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A NOTE ON TOPOLOGICAL COLLARS IN HOMOLOGY MANIFOLDS

BY

R. AYALA and A. QUINTERO

Basic definitions. A homology manifold ((HML, n)-manifold) is an euclidean polyhedron $M=|K|$ such that $\tilde{H}_*(\mathrm{lk}\,(x:K)$ is $\tilde{H}_*(S^{n-1})$ or 0, for each $x\in M$. The set of points with the latter property is called the boundary of M and is a (HML, $n-1$)-manifold, ∂M, without boundary which is the underlying space of a subcomplex $\partial K\subset K$.

A homology n-sphere is a closed (without boundary and compact) HML-manifold with the same homology as S^n. A homology n-ball is an acyclic compact (HML, n)-manifold. For a general reference of the usual properties of homology manifolds, see [4].

A pl-map $f: P\to Q$ is called a h-resolution if it is onto and $f^{-1}(x)$ is contractible for each $x\in Q$.

Given a metric space X, $A\subset X$ is locally 1-coconnected (1-LCC) in X if for each $x\in \overline{X-A}$ and $\varepsilon>0$ there is $k>0$ such that each loop in $(X-A)\cap B(x;k)$ is null-homotopic in $(X-A)\cap B(x;\varepsilon)$.

Topological collars in homology manifolds. 1. Definition. *A HML-manifold with boundary admits a topological collar* (TOP-*collar*) *if there exists an open $\partial M\subset U$ and a homeomorphism $h: U\to\partial M\times[0,1)$ such that $h(x)=(x,0)$ for each $x\in\partial M$.*

2. *Remark.* If G_n is a contractible Glaser manifold [3] and $\sigma\subset\partial G_n$ is a $(n-1)$-simplex, the cone $M=c*(\partial G_n-\mathring{\sigma})$ is a contractible homology ball which does not admit a TOP-collar because $\mathring{M}=M-\partial M=(\partial G_n-\sigma)\times\mathbf{R}$ is not simply connected.

By using results about ends of maps, F. Quinn gives in [5; 3.4.2] conditions for the existence of TOP-collars for generalized manifolds. Quinn's results in the case of HML-manifolds are included in

3. **Theorem.** *Let $M=|K|$ be a (HML, n)-manifold with boundary and $n\geqslant 5$. The following statements are equivalent*

a) *For each $\sigma\in\partial K$, $\mathring{\mathrm{lk}}(\sigma;K)=\mathrm{lk}(\sigma;K)-\mathrm{lk}(\sigma;\partial K)$ is simply connected.*

b) *∂M admits a TOP-collar in M*

c) *∂M is 1-LCC in M.*

d) *For each $x\in\partial M$, $\mathring{\mathrm{lk}}(x;K)=\mathrm{lk}(x;K)-\mathrm{lk}(x;\partial K)$ is simply connected.*

Proof. The condition $n\geqslant 5$ is only used in a)$\Rightarrow$b). The others are general. Specially, b)$\Rightarrow$c) is obvious.

a)⇒b). Let N be a regular neighbourhood of ∂M in M. If $\partial N=\dot{N}\cup\partial M$ and $r:\dot{N}\to\partial M$ is the restriction of the natural retraction of N onto ∂M, it is easy to prove that a) implies that r is a h-resolution. Because we work with ANR-spaces, r is actually a cell-like map. As $\partial M\times(0,1)$ and $N\times(0,1)$ are TOP-manifolds (see [1]), we can apply the Siebenmann Approximation Theorem ([7, A]) to $r\times\mathrm{id}$ and get a homeomorphism $f:\dot{N}\times(0,1)\to\partial M\times(0,1)$ such that $d(f, r\times\mathrm{id})<1/k$ over $\partial M\times(0,1/k)$ $(k=1,2,\dots)$. This condition lets us extend f to a homeomorphism $\tilde{f}: N-\dot{N}\to\partial M\times[0,1)$ with $\tilde{f}=\mathrm{id}$ in ∂M.

c)⇒d). If $x\in\partial M$ and $\mathrm{st}(x:K)=\mathrm{st}(x:K)-\mathrm{lk}(x:K)$, there exists a neighbourhood of x, U, such that each loop in $U-\partial M$ is null-homotopic in $\mathrm{st}(x:K)-\partial M\simeq\mathring{\mathrm{lk}}(x:K)\times\mathbf{R}$. Let K' be a subdivision of K with $\mathring{\mathrm{st}}(x:K')\subseteq U$, then $\mathrm{lk}(x:K')$ is a deformation retract of $\mathrm{st}(x:K)-\partial M$.

d)⇒a). It follows from $\mathrm{lk}(x:K)\simeq\mathrm{lk}(\sigma:K)\times\mathbf{R}^{n-k}$, where $x\in\sigma$ and $\dim\sigma=k$.

Now we are going to replace the condition a) of the above theorem by a condition a′) about the closed links.

4. Proposition. *If $M=|K|$ is a HML-manifold, the inclusion $\mathring{M}=M-\partial M\to M$ induces an onto morphism between the fundamental groups. So, $\pi_1(\mathring{M})=1$ implies $\pi_1(M)=1$.*

Proof. Let K' be the second barycentric subdivision of K. We can deform each $f:S^1\to M$ and obtain a map g, homotopic to f, with $g(S^1)$ lying in the 1-skeleton of the dual cell complex of K'. This skeleton admits a neighbourhood, N, which is TOP-manifold. The TOP-collar of ∂N lets us modify g and obtain a homotopic map missing ∂M.

5. *Remarks.* a) The HML-manifold M defined in Remark 2 shows that the converse of Proposition 4 is false.

b) Let M be the above HML-Manifold and G_n a Glaser manifold. We define V as the gluing of M with G_n by $\partial G_n-\mathring{\sigma}$. Whitehead's Theorem shows that V is contractible. Theorem 3 shows that ∂V does not admit a TOP-collar in V. So, $\mathring{V}$ has the same homotopy type as V although ∂V has not TOP-collar.

6. Lemma. *Let $M=|K|$ be a simply connected $(HML, 4)$-manifold with boundary such that $\pi_1(\mathrm{lk}(v:K))=1$ for each vertex $v\in\partial K$. Then $\mathring{M}=M-\partial M$ is also simply connected.*

Proof. M fails to be a PL-manifold only at some singular vertices of K, and we can assume that there only is a singular vertex $v\in\partial K$. Since $\mathrm{lk}(v:K)$ is PL-manifold we get $\pi_1(\mathring{\mathrm{lk}}(v:K))=\pi_1(\mathrm{lk}(v:K))$. Then $M'=M-\mathrm{st}(v:K)$ and $\tilde{M}=M'-\partial M$ are simply connected. Van Kampen's Theorem shows that $\mathring{M}=\tilde{M}\cup(\mathrm{st}(v:K)-\mathrm{st}(v:\partial K))$ is simply connected.

7. Proposition. *Let $M=|K|$ be a HML-manifold with boundary. The following are equivalent.*

a) *For each $\sigma\in\partial K$, $\pi_1(\mathring{\mathrm{lk}}(\sigma:K))=1$.*

a′) *For each $\sigma\in\partial K$, $\pi_1(\mathrm{lk}(\sigma:K))=1$.*

Proof. a)⇒a′) is just Proposition 4.

a′)⇒a). Let $\sigma^{n-k}\in\partial K$ be an $(n-k)$-simplex. If $k\leq 4$, $\dim\mathrm{lk}(\sigma^{n-k}:K)\leq 3$ and we deal with PL-manifolds.

The case $k=5$ follows from Lemma 6. If $k=6$ we use Theorem 3 and the latter case and we see that $\partial\mathrm{lk}(\sigma^{n-6}:K)$ admits a TOP-collar in $\mathrm{lk}(\sigma^{n-6}:K)$. If $k>6$ we use Theorem 3 inductively.

8. *Remarks.* 1) Although the condition a) can be substituted by a′) in Theorem 3, this is not possible with d) and

d′) For each $x\in\partial M$, $\pi_1(\mathrm{lk}(x:K))=1$.

Actually, if M is the HML-manifold of Remark 2, the double suspension $W=\Sigma^2M$ satisfies $\pi_1(\mathrm{lk}(x:K))=1$ for each $x\in\partial W$. But ∂W does not admit a TOP-collar because $\pi_1(W)=1$ whereas $\mathring{W}=M\times\mathbf{R}^2$ is not simply connected

2) The condition d) cannot be substituted by

d″) For each vertex $v\in\partial K$, $\pi_1(\mathring{\mathrm{lk}}(v:K))=1$.

The following example shows this fact: Let V be the HML-manifold of Remark 5.b) and W the gluing of the cones $c*V$ and $c*G_n$ by $c*(\partial G_n-\sigma)$. Then ∂W is not $1-LCC$ in W but, $\pi_1(\mathring{\mathrm{lk}}(v:J))=1$ for each vertex v of the natural triangulation, J, of W.

The previous results and the Edwards-Cannon Characterization [1, 3.5] give us

9. Theorem. *Let $M=|K|$ be a (HML, n)-manifold, $n\geq 6$ and $\partial M\neq 0$. Then M is a TOP-manifold if and only if*

a) *For each vertex $v\in K-\partial K$, $\pi_1(\mathrm{lk}(v:K))=1$.*

b) *For each vertex $w\in\partial K$, $\pi_1(\mathrm{lk}(w:\partial K))=1$.*

c) *Any condition of those included in Theorem 3 or Proposition 7.*

10. *Remarks.* a) [2, 1.5] is Theorem 9 with 3.b) in the place 9.c).

b) [6, C] is Theorem 9 with 3.a) in the place 9.c)

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Dpto. Geometria y Topologia
Facultad de Matematicas
Tarfia s/n 41012 — Sevilla, Spain