h(k)\exp\left((\alpha-1)\int_0^k (f(n)+g(n))\, dn\right) dk\right]^{(1-\alpha)^{-1}} d\tau\right]\right]. \tag{2.7}$$

Now integrating both sides of (2.7) from 0 to t and substituting the value of $m(t)$ in (2.1) we obtain the desired bound in (2.2).

Remark 1. The above theorem is an extension to the results of Candra and Davis [3], Beesack [2], and Reddy ([9], Th. 4.1).

Next we shall establish a new integrodifferential inequality of the Gronwall type which can be used as a tool in some applications.

Theorem 2.2. *Let $y(t)$, $\dot{y}(t)$, and $b(t)$ be real-valued nonnegative continuous functions defined on $I = [0, \infty)$, for which the inequality*

$$\dot{y}(t) \leq a + \int_0^t b(s)\ ((\dot{y}(s))^\alpha\ [y(s) + \dot{y}(s)]\ ds,\ \ t \in I, \tag{2.8}$$

holds, where a is a positive constant and $0 < \alpha < 1$. If

$$1 - \alpha\ \ [a + y(0)]^\alpha \int_0^t b(s)\ e^{\alpha s}\ ds > 0,\ \ t \in I.$$

then

$$\dot{y}(t) \leq [a^{1-\alpha} + (1-\alpha)\int_0^t b(s)\ Q(s)\ ds]^{1/(1-\alpha)} \tag{2.9}$$

for all $t \in I$, where

$$Q(t) = [a + y(0)]\ e^t / \{1 - \alpha(a + y(0))^\alpha \int_0^t b(s)\, e^{\alpha s} ds\}^{1/\alpha} \tag{2.10}$$

for all $t \in I$.

Proof. Define

$$m(t) = a + \int_0^t b(s)\ (\dot{y}(s))^\alpha\ [y(s) + \dot{y}(s)]\, ds,\ \ m(0) = a, \tag{2.11}$$

then (2.8) can be written in the form

$$\dot{y}(t) \leq m(t). \tag{2.12}$$

Integrating both sides of (2.12) from 0 to t we obtain

$$y(t) \leq y(0) + \int_0^t m(s)\, ds. \tag{2.13}$$

Differentiating (2.11) and using (2.12) and (2.13) we get

$$\dot{m}(t) \leq b(t)\ m^\alpha(t)\ [y_0 + m(t) + \int_0^t m(s)\ ds]. \tag{2.14}$$

Define

(2.15) $$u(t)=y(0)+m(t)+\int_0^t m(s)\,ds,\quad u(0)=y(0)+a.$$

then differentiating (2.15) and using (2.14) and the facts $y(0)+\int_0^t m(s)ds\leqslant u(t)$ and $m(t)\leqslant u(t)$ from (2.15) we see that the inequality

(2.16) $$\dot u(t)\leqslant b(t)\,u^{\alpha+1}(t)+u(t),$$

is satisfied for all $t\in I$, which implies the estimation for $u(t)$ such that

(2.17) $$u(t)\leqslant [a+y(0)]e^t/\{1-\alpha[a+y(0)]^{\alpha}\int_0^t b(s)\,e^{\alpha s}ds\}^{1/\alpha}=Q(t).$$

for all $t\in I$. Substituting this value of u in (2.14) we obtain

(2.18) $$\dot m(t)\leqslant b(t)Q(t)m^{\alpha}(t),$$

for all $t\in I$. Using (2.15) and (2.17) we can easily find the following estimation for $m(t)$

$$m(t)\leqslant [a^{1-\alpha}+(1-\alpha)\int_0^t b(s)\;Q(s)\;ds]^{1/(1-\alpha)}.$$

Now, substituting this value of $m(t)$ in (2.8) we obtain the desired bound in (2.9). This completes the proof of the theorem.

We next establish the following integrodifferential inequality which can be used in certain situations.

Theorem 2.2. *Let $y(t)$, $\dot y(t)$, and $b(t)$ be real-valued nonnegative continuous functions defined on I, for which the inequality*

(2.19) $$\dot y(t)\leqslant a+\int_0^t b(s)\;\dot y(s)\;(y^2(s)+\dot y(s))\,ds,\quad t\in I,$$

holds, where a is a positive constant. If

$$1-[a+y(0)+y^2(0)]\int_0^t [2+b(s)]\;e^s\,ds>0,\quad t\in I,$$

and

$$1-[a+y^2(0)]\int_0^t [b(s)-2]\;\exp\;\left(\int_0^s 2Q_1(r)dr\right)\,ds>0,\quad t\in I$$

then

(2.20) $$\dot y(t)\leqslant a\exp\left(\int_0^t b(s)\;Q_2(s)\;ds\right)$$

for all $t\in I$, where

(2.21) $$Q_1(t)=[a+y(0)+y^2(0)]\,e^t\,[1-(a+y(0)+y^2(0))\int_0^t (2+b(s))\,e^s ds]$$

and

(2.22) $$Q_2(t)=\frac{[a+y^2(0)]\exp\left(\int_0^t 2Q_1(s)\,ds\right)}{1-(a+y^2(0))\int_0^t (b(s)-2)\;\exp\left(\int_0^s 2Q_1(r)\,dr\right)ds}$$

for all $t\in I$.

Proof. Define

(2.23) $$m(t)=a+\int_0^t b(s)\;\dot y(s)\;[y^2(s)+\dot y(s)]\,ds,\quad m(0)=a.$$

Then (2.19) takes the form

(2.24) $$\dot y(t)\leqslant m(t).$$

Integrating from 0 to t, we obtain

(2.25) $$y(t)=y(0)+\int_0^t m(s)\,ds.$$

Differentiating (2.23) from 0 to t and using (2.24) and (2.25) we have

(2.26) $$\dot m(t)=b(t)\;\dot y(t)\;[y^2(t)+\dot y(t)]$$
$$\leqslant b(t)\,m(t)\;\{[y(0)+\int_0^t m(s)\,ds]^2+m(t)\}\leqslant b(t)m(t)\;u(t),$$

where

(2.27) $$u(t)=m(t)+[y(0)+\int_0^t m(s)ds]^2,\quad \text{with}\quad u(0)=a+y^2(0).$$

Differentiating (2.27) with respect to t from 0 to t, and after long but tedious calculations we find the estimate for $u(t)$ such that

$$u(t)\leqslant\frac{[a+y^2(0)]\exp\left(\int_0^t 2\,Q_1(s)ds\right)}{1-[a+y^2(0)]\int_0^t [b(s)-2]\exp\left(\int_0^s 2Q_1(r)dr\right)ds}=Q_2(t).$$

where

$$Q_1(t)=\alpha e^t\;(1-\alpha\int_0^t (2+b(s))\;e^s ds.$$

$$\alpha=[a+y(0)+y^2(0)]\;e^t,\quad t\in I.$$

Now substituting this estimate of $u(t)$ in (2.26), we obtain

$$\dot m(t)\;m(t)\leqslant b(t)\;Q_2(t).$$

Thus we have

$$m(t)=a\exp\int_0^t b(s)\;Q_2(s)\;ds.$$

Substituting this value of $m(t)$ in (2.19), we obtain the bound (2.20). This completes the proof of the theorem.

§3. Applications. In 1972, Imanaliev and Ved' [5], and Karpievic [6] have studied the stability of solutions of certain class of integrodifferential equations. In this section we indicate an application of our Theorem 2.2 to obtain the bound on the solution of a class of second order differential equations of the form

$$\ddot{y}=\dot{y}^{\alpha}(t)F(t,\ y(t),\ \dot{y}(t)),\ \dot{y}(0)=a;\ \dot{y}(0)=c. \tag{3.1}$$

where $F\in C\ [I\times R\times R,\ R]$, and a and C are given positive constants. Assume that the function F satisfies

$$|F(t,\ y(t),\ \dot{y}(t))|\leqslant b(t)\ [|\ y(t)\ |+|\ \dot{y}(t)\ |],\ t\in I, \tag{3.2}$$

where $b(t)$ is a real-valued nonnegative continuous function defined on I. Further, we assume that

$$1-\alpha[a+c]^{\alpha}\int_0^t b(s)\,e^{\alpha s}\,ds>0,\ t\in I. \tag{3.3}$$

Integrating (3.1) from 0 to t, $t\in I$, we obtain

$$\dot{y}(t)=a+\int_0^t \dot{y}^{\alpha}F(s,\ u(s),\ \dot{y}(s))\,ds. \tag{3.4}$$

Taking the absolute values on both sides and using the condition (3.2) in (3.4) we obtain

$$|\ \dot{y}(t)\ |\leqslant a+\int_0^t b(s)\ |\ \dot{y}^{\alpha}(s)\ |\ [|\ y(s)\ |+\dot{y}(s)]\ ds. \tag{3.5}$$

Now, an application of Theorem 2.2 yields

$$|\ \dot{y}(t)\ |\leqslant a^{1-\alpha}+(1-\alpha)\int_0^t b(s)\ Q_0(s)\,ds]^{1/(1-\alpha)}, \tag{3.6}$$

for all $t\in I$, where

$$Q_0(t)=[a+c]\,e^t\,[1-\alpha(a+c)^{\alpha}\int_0^t b(s)\ e^{\alpha s}ds]^{1/\alpha},$$

for all $t\in I$. Thus (2.6) gives us a bound on $|\ \dot{y}(t)\ |$ in terms of the known functions and hence integrating both sides of (2.6) from 0 to t we can obtain a bound on $|\ y(t)\ |$ for all $t\in I$, where $y(t)$ is a solution of (3.1).

Remark. We note that Theorem 2.3 can be used to establish the bound on the solutions of second order differential equations of the form

$$\ddot{y}=\dot{y}(t)\,g(t,\ y^2(t),\ \dot{y}(t)),\ \dot{y}(0)=a,\ y(0)=c, \tag{3.7}$$

under some suitable conditions on the function involved in (3.7).

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