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OVALOIDS OF CONSTANT WIDTH AND CONSTANT BRIGHTNESS

BY

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Let M be an ovaloid in the 3-dimensional Euclidean space with a continuous and positive curvature K. We suppose M is an oriented surface with the outward normal vector field N. Let $\varphi: M \to M$ be a mapping defined as follows

(1) $$N(\varphi)(x) = -N(x) \qquad \text{for each } x \in M.$$

Then we have

Lemma 1. *The mapping φ is a diffeomorphism.*

Proof. It is easy to see that $\varphi = L^{-1} \circ I \circ L$, where L denotes the Gauss mapping from M onto the unit sphere and I denotes the symmetry of the unit sphere with respect to its center.

For $x \in M$, let (u^1, u^2) denote the local coordinate system at x and (w^1, w^2) the local coordinate system at $\varphi(x)$. Then locally we have $N(\varphi^1(u^1, u^2), \varphi^2(u^1, u^2)) = -N(u^1, u^2)$, where $w^\alpha = \varphi^\alpha(u^1, u^2)$. Differentiating this equality with respect to u^α we get

(2) $$\varphi^\sigma_\alpha N_\sigma(\varphi) = -N_\alpha,$$

where

(3) $$N_\alpha = \frac{\partial N}{\partial u^\alpha}, \qquad N_\sigma(\varphi) = \frac{\partial N}{\partial w^\sigma} \circ \varphi, \qquad \varphi^\sigma_\alpha = \frac{\partial \varphi^\sigma}{\partial u^\alpha}.$$

Putting

(4) $$x_\rho = \frac{\partial x}{\partial u^\rho}, \qquad x_\xi(\varphi) = \frac{\partial x}{\partial w^\xi} \circ \varphi,$$
$$N_\alpha = l^\rho_\alpha x_\rho, \qquad N_\sigma(\varphi) = l^\xi_\sigma(\varphi) x_\xi(\varphi)$$

we get

(5) $$\varphi^\sigma_\alpha l^\xi_\sigma(\varphi) x_\xi(\varphi) = -l^\rho_\alpha x_\rho$$

and

(6) $$x_\nu(\varphi) = -\tilde{\varphi}^\alpha_\eta \tilde{l}^\eta_\nu(\varphi) l^\rho_\alpha x_\rho,$$

where

$$\tilde{l}^\eta_\nu(\varphi) l^\nu_\beta(\varphi) = \delta^\eta_\beta, \qquad \tilde{\varphi}^\sigma_\eta \varphi^\eta_\alpha = \delta^\sigma_\alpha.$$

Let us consider the vector field

(7) $$p = x - x \circ \varphi.$$

Putting

(8) $$p_\alpha = \partial p / \partial u^\alpha$$

we have

(9) $$p_\alpha = (\delta^\rho_\alpha + \varphi^\gamma_\alpha \tilde{\varphi}^\sigma_\eta \tilde{l}^\eta_\gamma(\varphi) l^\rho_\sigma) x_\rho .$$

Let

(10) $$P^\rho_\alpha = \delta^\rho_\alpha + \varphi^\gamma_\alpha \tilde{\varphi}^\sigma_\eta \tilde{l}^\eta_\gamma(\varphi) l^\rho_\sigma .$$

Then the determinant $\det (P^\rho_\alpha)$ is independent on the choice of the local coordinate systems and

$$\det (P^\rho_\alpha) = (1 + l^1_\sigma \tilde{\varphi}^\sigma_\eta \tilde{l}^\eta_\gamma(\varphi)\varphi^\gamma_1)(1 + l^2_\alpha \tilde{\varphi}^\alpha_\beta \tilde{l}^\beta_\gamma(\varphi)\varphi^\gamma_2) - l^1_\sigma \tilde{\varphi}^\sigma_\eta \tilde{l}^\eta_\gamma(\varphi)\varphi^\gamma_2 l^2_\alpha \tilde{\varphi}^\alpha_\beta \tilde{l}^\beta_\gamma(\varphi)\varphi^\gamma_1 =$$

$$= 1 + l^1_\sigma \tilde{\varphi}^\sigma_\eta \tilde{l}^\eta_\gamma(\varphi)\varphi^\gamma_1 + l^2_\alpha \tilde{\varphi}^\alpha_\beta \tilde{l}^\beta_\gamma(\varphi)\varphi^\gamma_2 + l^1_\sigma \tilde{\varphi}^\sigma_\eta \tilde{l}^\eta_\gamma(\varphi) l^2_\alpha \tilde{\varphi}^\alpha_\beta \tilde{l}^\beta_\gamma(\varphi)(\varphi^\gamma_1\varphi^\gamma_2 - \varphi^\gamma_2\varphi^\gamma_1) =$$

$$= 1 + \mathrm{tr}\,(l\tilde{\varphi}' \tilde{l}(\varphi)\varphi') + \det \varphi' [l^1_\sigma \tilde{\varphi}^\sigma_\eta l^2_\alpha \tilde{\varphi}^\alpha_\beta (\tilde{l}^\eta_1(\varphi)\tilde{l}^\beta_2(\varphi) - \tilde{l}^\eta_2(\varphi)\tilde{l}^\beta_1(\varphi))] =$$

$$= 1 + \mathrm{tr}\,(l\tilde{\varphi}' \tilde{l}(\varphi)\varphi') + \det \varphi' . \det \tilde{l}(\varphi) . (l^1_\sigma \tilde{\varphi}^\sigma_1 l^2_\alpha \tilde{\varphi}^\alpha_2 - l^1_\sigma \tilde{\varphi}^\sigma_2 l^2_\alpha \tilde{\varphi}^\alpha_1) =$$

$$= 1 + \mathrm{tr}\,(l\tilde{\varphi}' \tilde{l}(\varphi)\varphi') + \det \tilde{l}(\varphi) . \det l = 1 + \mathrm{tr}\,(l\tilde{\varphi}' \tilde{l}(\varphi)\varphi') + K/K(\varphi),$$

where

(11) $$l = (l^\alpha_\beta),\ l(\varphi) = (l^\alpha_\beta(\varphi)),\ \tilde{l}(\varphi) = (l(\varphi))^{-1},$$
$$\varphi' = (\varphi^\alpha_\beta),\ \tilde{\varphi}' = (\varphi^\alpha_\beta)^{-1}.$$

Thus we have

(12) $$\det (P^\beta_\alpha) = 1 + \mathrm{tr}\,(l\tilde{\varphi}' \tilde{l}(\varphi)\varphi') + K/K(\varphi).$$

Putting

(13) $$g_{\rho\sigma} = \langle x_\rho, x_\sigma \rangle,\quad g_{\xi\eta}(\varphi) = \langle x_\xi(\varphi), x_\eta(\varphi) \rangle,$$

where $<,>$ denotes the euclidean scalar product we get from (6):

$$\varphi^\sigma_\beta \varphi^\rho_\alpha l^\xi_\sigma(\varphi) l^\eta_\rho(\varphi) g_{\xi\eta}(\varphi) = l^\rho_\alpha l^\sigma_\beta g_{\rho\sigma}.$$

Knowing that $l^\eta_\rho = b_{\rho\alpha} g^{\alpha\eta}$ and $l^\eta_\rho(\varphi) = b_{\rho\alpha}(\varphi) g^{\alpha\eta}(\varphi)$, where $b = (b_{\rho\alpha})$ $(b(\varphi) = (b_{\rho\alpha}(\varphi)))$ is the matrix of the second fundamental form of M with respect to u^α (with respect to u^α composed with φ), we get

(14) $$l^\rho_\alpha b_{\beta\rho} = \varphi^\rho_\alpha \varphi^\sigma_\beta l^\xi_\sigma(\varphi) b_{\rho\xi}(\varphi).$$

The relation (14) implies that

$$\det^2 \varphi' . \det^2 b(\varphi) / \det g(\varphi) = \det^2 b / \det g$$

where

$$g = (g_{\alpha\beta}),\qquad g(\varphi) = (g_{\alpha\beta}(\varphi)).$$

Since φ is an orientation preserving diffeomorphism, so

(15) $$\det \varphi' = K\sqrt{\det g} / (K(\varphi)\sqrt{\det g(\varphi)}).$$

Theorem 2. *The following integral formula holds*

(16) $$\int_M \frac{K}{K(\varphi)}\, dM = S,$$

where S denotes the surface area of M and dM is the volume form of M.

Proof. It follows from (15) that

$$S = \int_M dM = \int_M \frac{K}{K(\varphi)}\, dM.$$

Definition 1. *A pair $(x, \varphi(x))$ will be called a π-pair if and only if the vectors $x\varphi(x)$ and $N(x)$ are parallel.*

Let us suppose that M is a convex body of a constant width h. In this case each point of M belongs to a π-pair, [2]. In our notations $h = \langle p, N \rangle$ and $0 = \langle p_\alpha, N \rangle + \langle p, N_\alpha \rangle = \langle p, l^\sigma_\alpha x_\sigma \rangle = \langle p, x_\sigma \rangle l^\sigma_\alpha$. Since $\det (l^\sigma_\alpha) \neq 0$, so $h =$ const. iff $\langle p, x_\sigma \rangle = 0$ for $\sigma = 1, 2$, but this exactly means that any point $x \in M$ belongs to a π-pair.

Theorem 3. *If M is a convex body of constant width h, then*

(17) $$Kh^2 = K/K(\varphi) + 2hH - 1.$$

Proof. We know that $p = hN$ and $p_\alpha = hN_\alpha = hl^\rho_\alpha x_\rho$. Using (9) we get

(18) $$\delta^\rho_\alpha + \varphi^\gamma_\alpha \tilde{\varphi}^\sigma_\eta \tilde{l}^\eta_\gamma(\varphi) l^\rho_\sigma = hl^\rho_\alpha .$$

But $\mathrm{tr}(l^\sigma_\alpha) = 2H$, so $\mathrm{tr}(l\tilde{\varphi}', \tilde{l}(\varphi)\varphi') = 2(hH - 1)$ and from (12) we get (17).

Theorem 4. *For a convex body M of constant width h the following relations hold*

(19) $$\int_M H dM = 2\pi h,$$ (20) $$H > 1/h,\quad S < 2\pi h^2.$$

Proof. From (17) we get

$$h^2 \int_M K dM = \int_M \frac{K}{K(\varphi)}\, dM + 2h \int_M H dM - \int_M dM.$$

Taking into account Theorem 2 we can see that the formula (19) holds. The equality (18) implies

$$\varphi^\gamma_\alpha \tilde{\varphi}^\sigma_\eta \tilde{l}^\eta_\gamma l^\rho_\sigma = hl^\rho_\alpha - \delta^\rho_\alpha ,$$

so

$$\mathrm{tr}(l\tilde{\varphi}' \tilde{l}(\varphi)\varphi') = 2hH - 2 > 0,$$

what gives the first inequality (20). Moreover, we have

$$0 < \int_M (hH - 1) dM = 2\pi h^2 - S.$$

Let us consider a family $LS(M)$ of all regular lines of shadow on an ovaloid M, see [3].

Theorem 5. *Let $C \in LS(M)$. We have*

(21) $$\oint k_g ds = 0,$$

where k_g denotes the geodesic curvature of C.

Proof. A line C divides M onto two regions. If one of them we denote by A, then the second is $\varphi(A)$. From the formula (15) we get

$$\int_A KdM = \int_{\varphi(A)} KdM.$$

Making use of the Gauss-Bonnet theorem we have

$$\int_A KdM + \oint k_g ds = 2\pi \text{ and } \int_{\varphi(A)} KdM - \oint k_g ds = 2\pi.$$

Comparing all three relations we get $\oint k_g ds = 0$.

Theorem 6. *A line of shadow on an ovaloid M of constant width divides M onto two regions with equal areas if and only if the total mean curvatures of these regions are equal.*

Proof. Let $C \in LS(M)$. We denote by $S(A)$ and $S(\varphi(A))$ the surface area of the regions A and $\varphi(A)$, respectively. From the proof of Theorem 5 it follows that $\int_A KdM = 2\pi$. The formulas (15), (17) and (19) imply the relation

$$S(A) - S(\varphi(A)) = \int_A (1 - K/K(\varphi))dM = 2h\int_A HdM - h^2\int_A KdM =$$

$$= 2h\int_A HdM - h^2 2\pi = h(2\int_A HdM - \int_M HdM) = h(\int_A HdM - \int_{\varphi(A)} HdM).$$

It ends the proof.

We recall a definition of constant brightness.

Definition 2. *An ovaloid M is said to be of constant brightness if the area of orthogonal projection of M onto the plane does not depend on the choice of the plane.*

G. Herglotz [1] has proved the following theorem. *An ovaloid is of constant brightness if and only if*

$$1/K + 1/K(\varphi) = c = \text{const}. \tag{22}$$

Theorem 7. *The surface area S of an avaloid M of constant brightness satisfying* (22) *is given by the formula*

$$S = 2\pi c, \tag{23}$$

Proof. We have $\int_M KdM = 4\pi$ and $\int_M \frac{K}{K(\varphi)} dM = S$ (Theorem 2). From (22) we get

$$Kc = 1 + K/K(\varphi). \tag{24}$$

The integration of both sides of (24) leads us to the formula (23).

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A PSEUDO-RIEMANNIAN STRUCTURE ON THE COTANGENT BUNDLE

BY

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In [2] the first author has studied the properties of a certain pseudo-Riemannian structure on the tangent bundle of a Lagrange space, using a nonlinear connection defined by the Euler-Lagrange equations. In [3] the authors have considered the same problem in the case of the cotangent bundle of a Lagrange space by using the image of the above nonlinear connection by the Legendre transformation. Remark that the obtained pseudo-Riemannian structure is very much similar to the Riemann extension considered in [4] (see also [5] pp. 258, [6]).

In the present paper we study the properties of a similar pseudo-Riemannian metric G on the cotangent bundle T^*M of a manifold M by using an arbitrary symmetric nonlinear connection on this bundle. We find the Levi Civita connection $\tilde{\nabla}$ of this pseudo-Riemannian metric. It follows that the vertical distribution on the cotangent bundle is preserved by the parallel transport defined by $\tilde{\nabla}$, while the deviation from the parallel transport of the horizontal distribution is strongly related to the nonholonomy of this distribution. We study some problems related to the curvature of the connection $\tilde{\nabla}$: Bianchi identities, the conditions under which $\tilde{\nabla}$ is flat or conformally flat and the conditions under which (T^*M, G) is a locally symmetric pseudo-Riemannian space. In all these cases we find that the connection coefficients of the nonlinear connection on T^*M are expressed as polynomials of degree at most two in the cotangential coordinates and we get the conditions which are fulfilled by the coefficients.

The manifolds, tensor fields and connections we consider in this paper are assumed to be smooth (i.e. C^∞). We use the well known summation convention, the range for the indices a, b, c, h, i, j, k, l being always $\{1, ..., n\}$. $\mathfrak{X}(M)$ denotes the set of the smooth vector fields defined on M.

1. The nonlinear connection on T^*M. Let M be an n-dimensional manifold and denote by $\pi : T^*M \to M$ its cotangent bundle. Then T^*M is a $2n$-dimensional manifold and some special local charts on T^*M, induced from local charts on M may be used. Namely, if (U, x^i), $i = 1, ..., n$, is a local chart on M then the local chart $(\pi^{-1}(U), (q, p)) = (\pi^{-1}(U), q^i, p_i)$; $i = 1, ..., n$, is induced on T^*M where $q^i = x^i \circ \pi$ and p_i are the vector space coordinates of an element from $\pi^{-1}(U)$ with respect to the local natural frame $(dx^1, ..., dx^n)$ of T^*M defined by the local chart (U, x^i). The tangent bundle TT^*M of T^*M has an integrable vector subbundle $VT^*M = \mathrm{Ker}\, \pi_*$, called the vertical