From the definition of G and the properties of f we obtain

$$G\left(\sum_{k=1}^{i} |\nabla u_k|\right) - G\left(\sum_{k=1}^{i-1} |\nabla u_k|\right) = \int_{\sum_{k=1}^{i-1}|\nabla u_k|}^{\sum_{k=1}^{i}|\nabla u_k|} \frac{1}{f(x)}\,dx \leq |\nabla u_i| / f\left(\sum_{k=1}^{i-1} |\nabla u_k|\right) = \tag{14}$$

$$= |\nabla u_i| / f\left(\sum_{k=1}^{i-1} |\nabla u_k|\right) - |\nabla u_i| / f\left(\sum_{k=1}^{i} |\nabla u_k|\right) + |\Delta u_i| / f\left(\sum_{k=1}^{i} |\nabla u_k|\right).$$

From (14) we have

$$|\nabla u_i| / f\left(\sum_{k=1}^{i} |\nabla u_k|\right) \geq G\left(\sum_{k=1}^{i} |\nabla u_k|\right) - G\left(\sum_{k=1}^{i-1} |\nabla u_k|\right) - \left[1/f\left(\sum_{k=1}^{i} |\nabla u_k|\right) - \right.$$
$$\left. - 1/f\left(\sum_{k=1}^{i} |\nabla u_k|\right)\right] |\nabla u_i|. \tag{15}$$

Now substituting $i=1, \ldots, n$ on both sides of the inequality (15) and summing up we obtain

$$\sum_{i=1}^{n} |\nabla u_i| / f\left(\sum_{k=1}^{i} |\nabla u_k|\right) \geq G\left(\sum_{k=1}^{n} |\nabla u_k|\right) - \sum_{i=1}^{n} \left[1/f\left(\sum_{k=1}^{i-1} |\nabla u_k|\right) - 1/f\left(\sum_{k=1}^{i} |\nabla u_k|\right)\right] |\nabla u_i|. \tag{16}$$

Replacing k by i in the first term on the right side in (16) and using this bound in (13) we obtain the required inequality in (6). This completes the proof of Theorem 3.

Remark 4. The discrete analogues of Opial's inequality or its variants are established in [1, 4, 7–9] by using different techniques. As noted earlier in Remarks 2 and 3, the steps in the proof of Theorem 2 there is a wastage at (9) and (10) and in particular in specializing Theorem 2 as mentioned in Remark 2 is there a further wastage while using a Schwarz inequality in order to obtain a net constant in inequality (5). Similarly, we can see that, there is a wastage at (13) and (14), in the general case of the proof of Theorem 3.

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THE GAUGE-COVARIANT EXTENSION $g_{pq}=\frac{1}{2}D_pD_qF^2$ OF FINSLERIAN METRIC TENSOR

BY

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The usual Finslerian metric tensor (7) and hence all the usual Finsler geometry are not covariant under the gauge transformations (1), for they are based on the condition $C_{i0j}=0$ which is not gauge-covariant. To remedy this, we make the note that the Miron's regularity condition $C_{00i}=0$ is gauge-covariant, which suggests us to propose the explicity gauge-covariant and attractive definition (9) of the metric tensor. Nearest implications of the definition are considered.

In the previous publications [1–5] the gauge transformations

$$y^q = y^q(x^i,\ y'^p) \tag{1}$$

were introduced to extend the Finslerian methods and to bring them near real applications in the modern theoretical physics. In (1), x^i, mean the local components of points of the base manifold, and y^q are the components of tangent vectors with respect to the local natural frame; $i, j, \cdots p, q, \cdots = 1, 2, \cdots N$. The conditions rank $(\partial y^p/\partial y'^q) = N$ together with the positive first-degree homogeneity,

$$y^q(x,\ ky') = ky^q(x,\ y'), \quad k>0 \tag{2}$$

will be assumed. Sufficiently smoothness of all the objects under consideration will be tacitly implied. The transformations mean the local representation of restricted automorphisms of the tangent bundle, that is, the automorphisms which act as the identity on the base manifold. In other words, (1) represent arbitrary, nonlinear in general, one-to-one mappings of tangent bundle onto tangent bundle subject to the condition that the mappings do not shift the points of support x^i. It can also be said that (1) represent the one-to-one mappings of tangent spaces onto tangent spaces provided that *the mapping depend on the background points in a sufficiently smooth way*. The last circumstance makes it legitimate to apply to (1) the epithet 'gauge' in accordance with the terminology adopted in the theoretical physics (see, for example, [6–8]). Since y^q are vectors under the general local coordinate transformations

$$x^i = x^i(\bar{x}^j), \tag{3}$$

that is, $y^q=\bar{y}^p B^q_p$, we must stipulate that the representation (1) must transform under (3) as follows:

(4) $$y^q(x^i(\bar{x}^j),\ \bar{y}^p B^s_p)=B^q_r\bar{y}^r(\bar{x}^j,\ \bar{y}'^s),$$

where $B^q_r=\partial x^q/\partial\bar{x}^r$. The condition (4) means in fact that the diagram

$$\begin{tikzcd} y \arrow[r] \arrow[d] & \bar{y} \arrow[d] \\ y' \arrow[r] & \bar{y}' \end{tikzcd}$$

is commutative, that is, $\bar{y}'=\bar{y}'$. Differentiating (2) with respect to y' and then, putting $k=1$, we get the relation

(5) $$y^q=y^q_r y'^r$$

where $y^q_r=\partial y^q/\partial y'^r$, which shows that *the tangent vector is covariant under the homogeneous gauge transformations* (1) — (2).

Let $F\ (x,\ y)$ be a scalar function under (2), that is,

(6) $$\bar{F}(\bar{x},\ \bar{y}^q)=F(x(\bar{x}),\ \bar{y}^p B^q_p(\bar{x})),$$

which simultaneously possesses the property of positive homogeneity of degree one with respect to y. The usual Finslerian metric tensor is given by the definition

(7) $$g_{pq}=\tfrac{1}{2}\partial_p\partial_q F^2$$

$((\partial_p=\partial/\partial y^p)$. If we assume naturally that F is a *gauge scalar*, that is, has the transformation law

(8) $$F'(x,\ y')=F(x,\ y(x.\ y'))$$

under (1) [the assumption (8) is consistent with (6) because of (4)], we observe immediately that *the usual Finslerian metric tensor* (7) *is not a gauge-covariant essence.* This circumstance suggests us to change (7) by

(9) $$g_{pq}=\tfrac{1}{2}D_p D_q F^2\equiv D_p y_q\equiv\partial_p y_q-C^r_{qp}y_r,$$

where

(10) $$y_q=\tfrac{1}{2}\partial_q F^2,$$

so that $\partial_p y_q=g_{pq}$. *The covariant tangent covector is gauge-covariant,* that is,

(11) $$y'_q=y_p y^p_q,$$

as it ensuies directly from (8) after differentiating (8) with respect to y'^q. The covariant derivative D_p is constructed by using some (vertical) connection coefficients $C^q_{p\ r}(x,\ y)$. Namely, we set forth the following two laws of transformations for $\mathbf{C}$:

(12) $$B^s_q C^q_{pr}(\bar{x},\ \bar{y})=B^t_p B^u_r C^s_{tu}(x(\bar{x}),\ \bar{y}B(\bar{x}))$$

and

(13) $$\partial y^p_q/\partial y'^r+y^t_r y^s_q C^p_{st}(x,\ y(x,\ y'))-y^p_s C'^s_{qr}(x,\ y')=0$$

Here, (12) claims that $C_p{}^q{}_r$ is a tensor under (2), while (13) reads that $C_p{}^q{}_r$ are connection coefficients under (1). The gauge-covariant definitions of the form (9) — (10) were proposed on p. 255 —256 of [1] and used in the end of Sec. 6 of [4]. Below, the homogeneity $C_p{}^r{}_q(x,\ ky)=k^{-1}C_p{}^r{}_q(x,\ y)$ and the torsionless $C_p{}^r{}_q=C_q{}^r{}_p$ will be implied.

Contracting (13) by y'^r and noting that $y'^r\partial'_r y^p_q=0$ and $y'^r y^t_r=y^t$ because of the homogeneity assumed, we find that

(14) $$C_0{}^p{}_q\overset{\text{def}}{=\!=}y^r C_r{}^p{}_q$$

is a *gauge tensor*. As the same time, $C_p{}^r{}_q y_r$ is not a gauge tensor. The gauge-covariant nature of (14) is also obvious from the relation

(15) $$C_0{}^p{}_q=D_q y^p-\delta^p_q$$

(because of (5)). Contracting (9) by y^p and by $y^p y^q$, we get two gauge-covariant relations

(16) $$g_{pq}y^p=y_q-C_0{}^q{}_q y_p$$

and

(17) $$g_{pq}y^p y^q=F^2-C_0{}^p{}_q y_p y^q$$

We see that the relations

(18) $$g_{pq}y^q=y_p$$

and

(19) $$g_{pq}y^p y^q=F^2$$

of the usual Finslerian form will hold when *the Miron's regularity condition*

(20) $$C_0{}^p{}_q y_p=0$$

([9]; see also [3, 10] where the regularity condition was used) is postulated. The condition (20) will be assumed henceforth, so that the relations (18) and (19) will take place. On applying the operator D_q to (18), we can readily represent (14) as

(21) $$C_0{}^p{}_p=g^{pt}y^r D_q g_{rt}.$$

Next, we set forth the metric condition.

(22) $$D_q g_{rt}=0$$

obtaining

(23) $$C_0{}^p{}_q=0$$

from (21). The condition (23) is stronger than the proper regularity condition (20). From (22) and (9) we find that

$$C_{prq}=\overset{0}{C}_{prq}-\frac{1}{2}(\partial_p C_{r0q}+\partial_q C_{r0p}-\partial_r C_{p0q}), \tag{24}$$

where

$$\overset{0}{C}_{prq}=\frac{1}{2}\partial_p g_{rq} \tag{25}$$

is the ordinary Cartan torsion tensor and $C_{r0q}=y^p C_{rpq}$. So, *the metric and torsionless connection coefficients* $C_p{}^r{}_q$ *are constructed explicitely from* F *and* C_{r0q}, where the functions C_{r0q} are entirely independent of F in general. The associated gauge-covariant curvature tensor

$$F^{-2}S_r{}^s{}_{pq}=\partial_q C_r{}^s{}_p-\partial_p C_r{}^s{}_q-C_r{}^t{}_q C_t{}^s{}_p+C_r{}^t{}_p C_t{}^s{}_q \tag{26}$$

will distinguish from the Cartan's classical ansatz

$$F^{-2}\overset{0}{S}_r{}^s{}_{rq}=\overset{0}{C}_t{}^s{}_p\overset{0}{C}_r{}^t{}_q-\overset{0}{C}_t{}^s{}_q\overset{0}{C}_r{}^t{}_p. \tag{27}$$

The definition (26) *is a gauge-covariant tensor*, that is, posesses the tensor law of transformation under (1), while (27) is not.

An attractive particular choice of C_{r0q} is

$$C_{r0q}=a\overset{0}{h}_{pq} \tag{28}$$

where a is a constant and $\overset{0}{h}_{pq}=g_{pq}-l_pl_q$ with $l_p=y_p/F$. The substitution of (28) in (9) and (24) yields

$$g_{pq}=(l-a)\overset{0}{g}_{pq}+al_pl_q \tag{29}$$

and

$$C_{prq}=(l-a)\overset{0}{C}_{prq}+aF^{-1}l_r g_{pq} \tag{30}$$

respectively. The inverse of (29) will read

$$g^{pq}=\frac{1}{l-a}(\overset{0}{g}{}^{pq}-al^pl^q) \tag{31}$$

entailing

$$C_p{}^r{}_q=\overset{0}{C}_p{}^r{}_q+aF^{-1}l^r\overset{0}{g}_{pq} \tag{32}$$

and then

$$S_s{}^r{}_{pq}=\overset{0}{S}_s{}^r{}_{pq}+a(\overset{0}{h}_{sp}\overset{0}{h}{}^r_q-\overset{0}{h}_{sq}\overset{0}{h}{}^r_p). \tag{33}$$

The theory of indicatrix may be developed for the present approach by following the patterns of the usual theory. Indeed, let us introduce the projection factors l and u of the indicatrix $F(x, y)=1$ by the same way as was used in Sec. 1.2. of [1]. The calculations just similar to those performed in Sec. 1.2 of [1] will give the relation

$$F^{-2}S_{pqrs}=(R^*_{abcd}-g^*_{ac}g^*_{bd}+g^*_{ad}g^*_{bc})u^a_p u^b_q u^c_r u^d_s. \tag{34}$$

where $g^*_{ab}=g_{pq}u^p_a u^q_b$ is the induced Riemannian metric tensor of the indicatrix and R^*_{abcd} its associated Riemannian curvature tensor. The difference from the classical relation (1.1.122) of [1] is only that the tensor (26), instead of the classical expression (27), enters the left-hand side of (34).

In the particular case (28), any Finslerian metric function which was S3-like [1, 11] in sense of the classical definition (7) of the Finslerian metric tensor will be S3-like also in sense of the gauge-covariant definition (9). The assertion ensues immediately from (33).

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