ANALELE ȘTIINȚIFICE ALE UNIVERSITĂȚII „AL. I. CUZA" DIN IAȘI

Matematica, 36 (1990) fasc. 3

SUMAR

An. șt. Univ. „Al. I. Cuza" Iași
Matematica, 36 (1990)

ON ORDER PRINCIPAL MODULES

BY

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1. Introduction. Let M be a lattice ordered module over an Archimedean f-algebra A with unit. M has a natural Riesz space structure, and we assume throughout this paper that it is Archimedean. M is said to be *order principal A-module*, if for every $x \in M^+$ the Riesz subspace $Ax=\{ax : a \in A\}$ of M is order dense in the principal band B_x generated by x. The notion originated under the influence of [10]. We shall characterize order principal A-modules, and extend some Riesz space results to this more general framework.

It is well known that every order bounded linear operator on a Riesz space with the principal projection property commutes with all band projections of L if and only if it is an orthomorphism. As a consequence of a module characterization of principal bands in order principal modules we get a generalization of this result.

Let M be an order principal A-module and N a Dedekind complete Riesz space. It will be shown that the order continuous component T_n of a positive operator $T : M \to N$ can be given by formula

$$T_n u = \inf \sup \{T a_\alpha u : a_\alpha u \uparrow u,\ a_\alpha \in A^+\},\quad u \in M^+.$$

The largest ideal in M on which T is order continuous will be described in a similar manner.

For the terminology concerning Riesz spaces we refer the reader to [6], [1], and for elementary f-algebra theory we refer to [11] and [2].

2. Preliminaries. Let L be an Archimedean Riesz space. If a nonempty subset D of L is directed upward we shall write $D\uparrow$, while $D\uparrow u$ (with $u \in L$) means that also $u=\sup D$ holds. The meanings of $D\downarrow$ and of $D\downarrow u$ are analogous. Let us denote by $D^\uparrow$ the set of all elements $u \in L$ such that there exists some $C \subset D$ with $C\uparrow u$. Given the element $u \in L^+$, $\mathcal{B}_u$ will denote the Boolean algebra $\{v \in L : v \wedge (u-v)=0\}$ of all components of u, and $\mathcal{M}_u$ the Riesz subspace of L generated by $\mathcal{B}_u$.

Let K and L be Archimedean Riesz spaces. The ordered vector space of all order bounded linear maps from K to L will be denoted by $\mathcal{L}_b(K, L)$. A band preserving operator $T \in \mathcal{L}_b(L)$ is called an orthomorphism.

A Riesz algebra A is said to be an *f-algebra* whenever $a, b, c \in A^+$, $a \wedge b = 0$ implies $ac \wedge b = ca \wedge b = 0$. Any Archimedean f-algebra is commutative.

The set $\mathrm{Orth}(L)$ of all orthomorphisms of an Archimedean Riesz space L is (under pointwise defined operations) an Archimedean f-algebra with unit $I=\mathrm{id}$. The same holds for the order ideal $Z(L)$ generated by I in $\mathrm{Orth}(L)$ (and called the center of L), and for its subalgebra $\mathcal{M}(L)$ generated by the Boolean algebra $\mathcal{P}(L)$ of all band projections of L.

Let A be an Archimedean f-algebra with unit e. A is said to be *zero-dimensional*, if $\mathcal{M}_e$ is e-uniformly dense in the principal order ideal A_e generated by e. A is zero-dimensional if and only if the Kakutani representation space of A_e is a zero-dimensional topological space. See [5], where some equivalent descriptions are given.

In this work M will be an Archimedean lattice ordered module over an Archimedean f-algebra with unit e. In an abvious way M inherits a Riesz space structure. Consider the map

$$H: A \to \mathrm{Orth}(M), \qquad H(a)=ax, \qquad a\in A,\ x\in M.$$

To see that H is well defined observe first that $H(a)$ is an order bounded linear map. Then use the estimate

$$0\le a-a\wedge ne\le (1/n)a^2, \qquad a\in A^+, \qquad n\in \mathbf{N}$$

to prove that

$$ax\wedge y\le ax\wedge (y+(1/n)a^2x)\le (1/n)a^2x$$

holds for any pair of disjoint elements $x, y\in M^+$. It follows that $x\wedge y=0$ implies $ax\wedge y=0$ for any $a\in A^+$, therefore $H(a)$ is an orthomorphism. Note that H is a positive algebra homomorphism such that $H(e)=\mathrm{id}_M$. By [3, Corollary 5.5] H is a Riesz homomorphism, thus the map $(a, x)\to ax$ from $A\times M$ to M is a Riesz bimorphism (compare [10, p. 309]). Order continuity of the multiplication by $a\in A^+$ in M is implied by the same property of orthomorphism $H(a)$.

3. Order principal modules. Let M be an Archimedean lattice ordered A-module. Recall that M is said to be order principal, if for any $x\in M^+$ the subspace Ax of M is order dense in the principal band B_x.

Theorem 3.1. *The following statements are equivalent.*

(i) *M is order principal A-module.*

(ii) *M is order principal A_e-module.*

(iii) *$([0, e]x)^\uparrow=[0, x]$ holds for every $x\in M^+$.*

(iv) *For any pair of disjoint nonzero elements $x, y\in M^+$ there exists a set $D\subset[0, e]$ such that $D\uparrow$, $Dx\uparrow x$ and $Dy=\{0\}$.*

(v) *For any pair of disjoint nonzero elements $x, y\in M^+$ there exists $a\in A^+$ (or $a\in[0, e]$) such that $ax>0$ and $ay=0$.*

Proof. (i)$\Rightarrow$(ii). Let $0<y\in B_x$, $x\in M^+$, and take an element $a\in A$ such that $0<ax\le y$. Then $0<(|a|\wedge e)x\le y$ (since $ax=H(a)x\in B_x$), therefore (ii) holds.

(ii)$\Rightarrow$(iii). Take $y\in[0, x]$, choose a net a_α in A_e^+ with $a_\alpha x\uparrow y$, and note that $(a_\alpha\wedge e)x\uparrow y$.

(iii)$\Rightarrow$(iv). Let $x, y>0$ be disjoint elements of M. Put $u=x+y$, and take a subset C of $[0, e]$ such that $Cu\uparrow x$. The set D of all finite suprema of C

is upward directed and satisfies $Du\uparrow x$. Since $x\wedge y=0$, we get $Dy=\{0\}$ and $Dx\uparrow x$.

(iv)$\Rightarrow$(v). Obvious.

(v)$\Rightarrow$(i). Let $0<y\le x$. Take a real $\varepsilon>0$ such that $(y-\varepsilon x)^+>0$, and choose $a\in[0, e]$ satisfying $a(y-\varepsilon x)^+>0$ and $a(y-\varepsilon x)^-=0$. It follows from $\varepsilon x\le (y-\varepsilon x)^-+y$ that

$$(\varepsilon a)x\le a(y-\varepsilon x)^-+ay=ay\le y$$

and consequently $(\varepsilon a)x\le y$. On the other hand $(y-\varepsilon x)^+\le x$ implies $(\varepsilon a)x>0$, thus (i) follows.

If A is zero-dimensional, we may add some equivalent descriptions of an order principal A-module.

Theorem 3.2. *Let A be a zero-dimensional f-algebra and M a lattice ordered A-module. Then the following assertions are equivalent.*

(i) *M is order principal A-module.*

(ii) *For any pair of disjoint nonzero elements $x, y\in M^+$ there exists an idempotent $p\in\mathcal{B}_e$ such that $px>0$ and $py=0$.*

(iii) *For any pair of disjoint nonzero elements $x, y\in M^+$ there exists an upward directed set $D\subset\mathcal{B}_e$ such that $Dx\uparrow x$ and $Dy=\{0\}$.*

(iv) *M is order principal $\mathcal{M}_e$-module.*

Proof. (i)$\Rightarrow$(ii). For given disjoint elements $x, y\in M^+$ with $x>0$ pick $a\in[0, e]$ such that $ax>0$ and $ay=0$. Since A is zero-dimensional, we can find an increasing sequence of positive elements $a_n\in\mathcal{M}_e$ satisfying $0\le a-a_n\le(1/n)e$ for every $n\in\mathbf{N}$ (See [4]). Hence $0\le ax-a_nx\le(1/n)x$, and there exists $a_n\in\mathcal{M}_e$ such that $a_nx>0$, $a_ny=0$. Clearly this implies (ii).

(ii)$\Rightarrow$(iii). Let $x, y>0$ be disjoint elements of M. Put $D=\{p\in\mathcal{B}_e : py=0\}$ and assume that $z<x$ is an upper bound of Dx. Then we have

$$0\le(x-z)\wedge px\le(x-px)\wedge px=0 \text{ for every } p\in D.$$

By (ii) there exists an element $q\in D$ with $q(x-z)>0$, and it follows that $(x-z)\wedge qx=0$. This is a contradiction, thus $Dx\uparrow x$, and consequently (iii) holds.

(iii)$\Rightarrow$(iv). This follows from Theorem 3.1.

(iv)$\Rightarrow$(i). Evident. □

Let L be an Archimedean Riesz space. Observe that L is a lattice ordered module over zero-dimensional f-algebra $\mathcal{M}(L)$. By Theorem 3.2 L is an order principal $\mathcal{M}(L)$-module if and only if it possesses the following projection property (which can be added in a natural way to the properties considered in [9] and [4]).

(B) For every pair of disjoint nonzero elements $u, v\in L^+$ there exists a band projection P such that $Pu>0$ and $Pv=0$.

Since any Riesz space with sufficiently many projections possesses this property (See [6, p. 174]), an Archimedean Riesz space satisfying (B) will be called a *weak SMP-space*. It should be noted that an order principal module over a zero-dimensional f-algebra is a weak SMP-space.

An example of a weak SMP-space without sufficiently many projections is presented in [9, 3.4]. On the other hand, we shall see that if a weak

SMP-space possesses a weak order unit, then it has sufficiently many projections. A more general treatment is given next.

First, we extend the notion of algebraically rich center [7, Def. 1.14] to Archimedean lattice ordered A-module M. We shall say that A is algebraically rich for M if every nontrivial band B of M contains a nontrivial subset aM, $a\in A$. It can be easily seen that A is algebraically rich for M if and only if for any $x>0$, $x\in M$, there exists an $a\in[0, e]$ (where e is the unit of A) such that $\{0\}\neq aM\subset B_x$.

Lemma 3.3. *Let M be an Archimedean lattice ordered A-module with a weak order unit $w\in M^+$. If $a\in A$ and $x\in M$ satisfy the relation $aw\in B_x$, then $aM\subset B_x$.*

Proof. The preimage of the band B_x under the automorphism $H(a)$ is a band in M which contains w, hence equals M. $\square$

Proposition 3.4. *Let M be an Archimedean lattice ordered A-module. Consider the following conditions.*

(i) *A is algebraically rich for M.*

(ii) *M is an order principal A-module.*

Then (i) implies (ii). If M has a weak order unit, then (i) and (ii) are equivalent.

Proof. (i) implies (ii) by Theorem 3.1. For the converse assume that $w\in M^+$ is a weak order unit. If (ii) holds, the Riesz subspace Aw is order dense in M, hence for any $x>0$ from M there exists $a\in A^+$ such that $0<aw\leq x$. Therefore aw is contained in B_x, and by Lemma 3.3 $aM\subset B_x$. $\square$

An Archimedean Riesz space L has sufficiently many projections if and only if $\mathscr{M}(L)$ is algebraically rich for $\mathscr{M}(L)$-module L, so we get the following result.

Corollary 3.5. *An Archimedean Riesz space with a weak order unit has sufficiently many projections if and only if it is a weak SMP-space.* $\square$

4. Applications. The following module characterization of a principal band in an order principal A-module can be compared with [10, Theorem 4.1].

Theorem 4.1. *Let M be an order principal A-module. Then the following conditions are equivalent for any pair $x, y\in M^+$.*

(i) *$y\in B_x$.*

(ii) *For any $a\in A$ (or $a\in[0, e]$) $ax=0$ implies $ay=0$. If A is zero-dimensional, we may add condition*

(iii) *For any $p\in\mathscr{B}_e$ $px=0$ implies $py=0$.*

Proof. (i)$\Rightarrow$(ii). If $ax=0$ and $y\in B_x$ the order continuity of multiplication implies $0=a(y\wedge nx)\uparrow ay$ and (ii) follows.

(ii)$\Rightarrow$(i). Assume $0<y\in B_x$ and take $z>0$ such that $z\wedge x=0$, $z\leq y$. Since M is order principal, there exists $a\in[0, e]$ with $az>0$, $ax=0$. Thus $ay>0$, and (ii) does not hold.

To prove the remaining part of the proposition use Theorem 3.2. $\square$

Corollary 4.2. *Let M be an order principal A-module and let $T: M\to M$ be an order bounded linear map. Then T is a module homomorphism if and only if T is an orthomorphism.*

Proof. If T is A-linear, then $T(ax)=aTx$ for any $a\in A$ and $x\in M$, and therefore $ax=0$ implies a $Tx=0$. By Theorem 4.1 $Tx\in B_x$, hence T is an orthomorphism. If conversely, T is an orthomorphism, then $T(ax)=TH(a)x=$ $=H(a)Tx=aTx$, and the reverse implication follows. $\square$

As an easy consequence we get a generalization of a result well known for Riesz spaces with the principal projection property.

Corollary 4.3. *Let L be a weak SMP-space. Then an order bounded linear map $T\in\mathscr{L}_b(L)$ is an orthomorphism if and only if it commutes with all band projections of L.*

Proof. If T commutes with all band projections of L, then T is $\mathscr{M}(L)$-linear. $\square$

In further applications we need the following lemma.

Lemma 4.4. *Let M be an order principal A-module, $x\in M^+$, $x\geq y_\alpha\downarrow 0$ and $\varepsilon\in\mathbf{R}$, $\varepsilon>0$. Then the subset*

$$B=\{a\in[0, e]\cap A: (e-a)(y_\alpha-\varepsilon x)^+=0 \ \textit{for some} \ \alpha\}$$

of A is downward directed and satisfies $Bx\downarrow 0$. If A is zero-dimensional, we may replace B by $B\cap\mathscr{B}_e$.

Proof. Evidently B is downward directed. Let $w>0$ be a lower bound of Bx. Take y_α such that $(\varepsilon w-y_\alpha)^+>0$. By Theorem 3.1 there exists a decreasing net a_τ in $[0, e]$ satisfying

$$(e-a_\tau)(y_\alpha-\varepsilon x)^+=0 \ \text{and} \ a_\tau(y_\alpha-\varepsilon x)^-\downarrow_\tau 0.$$

Since $a_\tau\in B$, we have $a_\tau x\geq w$ for all τ, hence

$$0<(\varepsilon w-y_\alpha)^+\leq(\varepsilon w-a_\tau y_\alpha)^+\leq a_\tau(y_\alpha-\varepsilon x)^-\downarrow_\tau 0.$$

This contradiction establishes $Bx\downarrow 0$.

The remaining part of lemma can be proved in the same way using Theorem 3.2. $\square$

Let M be an Archimedean lattice ordered A-module, N a Dedekind complete Riesz space, and $T\in\mathscr{L}_b(M, N)$ a positive map. The proof of Theorem 88.6 in [11] shows that

$$M^{an}(T)=\{x\in M: |x|\geq y_\alpha\downarrow 0 \ \text{implies} \ Ty_\alpha\downarrow 0\}$$

is the largest ideal of M on which T is order continuous. Put

$$M^{an}_A(T)=\{x\in M: |a_\alpha x|\downarrow 0, \ a_\alpha\in A^+ \ \text{implies} \ T|a_\alpha x|\downarrow 0\}.$$

Proposition 4.5. *If M is an order principal A-module, then $M^{an}(T)=$ $=M^{an}_A(T)$. If in addition A is zero-dimensional, then*

$$M^{an}(T)=\{x\in M: |p_\alpha x|\downarrow 0, \ p_\alpha\in\mathscr{B}_e \ \text{implies} \ T|p_\alpha x|\downarrow 0\}.$$

Proof. Let $0\leq x\in M^{an}_A(T)$, $x\geq y_\alpha\downarrow 0$ and $\varepsilon>0$, $\varepsilon\in\mathbb{R}$. Assume $Ty_\alpha\downarrow\geq$ $\geq z\geq 0$. Take an arbitrary $a\in B$ (B from Lemma 4.4), and choose y_α such that $(e-a)(y_\alpha-\varepsilon x)^+=0$. It follows that

$$z\leq Ty_\alpha=Tay_\alpha+T(e-a)y_\alpha\leq Tax+\varepsilon Tx$$

and by Lemma 4.4 we get $Tax\downarrow_{a\in B}0$. Hence $0\leq z\leq\varepsilon Tx$ holds for any $\varepsilon>0$, therefore $z=0$. $Ty_\alpha\downarrow 0$, and consequently $M^{an}_A(T)\subset M^{an}(T)$. The reverse inclusion is clear, and the remaining part of the proof is similar. $\square$

As a special case we get the following generalization of [11, 88.21].

Corollary 4.6. *Let L be a weak SMP-space, N a Dedekind complete Riesz space, and $T \in \mathcal{L}_b(L, N)$ a positive map. Then*

$$\{f \in L : P_\alpha |f| \downarrow 0, \ P_\alpha \in \mathcal{P}(L) \text{ implies } TP_\alpha |f| \downarrow 0\}$$

is the largest ideal on which T is order continuous. □

Similarly we may generalize [11, 102.17] on locally convex solid Hausdorff weak SMP-spaces.

Also the following formula for an order continuous component T_n of a positive linear map T can be established in a similar way using for example [8] and Lemma 4.4.

Proposition 4.6. *Let M be an order principal A-module, N a Dedekind complete Riesz space, and $T \in \mathcal{L}_b(M, N)$ a positive operator. Then for every $x \in M^+$,*

$$T_n x = \inf \sup \{Ta_\alpha x : a_\alpha \in A^+, \ a_\alpha x \uparrow x\}.$$

If A is zero-dimensional, the elements a_α can be taken from $\mathcal{B}_e$. □

Acknowledgement. The author is indebted to the referee for suggesting a simplification of the proof of Lemma 3.3. which appeared in the original version of this paper.

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Received 14. IV. 1989

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Matematica, 36 (1990)

A NOTE ON ADJOINT ABELIAN OPERATORS ON $C_0(X)$

BY

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1. Introduction. The concept of adjoint abelian operators on Banach spaces was introduced by Stampfli [12] and further developed by Fleming ([3], [4]) and Sen ([8]—[11]) by semiinner product techniques. In Fleming [4], a characterization of adjoint abelian operators on $C(X)$, X a compact metric space, is given. This note extends this result to $C_0(X)$, where X denotes a locally compact Hausdorff space (throughout this note) and $C_0(X)$ is the Banach space of all continuous complex functions on X that vanish at infinity. For literature on semiinner products (s.i.p.) we refer to Bonsall ([1], [2]).

2. Adjoint Abelian Operators. For our characterization of adjoint abelian operators on $C_0(X)$, The following result plays a central role.

Theorem 2.1. *Let $A \neq 0$ be an adjoint abelian operator on $C_0(X)$. Then there exists a positive constant λ such that $A^2 = \lambda I$.*

Proof. For the proof we need the following lemmas.

Lemma 1. ([5], p. 218). *If C is compact, U is open, and $C \subset U$, then there exist sets C_0 and U_0 such that C_0 is a compact G_δ, U_0 is a σ-compact open set, and*

$$C \subset U_0 \subset C_0 \subset U.$$

Lemma 2. *If K_1 is a compact G_δ, K_2 is compact with $K_1 \cap K_2 = \varnothing$, and $0 < a < 1$, then there exists a function $h \in C_0(X)$ such that*

$$K_1 = \{t : h(t) = 1\}, \ K_2 \subset \{t : h(t) = a\}$$

and $0 \leq h(t) \leq 1$.

Lemma 2 is an easy consequence Theorems B and C of Halmos ([5], p. 216).

Proof of Lemma 2. Since X is a locally compact Hausdorff space there exist compact neighborhoods V_1 and V_2 of K_1 and K_2 respectively (i.e. $K_i \subset$ int V_i, $i = 1, 2$). such that $V_1 \cap V_2 = \varnothing$. By Theorem C of Halmos ([5], p. 217) there exists a continuous function g such that $0 \leq g \leq 1$, $K_1 = \{x : g(x) = 1\}$. Again, by Theorem B of Halmos ([5], p. 216), there exists a continuous function f such that $0 \leq f \leq 1$, $f(x) = 1$ for $x \in K_1$ and $f(x) = 0$ for $x \in X \setminus V_1$. Let $h_1(x) = \min(f(x), g(x))$. Then $h_1 \in C_0(X)$ and $K_1 = \{x : h_1(x) = 1\}$. As K_2 is compact, by Theorem B of Halmos there exists a continuous function h_2 such